

Pacific Journal of Mathematics

EXAMPLES OF HEREDITARILY l^1 BANACH SPACES FAILING THE SCHUR PROPERTY

PARVIZ AZIMI AND JAMES NEIL HAGLER

EXAMPLES OF HEREDITARILY l^1 BANACH SPACES FAILING THE SCHUR PROPERTY

PARVIZ AZIMI AND JAMES N. HAGLER

A class of separable Banach sequence spaces is constructed. A member X of this class (i) is a hereditarily l^1 dual space which fails the Schur property, and (ii) is of codimension one in its first Baire class. A consequence of (ii) is that X is not isomorphic to the square of any Banach space Y .

Introduction. In this paper we introduce and study a new class of Banach sequence spaces, the X_α spaces. The definition of the norm in a particular X_α space depends on the action of special sequences of intervals of integers on a vector $x = (t_1, t_2, \dots)$ (as in the definition of the James space J [6]) in conjunction with a fixed sequence of weighting factors (as in the Lorentz sequence spaces [7].)

Let X denote a specific X_α space, and let (e_i) denote the sequence of usual unit vectors in X (i.e. $e_i(j) = \delta_{ij}$ for integers i and j). Our main result is the following:

THEOREM 1. (1) X is hereditarily l^1 .

(2) The sequence (e_i) is a normalized boundedly complete basis for X . Thus, X is a dual space.

(3) (i) The sequence (e_i) is a weak Cauchy sequence in X with no weak limit in X . In particular, X fails the Schur property. (ii) There is a subspace X_0 of X which fails the Schur property, yet which is weakly sequentially complete.

(4) Let $B_1(X)$ denote the first Baire class of X in its second dual, i.e.,

$$B_1(X) = \{x^{**} \in X^{**} : x^{**} \text{ is a weak}^* \text{ limit of a sequence } (x_n) \text{ in } X\}$$

Then $\dim B_1(X)/X = 1$.

Part (4) shows that the space X has properties analogous to those of the quasireflexive spaces of James. Since $\dim B_1(X)/X$ is an isomorphism invariant, we have the following immediate consequences of the Theorem.

COROLLARY 2. (1) *For any n and any Banach space Y , X is not isomorphic to Y^n . In particular, X is not isomorphic to its square.*

(2) *For any $n > 1$, X^n does not imbed isomorphically in X .*

(3) *Let $X = A \oplus B$. Then exactly one of A or B is weakly sequentially complete and the other is of codimension one in its first Baire class.*

The properties of the X_α spaces provide an interesting contrast to the work in the paper [5], where an example of a separable Banach space which has the Schur property yet fails the Radon-Nikodym property is given. The spaces presented here were designed (in part) so that the combinatorial considerations encountered in [5] could be avoided.

In addition to the James space and the Lorentz sequence spaces mentioned above, the X_α spaces owe their origin to the space of Maurey and Rosenthal [8]. A class of examples (unpublished), similar to the X_α spaces, was constructed independently by E. Odell.

The existence of hereditarily l^1 Banach spaces failing the Schur property was shown first by Bourgain [3]. However, the analysis of the X_α spaces is self contained and particularly straightforward. For example, the basic sequences which are equivalent to the usual basis of l^1 are explicitly constructed, and there is no use of Rosenthal's characterization [9] of Banach spaces containing l^1 .

Except as indicated below, our terminology and notation are standard. The reader is referred to the books of Day [4] and Lindenstrauss and Tzafriri [7] for standard reference material on Banach spaces.

The authors would like to thank S. Bellenot, E. Odell, and H. P. Rosenthal for suggestions and discussions regarding the current paper.

Preliminaries. In this section the definition of the X_α spaces is given. First, by a block we mean an interval F (finite or infinite) of integers. For a block F and $x = (t_1, t_2, \dots)$ a sequence of scalars such that $\sum_j t_j$ converges, define $\langle x, F \rangle = \sum_{j \in F} t_j$.

To define the norm, we consider special sequences of blocks and special sequences of nonnegative reals. Specifically, we call a sequence (finite or infinite) $F_1, F_2, \dots, F_n, \dots$ (where each F_i is a finite block) *admissible* if

$$\max F_i < \min F_{i+1} \quad \text{for } i = 1, 2, 3, \dots$$

Let us now consider a sequence α of nonnegative reals (α_i) (whose terms are used as weighting factors in the definition of the norm) which

satisfies the following properties:

- (1) $\alpha_1 = 1$ and $\alpha_{i+1} \leq \alpha_i$ for $i = 1, 2, \dots$
- (2) $\lim_{i \rightarrow \infty} \alpha_i = 0$.
- (3) $\sum_{i=1}^{\infty} \alpha_i = \infty$.

For $x = (t_1, t_2, t_3, \dots)$ a finitely nonzero sequence of scalars, define

$$\|x\| = \max \sum_{i=1}^n \alpha_i |\langle x, F_i \rangle|$$

where the max is taken over all n , and admissible sequences F_1, F_2, \dots, F_n . Let $X (= X_{(\alpha_i)})$ be the completion of the finitely non zero sequences of scalars $x = (t_1, t_2, \dots)$ in this norm. An X_α space is a Banach space constructed in this fashion from some sequence α satisfying (1)–(3) above.

REMARK. Property (3) of the sequence (α_i) is introduced to insure a new class of spaces. Indeed, if we consider sequences (α_i) which satisfy (1) and

(2') there is a $\delta > 0$ such that $\alpha_i > \delta$ for all i , then the spaces X we obtain are all isomorphic to l^1 . If we require (1), (2) and

(3') $\sum_{i=1}^{\infty} \alpha_i < \infty$,

then the spaces X are all isomorphic to c_0 .

Proofs of the results. For the remainder of the paper let us pick and fix a sequence (α_i) satisfying (1)–(3) above, and let $X = X_{(\alpha_i)}$. This section contains the analysis of the stucture of the space X .

What we will show in the proof of Theorem 1 is that an l^1 subspace of X is obtained by considring block basic subsequences (u_i) of (e_i) which have the property (roughly) that the number of sets m in an admissible sequence F_1, F_2, \dots, F_m needed to norm u_n goes to ∞ as $n \rightarrow \infty$.

Before beginning our detailed analysis, we collect some basic facts about the space X into the following lemma:

LEMMA 3. (a) *The sequence (e_i) forms a monotone, subsymmetric basis for the space X . (Recall that a basic sequence is subsymmetric if it is equivalent to each of its subsequences.)* (b) *For each integer n ,*

$$\left\| \sum_{i=1}^n (e_{2i-1} - e_{2i}) \right\| = \sum_{i=1}^{2n} \alpha_i.$$

The proof of part (a) of the lemma follows immediately from the definition of the norm in X . Part (b) follows from the obvious selection of the admissible sequence $F_i = \{i\}$ for $i = 1, 2, \dots, 2n$.

This next simple lemma provides the key to the analysis of the space X .

LEMMA 4. *Let the sequence (α_i) be as above, let $n_0 > 0$ be an integer and let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that, if b_1, b_2, \dots, b_n are ≥ 0 , $b_i < \delta$ for all i , and $\sum_{i=1}^n \alpha_i b_i = 1$, then $\sum_{i=1}^n \alpha_{i+n_0} b_i \geq 1 - \varepsilon$.*

Proof. The series of nonnegative reals $\sum_{i=1}^\infty [\alpha_i - \alpha_{i+n_0}]$ converges, say to c . So, for any n , $\sum_{i=1}^n [\alpha_i - \alpha_{i+n_0}] \leq c$. Thus,

$$\sum_{i=1}^n [\alpha_i - \alpha_{i+n_0}] b_i \leq [\max b_i] \cdot c < \varepsilon$$

if $\max b_i$ is small enough.

Lemma 4 provides us with a tool for calculating the norm of linear combinations of vectors in terms of the norms of the individual components. We apply this to obtain a criterion for a sequence of vectors to have a subsequence which is equivalent to the usual basis of l^1 .

For $x \in X$, put $s(x) = \max | \langle x, G \rangle |$ where the max is taken over all blocks G .

LEMMA 5. *Let (u_i) be a sequence of norm one vectors in X and (G_i) an admissible sequence of blocks such that $\{j: u_i(j) \neq 0\} \subset G_i$. For each i , put $s_i = s(u_i)$. If $\lim_{i \rightarrow \infty} s_i = 0$, then a subsequence (v_k) of (u_k) is equivalent to the usual basis of l^1 .*

Proof. We select the sequence (v_k) by induction. Let $v_1 = u_1$. Pick n_1 and admissible blocks F_1, F_2, \dots, F_{n_1} satisfying $\max F_{n_1} = \max G_1$ and $\sum_{i=1}^{n_1} \alpha_i | \langle v_1, F_i \rangle | = \|v_1\| = 1$. Let δ_1 be any δ guaranteed by Lemma 4 for the integer n_1 and $\varepsilon = 1/2$. (To simplify notation in the remainder of the proof, let $n_0 = 0$.)

Assume now that we have selected for $k = 1, \dots, p - 1$

- (1) an integer $m_k (> m_{k-1})$ so that $v_k = u_{m_k}$.

(2) an integer $n_k (> n_{k-1})$, blocks $F_{n_{k-1}+1}, \dots, F_{n_k}$ and $\delta_k > 0$ such that

(a) $\max F_{n_k} = \max G_{m_k}$.

(b) The sequence $F_1, F_2, \dots, F_{n_1}, \dots, F_{n_2}, \dots, F_{n_k}$ is admissible.

(c) $\sum_{i=1}^{n_k-n_{k-1}} \alpha_i |\langle v_k, F_i \rangle| = \|v_k\| = 1$.

(d) δ_k is any δ guaranteed by Lemma 4 for the integer n_{k-1} and $\varepsilon = 1/2$.

Now let $\delta_p > 0$ be any δ guaranteed by Lemma 4 for the integer n_{p-1} and $\varepsilon = 1/2$. Pick $m_p (> m_{p-1})$ so that $s_{m_p} < \delta_p$ and let $v_p = u_{m_p}$. Finally, pick blocks $F_{n_{p-1}+1}, \dots, F_{n_p}$ such that (a), (b) and (c) above are satisfied for v_p and G_{m_p} . This completes the induction process.

Observe that $|\langle v_k, F_{i+n_{k-1}} \rangle| < s_{n_k} < \delta_k$ for $i = 1, \dots, n_k - n_{k-1}$. By Lemma 4,

$$\sum_{i=1}^{n_k-n_{k-1}} \alpha_{i+n_{k-1}} |\langle v_k, F_{i+n_{k-1}} \rangle| > \frac{1}{2}.$$

This inequality can be rewritten as

$$\sum_{i=n_{k-1}+1}^{n_k} \alpha_i |\langle v_k, F_i \rangle| > \frac{1}{2}.$$

Now, let scalars t_1, t_2, \dots, t_k be given. Since the sequence F_1, \dots, F_{n_k} is admissible, it follows from the observation above that

$$\begin{aligned} \left\| \sum_{j=1}^n t_j v_j \right\| &\geq \sum_{j=1}^{n_n} \alpha_j \left| \left\langle \sum_{j=1}^k t_j v_j, F_j \right\rangle \right| \\ &= \sum_{j=1}^k |t_j| \sum_{i=n_{j-1}+1}^{n_j} \alpha_i |\langle v_j, F_i \rangle| > \frac{1}{2} \sum_{j=1}^k |t_j|. \end{aligned}$$

Thus, the sequence (v_k) is equivalent to the usual basis of l^1 .

Proof of Theorem 1 (1). By standard perturbation arguments, we need only establish the result for norm one vectors (u_i) and blocks (G_i) with $\max G_i < \min G_{i+1}$ such that $\{j: u_i(j) \neq 0\} \subset G_i$.

Let (s_i) be as in the statement of Lemma 5. If some subsequence of $(s_i) \rightarrow 0$, then we're done. If not, then there is a $\delta > 0$ such that, for each i , there is a block F_i with $F_i \subset G_i$ and $|\langle u_i, F_i \rangle| > \delta$.

Select a sequence of (u_i) (which we don't rename) so that $\lim_{i \rightarrow \infty} \langle u_i, N \rangle$ exists. Put $v_i = u_{2i-1} - u_{2i}$. Then $\|v_i\| \leq 2$ and $\lim_{i \rightarrow \infty} \langle v_i, N \rangle = 0$. By passing to a subsequence of (v_i) and again not

renaming, we may assume that

$$\sum_{j=1}^{\infty} |\langle v_j, N \rangle| \leq 1.$$

Thus, if F is any block, and $m \leq n$, it follows that

$$\left| \sum_{j=m}^n \langle v_j, F \rangle \right| \leq 5.$$

To see this, suppose that H_1, H_2, \dots is an admissible sequence of blocks, so that each v_i is supported in H_i (i.e. $\{j: v_i(j) \neq 0\} \subset H_i$.) Pick i_0 and j_0 so that $\text{inf } F \in H_{i_0}$ and $\text{sup } F \in H_{j_0}$. Then (since $|\langle x, F \rangle| \leq \|x\|$ for any block F) it follows that

$$\begin{aligned} \left| \sum_{j=m}^n \langle v_j, F \rangle \right| &\leq |\langle v_{i_0}, F \rangle| + \sum_{j=i_0+1}^{j_0-1} |\langle v_j, F \rangle| + |\langle v_{j_0}, F \rangle| \\ &\leq \|v_{i_0}\| + 1 + \|v_{j_0}\| \leq 5. \end{aligned}$$

Finally, we show that for any subsequence (z_i) of (v_i) , $\|z_1 + \dots + z_n\| \rightarrow \infty$ as $n \rightarrow \infty$. For each i pick a block $F_i \subset H_i$ such that $|\langle z_i, F_i \rangle| > \delta$ and $\langle z_j, F_i \rangle = 0$ if $j \neq i$. Clearly, the sequence F_1, F_2, \dots is admissible. So, if $z^n = z_1 + \dots + z_n$,

$$\|z^n\| \geq \sum_{i=1}^n \alpha_i |\langle z^n, F_i \rangle| \geq \sum_{i=1}^n \alpha_i |\langle z_i, F_i \rangle| \geq \delta \sum_{i=1}^n \alpha_i.$$

Thus, $\|z^n\| \rightarrow \infty$ as $n \rightarrow \infty$.

Now, observe that if F is any block,

$$\left| \left\langle \frac{z^n}{\|z^n\|}, F \right\rangle \right| = \frac{1}{\|z^n\|} |\langle z^n, F \rangle| \leq \frac{5}{\|z^n\|} \rightarrow 0$$

as $n \rightarrow \infty$.

At last we are ready to select a sequence (x_i) equivalent to the usual basis of l^1 . Let $n_1 = 1$. Inductively pick n_{k+1} so that $\|v_{n_{k+1}} + \dots + v_{n_{k+1}}\| \geq 5 \cdot 2^k$.

Let $x_1 = v_1/\|v_1\|$ and, for $k > 1$, let

$$x_{k+1} = \frac{v_{n_{k+1}} + \dots + v_{n_{k+1}}}{\|v_{n_{k+1}} + \dots + v_{n_{k+1}}\|}$$

Then $\|x_k\| = 1$, and the sequence (x_k) satisfies the hypotheses of Lemma 5 for some admissible sequence G_1, G_2, \dots , so a subsequence of (x_k) is equivalent to the usual basis of l^1 .

Proof of Theorem 1 (2). Suppose that (t_j) is a sequence of scalars such that, for each integer n , $\|\sum_{j=1}^n t_j e_j\| \leq 1$, yet $\sum_{j=1}^\infty t_j e_j$ does not converge.

Without loss of generality, we may assume that

(i) $\sup \|\sum_{j=1}^n t_j e_j\| = 1$.

(ii) There exists an $\epsilon > 0$, such that if m is any integer, there is a $k > m$ with $\|\sum_{j=m}^k t_j e_j\| > \epsilon$.

We claim that for every $\delta > 0$, there is an integer n such that, if F is a block with $\min F > n$, then $|\langle \sum_{j=1}^\infty t_j e_j, F \rangle| < \delta$. Let us assume for the moment that the claim has been established and finish the proof of (2).

Using property (i), we first find an integer p_0 such that, if $x = \sum_{j=1}^{p_0} t_j e_j$, then $\|x\| > 1 - \epsilon/4$. Now pick an admissible sequence F_1, F_2, \dots, F_{n_0} such that

$$\|x\| = \sum_{i=1}^{n_0} \alpha_i |\langle x, F_i \rangle|.$$

Let $\delta > 0$ be any δ guaranteed by Lemma 4 for $\epsilon = 1/2$ and the integer n_0 . Using the claim, pick $p_1 > p_0$ so that if F is any block with $\min F \geq p_1$, then $|\langle \sum_{j=1}^\infty t_j e_j, F \rangle| < \delta$.

Let $y = \sum_{j=p_1}^k t_j e_j$ be chosen so that $\|y\| > \epsilon$, as guaranteed by (ii). Pick blocks G_1, G_2, \dots, G_s such that $\min G_1 \geq p_1$ and

$$\|y\| = \sum_{i=1}^s \alpha_i |\langle x, G_i \rangle|.$$

Observe that $|\langle x, G_i \rangle| < \delta$ for all $i = 1, \dots, s$. Thus, by the choice of δ ,

$$\sum_{i=1}^s \alpha_{i+n_0} |\langle x, G_i \rangle| \geq \frac{\epsilon}{2}.$$

Then the sequence $F_1, F_2, \dots, F_{n_0}, G_1, \dots, G_s$ is admissible, and

$$\begin{aligned} \left\| \sum_{i=1}^k t_i e_i \right\| &\geq \sum_{i=1}^{n_0} \alpha_i |\langle x, F_i \rangle| + \sum_{i=n_0+1}^{n_0+s+1} \alpha_i |\langle y, G_{i-n_0} \rangle| \\ &\geq 1 - \epsilon/4 + \sum_{i=1}^s \alpha_{i+n_0} |\langle x, G_i \rangle| \geq 1 - \epsilon/4 + \frac{\epsilon}{2} > 1. \end{aligned}$$

which is a contradiction. Thus, the basis (e_i) is boundedly complete.

It remains to prove the claim. If the claim were false, we could find blocks G_1, G_2, \dots such that $\max G_i < \min G_{i+1}$ for all i and

$$\left| \left\langle \sum_{j=1}^\infty t_j e_j, G_i \right\rangle \right| > \delta$$

for each i . But then, if $m > \max G_{i(m)}$, and $x^m = \sum_{j=1}^m t_j e_j$,

$$\|x^m\| > \sum_{i=1}^{i(m)} \alpha_i |\langle x^m, G_i \rangle| > \delta \sum_{i=1}^{i(m)} \alpha_i.$$

Since we can choose $i(m)$ so that $i(m) \rightarrow \infty$ as $m \rightarrow \infty$, it follows that $\|x^m\| \rightarrow \infty$ as $m \rightarrow \infty$, a contradiction. This establishes the claim and finishes the proof of part (2).

The following result is crucial to the proof of parts 3 (ii) and 4 of the Theorem:

LEMMA 6. *Let (u_i) be a bounded sequence in X and (G_i) an admissible sequence of blocks such that*

- (i) $\{j: u_i(j) \neq 0\} \subset G_i$.
- (ii) $\langle u_i, N \rangle = 0$ for each i .
- (iii) (u_i) is a weak Cauchy sequence in X .

Then $(u_i) \rightarrow 0$ weakly in X .

Proof. First observe that (u_i) is an unconditional basic sequence in X . This follows easily from the fact that, for any scalars (t_i) , and any j , $\|\sum_{i \neq j} t_i u_i\| \leq \|\sum_i t_i u_i\|$. See [7] (Proposition 1.c.6, page 18).

Now, assume that (u_i) does not converge weakly to 0. Then, there exists an $f \in X^*$, $\|f\| = 1$, and a $\delta > 0$ such that (passing to a subsequence of (u_i) and not renaming) $f(u_i) > \delta$ for all i . On the other hand, since (u_i) is unconditional and not equivalent to the usual basis of l^1 , there are an N and non-negative scalars t_1, \dots, t_N such that

$$\sum_{i=1}^N t_i = 1 \quad \text{and} \quad \left\| \sum_{i=1}^N t_i v_i \right\| < \frac{\delta}{2}.$$

Thus,

$$\frac{\delta}{2} > f\left(\sum_{i=1}^N t_i v_i\right) > \sum_{i=1}^N t_i f(v_i) > \delta,$$

which contradicts the assumption that (u_i) does not converge weakly to 0. This completes the proof of Lemma 6.

Proof of Theorem 1 (3–i). If the sequence (e_i) were not weak Cauchy, we could find $n_1 < m_1 < n_2 < m_2 < \dots$, a $\delta > 0$, and an $f \in X^*$ with $\|f\| = 1$ and $f(e_{n_i} - e_{m_i}) > \delta$ for all i . Thus,

$$\left\| \frac{1}{N} \sum_{i=1}^N (e_{n_i} - e_{m_i}) \right\| > \delta \quad \text{for all } N.$$

But since the basis (e_i) of X is subsymmetric, it follows from Lemma 3 that

$$\left\| \frac{1}{N} \sum_{i=1}^N (e_{n_i} - e_{m_i}) \right\| = \frac{1}{N} \sum_{i=1}^{2N} \alpha_i \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Thus, the sequence (e_i) is weak Cauchy.

Suppose that this sequence has a weak limit $x \in X$. If $x = (t_j)$, then

$$t_j = \langle x, \{j\} \rangle = \lim_{i \rightarrow \infty} \langle e_i, \{j\} \rangle = 0,$$

so $x = 0$. On the other hand,

$$\langle x, N \rangle = \lim_{i \rightarrow \infty} \langle e_i, N \rangle = 1,$$

which is a contradiction.

Proof of Theorem 1. (3-ii). For each integer i , let $x_i = e_{2i} - e_{2i-1}$, and let X_0 be the closed subspace of X generated by the sequence (x_i) . Since (x_i) is an unconditional basic sequence (see the proof of Lemma 6) and since X_0 contains no isomorph of c_0 , it follows from [4] (Theorem 2, page 74) that X_0 is weakly sequentially complete. On the other hand, $\|x_i\| > 1$ for all i and, as was shown in the proof of part (3-i), $x_i \rightarrow 0$ weakly. Thus, X_0 fails the Schur property.

REMARK. Since the space X contains no isomorph of c_0 and fails to be weakly sequentially complete, it follows from a result of Bessaga and Pelczynski [2] that X does not imbed isomorphically in a space with an unconditional basis. (See also [4], page 74.) H. Rosenthal has observed that, in fact, X does not have local unconditional structure.

Proof of Theorem 1 (4). Let $\theta_0 \in X^{**}$ be the weak* limit of the sequence (e_i) in X . We will show that if (v_i) is a weak Cauchy sequence in X , then $v_i \rightarrow x + \alpha \cdot \theta_0$, where $x \in X$ and $\alpha = \lim_{i \rightarrow \infty} \langle v_i, N \rangle$.

For each i , let $f_i \in X^*$ be defined by $f_i(e_j) = \delta_{ij}$. First, observe that if $u_i \rightarrow x^{**}$ weak*, then $x^{**} = x + \theta$, where $x \in X$ and $\theta(f_i) = 0$ for each i . (This follows from the fact that X is a dual space and the usual duality arguments.) Let $w_i = v_i - x$. Then $w_i \rightarrow \theta$ weak*. From this it follows that $f_j(w_i) \rightarrow \theta(f_j) = 0$ as $i \rightarrow \infty$. By standard perturbation arguments, we can assume that a subsequence of the (w_i) (which we don't

rename) satisfies the following:

There is an admissible sequence (G_i) of blocks with $\max G_i + 1 < \min G_{i+1}$ and $\{j: w_i(j) \neq 0\} \subset G_i$.

Let $m_i = \max G_i + 1$, and $u_i = w_i - \langle w_i, N \rangle \cdot e_{m_i}$. By Lemma 6, $u_i \rightarrow 0$ weakly in X . On the other hand,

$$u_i = w_i - \langle w_i, N \rangle \cdot e_{m_i} \rightarrow \theta - \alpha \cdot \theta_0$$

weak* in X^{**} , where $\alpha = \lim_{i \rightarrow \infty} \langle w_i, N \rangle$. Thus, $\theta = \alpha \cdot \theta_0$. This shows that $x^{**} = x + \alpha \cdot \theta_0$ and completes the proof of part 4 and of Theorem 1.

Final remarks. There are a number of possible future directions that one might take in studying further the structure of the X_α spaces. We briefly list some of them:

(1) Determine the isomorphism types of the spaces X_α in terms of the sequence $\alpha = (\alpha_i)$.

(2) If X is isomorphic to $A \oplus B$, must one of A or B be isomorphic to X ? (Corollary 2 shows that the usual decomposition techniques do not apply to the space X .)

(3) Since each X is a dual space, $X = Y^*$ for some Banach space Y . What is the subspace structure of Y ? In particular, is Y hereditarily c_0 ?

(4) Is X hereditarily complementably l^1 ?

Added in proof. A. Andrew (Rocky Mountain J., to appear) has shown that X_α and X_β are isomorphic if and only if they are equal as sets, answering question (1). He also has shown that if X is isomorphic to $A \oplus B$, then one of A or B contains a complemented isomorph of X . The second named author (in preparation) has shown that the answer to question (4) is yes, and that, if $Y^* = X$, there are many subspaces of Y isomorphic to c_0 .

REFERENCES

- [1] P. Azimi, *Two Classes of Separable Banach Spaces*, Ph.D. Thesis, 1982.
- [2] C. Bessaga and A. Pelczynski, *A generalization of results of R. C. James concerning absolute basis in Banach spaces*, *Studia Math.*, **17** (1958), 165–174.
- [3] J. Bourgain, *l^1 subspaces of Banach spaces*, Lecture notes, Free University of Brussels.
- [4] M. M. Day, *Normed Linear Spaces*, Springer Verlag, Berlin.
- [5] J. Hagler, *A counterexample to several questions about Banach spaces*, *Studia Math.*, **60** (1977), 289–307.

- [6] R. C. James, *A non-reflexive Banach space isometric with the second conjugate*, Proc. Nat. Acad. Sci. U.S.A., **37** (1951), 174–177.
- [7] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, Vol I Sequence Spaces*, Springer Verlag, Berlin.
- [8] B. Maurey and H. P. Rosenthal, *Normalized weakly null sequences with no unconditional subsequence*, Studia Math., **61** (1977), 77–98.
- [9] H. P. Rosenthal, *A characterization of Banach spaces containing l^1* , Proc. Nat. Acad. Sci. U.S.A., **71** (1974), 2411–2413.

Received October 5, 1984. Portions of this paper first appeared in the dissertation of the first author written under the direction of the second. Research of the second author was partially supported by The National Science Foundation.

CATHOLIC UNIVERSITY OF AMERICA
WASHINGTON, D. C. 20064
AND
TEACHERS TRAINING UNIVERSITY
ZAHEDAN, IRAN

CATHOLIC UNIVERSITY OF AMERICA
WASHINGTON, D. C. 20064
AND
UNIVERSITY OF DENVER
DENVER, CO 80208

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

V. S. VARADARAJAN (Managing Editor)
University of California
Los Angeles, CA 90024

HEBERT CLEMENS
University of Utah
Salt Lake City, UT 84112

CHARLES R. DEPRIMA
California Institute of Technology
Pasadena, CA 91125

R. FINN
Stanford University
Stanford, CA 94305

HERMANN FLASCHKA
University of Arizona
Tucson, AZ 85721

RAMESH A. GANGOLLI
University of Washington
Seattle, WA 98195

ROBION KIRBY
University of California
Berkeley, CA 94720

C. C. MOORE
University of California
Berkeley, CA 94720

H. SAMELSON
Stanford University
Stanford, CA 94305

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

ASSOCIATE EDITORS

R. ARENS E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSHIDA
(1906–1982)

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$190.00 a year (5 Vols., 10 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) publishes 5 volumes per year. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1986 by Pacific Journal of Mathematics

Gideon Amit and David Chillag, On a question of Feit concerning character values of finite solvable groups	257
Constantin Gelu Apostol and Frank Larkin Gilfeather, Isomorphisms modulo the compact operators of nest algebras	263
Parviz Azimi and James Neil Hagler, Examples of hereditarily l^1 Banach spaces failing the Schur property	287
Brian Evan Blank, Boundary behavior of limits of discrete series representations of real rank one semisimple groups	299
Jeffrey Carroll, Some undecidability results for lattices in recursion theory	319
Gerald Howard Cliff and Alfred Rheinhold Weiss, Crossed product and hereditary orders	333
Ralph Cohen, Realizing transfer maps for ramified coverings	347
Ronald James Evans, Hermite character sums	357
C. L. Frenzen and Roderick Sue-Chuen Wong, Asymptotic expansions of the Lebesgue constants for Jacobi series	391
Bruno Iochum, Nonassociative L^p -spaces	417
John McDonald, Unimodular approximation in function algebras	435
John Robert Quine, Jr., Ramification and unintegrated value distribution ...	441
Marc Raphael, Commutants of quasisimilar subnormal operators	449
Parameswaran Sankaran and Peter Zvengrowski, On stable parallelizability of flag manifolds	455
Helga Schirmer, A relative Nielsen number	459
Barry Simon, Schrödinger semigroups on the scale of Sobolev spaces	475
Viakalathur Shankar Sunder, Stochastic integration in Fock space	481
Jan de Vries, A note on the G -space version of Glicksberg's theorem	493