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We consider the action of semigroups e^{-tH}, with $H = -\Delta + V$ on $L^2(R^\nu)$, on the scale of Sobolev spaces $\mathscr{H}^\alpha$. We show that while e^{-tH} maps $L^2 = \mathscr{H}^0$ to $\mathscr{H}^2$ under great generality, there exist bounded V so that, for all $\beta > 0$, $e^{-tH}[\mathscr{H}^\beta]$ is not contained in any $\mathscr{H}^\alpha$ with $\alpha > 2$.

1. Introduction. This note represents a modest contribution to the issue of smoothing properties of Schrödinger semigroups, e^{-tH}, $H = -\Delta + V$ on $L^2(R^\nu)$ [**12**]. It has been shown [**3, 8, 2, 11, 12**] under fairly great generality (i.e. assumptions on V) that e^{-tH} is smoothing on the scale of L^p spacses, i.e. e^{-tH} maps L^p into any L^q with $q \geq p$. Kon [**7**] asked the question of smoothing properties on the scale of Sobolev spaces $\mathscr{H}^\alpha$. Below we will exploit their L^q analogs, so we define them: $f \in L^q(R^\nu)$ is said to lie in L^q_α ($\alpha \geq 0$) if there exists $g \in L^q(R^\nu)$ so that $\hat{g}(p) = (1 + |p|^2)^{\alpha/2}\hat{f}(p) \cdot L^2_\alpha \equiv \mathscr{H}^\alpha$. We will also require the spaces K_ν defined initially by Kato [**5**]: If $\nu = 1$,

$$K_\nu = \left\{ f \,\middle|\, \sup_x \left[\int_{x-1}^{x+1} |f(y)|\, dy \right] < \infty \right\},$$

otherwise

$$K_\nu = \left\{ f \,\middle|\, \lim_{\alpha \downarrow 0} \left[\sup_x \int_{|x-y| \leq \alpha} B_\nu(x-y) |f(y)|\, d^\nu y \right] = 0 \right\}$$

where $B_\nu(x) = |x|^{-(\nu-2)}$ if $\nu \geq 3$ and $B_2(x) = -\ln|x|$. For any of these spaces χ, we define $\chi_{\text{loc}} = \{ f \mid f\varphi \in \chi \text{ for all } \varphi \in C_0^\infty(R^\nu)\}$. We summarize properties of these spaces needed below in an appendix.

Consider for a moment $\nu = 3$. It is well known [**6**] that if $V \in (L^2 + L^\infty)(R^3)$, then $D(H) = \mathscr{H}^2$, and thus obviously e^{-tH} maps $\mathscr{H}^0 = L^2$ to $\mathscr{H}^2$. Since there is lots of room between L^2 and L^∞, one might hope that for any $V \in L^\infty$, L^2 is mapped into some $\mathscr{H}^\alpha$ with $\alpha > 2$. Our main result in §2 will be to show there are $V \in L^\infty$ with compact support, so that $\operatorname{Ran}(e^{-tH})$ is not in any $\mathscr{H}^\alpha$ with $\alpha > 2$. Indeed, we will prove:

THEOREM 1. *Suppose that* $V_+ = \max(V, 0) \in K_\nu^{\text{loc}}$ *and* $V_- = \max(-V, 0) \in K_\nu$ *and that* $He^{-tH}\varphi$ *and* $e^{-tH}\varphi$ *lie in* $\mathscr{H}^\alpha_{\text{loc}}$ *for some* $\alpha > 2$ *and for one* $\varphi \geq 0$ ($\varphi \not\equiv 0$). *Then for* $\beta = \min(\alpha - 2, 1)$, $V \in L^{4/3}_{\beta,\text{loc}}$.

The example above will come from the fact that there exist $V \in L^\infty$ of compact support but in no $L^{4/3}_\beta$ with $\beta > 0$.

In the above motivating discussion, we obtain mapping onto $\mathscr{H}^2$ by using $D(H) = D(-\Delta)$. This is not necessary; $V \in L^2_{\text{loc}}$ suffices if we only want local results, for in §3, we prove

THEOREM 2. *If $V_- \in K_\nu$, $V_+ \in L^1_{\text{loc}}$ and $V \in L^p_{\text{loc}}$ on some open set S, then e^{-tH} maps L^2 to functions in $L^p_{\alpha=2,\text{loc}}$ on S.*

Of course, to get additional smoothing, one needs more smoothness on V. In §3, we also prove

THEOREM 3. *If $V \in L^\infty_{\alpha,\text{loc}}$, then e^{-tH} maps L^2 to $\mathscr{H}^{\alpha'+2}_{\text{loc}}$ for all $\alpha' < \alpha$.*

We note that Sobolev imbedding theorems imply that $L^p_{\beta,\text{loc}} \subset L^\infty_{\alpha,\text{loc}}$ if $0 < \alpha < \alpha' \equiv \beta - \nu p^{-1}$.

The astute reader will note the presence of L^p conditions on the potential with $p < 2$ in Theorem 1 and $p > 2$ in Theorem 3. One might hope that with more clever arguments one could get away with sharp L^2 conditions. We do not have enough evidence to call the statement below a conjecture, although it would be pretty if true:

Open question. Is it true that e^{-tH} maps L^2 to $\mathscr{H}^\alpha_{\text{loc}}$ with $\alpha > 2$ if and only if $V \in L^2_{\alpha-2,\text{loc}}$?

We remark that there are earlier results of Hunziker [**4**] on smooth V's yielding e^{-tH} mapping the Schwartz space, $\mathscr{S}$, to itself. Indeed, by using the ideas of [**10**], one can prove that if $D^\alpha V \in L^\infty$ for all α, then e^{-tH} maps $\mathscr{S}$ to itself. From this, it immediately follows that e^{-tH} maps $\mathscr{S}$ to itself.

The counterexample in this paper involves the most regular of elliptic operators with mildest noncontinuity possible, namely, a noncontinuity in the lowest order term. We show that the semigroup for the operator which is normally the most regularizing function of the operator already does not act particularly well on the scale of Sobolev spaces. This illustrates that the Sobolev spaces are not really well suited to the study of partial differential operators with noncontinuous coefficients.

2. Negative results.

Proof of Theorem 1. Let $\psi = e^{-tH}\varphi$. By hypothesis, $\psi \in \mathscr{H}^\alpha_{\text{loc}}$. Moreover, by general principles (see Cor. B.3.2, Thm. B.7.1 and the proof of

Lemma B.7.7 in [**12**]), ψ is continuous and everywhere strictly positive. Since $\psi \in \mathscr{H}^2 \subset \mathscr{H}^\alpha$, we have, by Theorem A.6 that $\psi \in L_1^4$, and then by Theorem A.5 that $\psi^{-1} \in L_1^4$. But since $H\psi \in \mathscr{H}_{\text{loc}}^\alpha$ by hypothesis and $-\Delta\psi \in \mathscr{H}_{\text{loc}}^{\alpha-2}$ by hypothesis, $V\psi = H\psi + \Delta\psi \in \mathscr{H}_{\text{loc}}^{\alpha-2}$, so by Theorem A.4, $V = \psi^{-1}(V\psi) \in L_\beta^{4/3}$ where $\beta = \min(\alpha - 2, 1)$. This completes the proof of Theorem 1. □

REMARKS (1) As noted in the introduction, there exist $V \in L^\infty$ with $V \notin L_{\beta,\text{loc}}^{4/3}$ for all $\beta > 0$. Thus there exist $V \in L^\infty$ so e^{-tH} does not map L^2 to $\mathscr{H}^\alpha$ ($\alpha > 2$) for any $s > 0$. For if it does, then for $t > s$, e^{-tH} and $He^{-tH} = e^{-sH}He^{-(t-s)H}$ map L^2 to $\mathscr{H}^\alpha$.

(2) It cannot even be true if $V \notin L_\beta^{4/3}$, that e^{-tH} ($t \geq t_0$) maps $\mathscr{H}^\gamma$ to $\mathscr{H}^\alpha$ with $\alpha > 0$. For if it did, by applying the Stein interpolation theorem, e^{-tH} would map $\mathscr{H}^{\gamma-\varepsilon}$ to $\mathscr{H}^{\alpha-\varepsilon'}$ holomorphically in s for s near t_0, and so He^{-sh} would also map $\mathscr{H}^{\gamma-\varepsilon}$ to $\mathscr{H}^{\alpha-\varepsilon'}$.

3. Positive results.

Proof of Theorem 2. By general principles (Theorem B.1.1 of [**22**]), e^{-tH} maps L^2 to L^∞, so if $\varphi \in L^2$, $Ve^{-tH}\varphi \in L_{\text{loc}}^p$ and $He^{-tH}\varphi \in L^\infty \subset L_{\text{loc}}^p$, so $-\Delta(e^{-tH}\varphi) = (H - V)e^{-tH}\varphi \in L_{\text{loc}}^p$. Since $e^{-tH}\varphi \in L_{\text{loc}}^p$, we have, by Theorem A.8, that $e^{-tH}\varphi \in L_{\alpha=2,\text{loc}}^p$.

REMARK. It is known (Theorem B.2.1 of [**12**]) that $(H + c)^{-a}$: $L^2 \to L^\infty$ if $a > n/4$. The above proof shows that if $\tilde{a} > 1 + n/4$, then $(H + c)^{-\tilde{a}}$ maps L^2 to $L_{\alpha=2,\text{loc}}^p$.

Proof of Theorem 3. Suppose first $S = R^\nu$. By Theorem 2, $f \equiv e^{-tH}\varphi \in L_{2,\text{loc}}^\infty \subset L_{2,\text{loc}}^p$, all $p < \infty$ (by Theorem A.2). Since $V \in L_{\alpha,\text{loc}}^\infty$, it follows by Theorem A.1 that $Vf \in L_{\beta,\text{loc}}^p$ with $\beta = \min(\alpha, 2)$, $p < \infty$ and so, by a Sobolev estimate, $Vf \in L_{\beta',\text{loc}}^\infty$, all $\beta' < \beta$. Thus, since $He^{-tH}\varphi \in L_{2,\text{loc}}^\infty$, we see that $\Delta f \in L_{\beta',\text{loc}}^\infty$, i.e. by Theorem A.8, $f \in L_{\beta'+2,\text{loc}}^\alpha$. This completes the proof if $\alpha \leq 2$. If $\alpha > 2$, we have f, $Hf \in L_{\eta,\text{loc}}^\infty$, all $\eta < 2$, and we can repeat the argument above to learn $f \in L_{\beta',\text{loc}}^\infty$ with $\beta' < \beta = \min(\alpha, 4)$. By iteration, we can obtain the result for any α. Since the above proof is local, it works for any S. □

APPENDIX

Some properties of L^p-Sobolev spaces. In this appendix, we provide, for the reader's convenience, a summary of various facts about L^p-Sobolev spaces used in this paper. L_α^p is defined to be the set of functions, f, in L^p so that there exists $g \in L^p$ with $\hat{f}(k) = (1 + k^2)^{\alpha/2}\hat{g}(k)$. If β is a multi-index with $|\beta| < \alpha$, then $k^\beta(1 + k^2)^{-\alpha/2}$ is the Fourier transform of a function in L^1, so since convolution with L^1 functions leaves L^p invariant, we see that if $\alpha < \gamma$, then $L^p_\alpha \subset L^p_\gamma$, and if $\alpha > n$ is an integer, all $D^\beta f \in L^p$ for $|\beta| \leq n$. Only for $n = 1$ will we require the more subtle fact **[13]** that this holds if $\alpha = n$ when $p \neq 1, \infty$ (see Theorem A.3). In particular, it is obvious that if $\alpha = 2l$ is an even integer, then $f \in L^p_\alpha$ if and only if $f, \Delta f, \ldots, \Delta^k f \in L^p$.

We will exploit the theory of complex interpolation **[1]** for the L^p_α's. The key fact **[1]** is that if $1 < p_0, p_1 < \infty$, then $(L^{p_0}_{\alpha_0}, L^{p_1}_{\alpha_1})_\theta = L^{p_\theta}_{\alpha_\theta}$ where $p_\theta^{-1} = \theta p_1^{-1} + (1 - \theta)p_0^{-1}$ and $\alpha_\theta = \theta\alpha_1 + (1 - \theta)\alpha_0$.

We will occasionally state the results in a less general form than that which is valid if the proof of the less general result is easier, and the less general result is all that is needed in the text.

THEOREM A.1. *Let f be C^{2k} for k, an integer. Then the map M_f: $g \mapsto fg$ maps L^p_α to itself for all $p \neq 1, \infty$ and all $\alpha \in [0, 2k]$.*

PROOF. By interpolation, we need only consider the cases $\alpha = 0, 2k$, $\alpha = 0$ is trivial. For $\alpha = 2k$, note that $\Delta^k(fg) = f(\Delta^k g) + g(\Delta^k f) + R$ where R involves products of derivative of degree $\beta < 2k$. Since $g \in L^p_{2k}$, $D^\beta g \in L^p$ for all $\beta < 2k$ and $\Delta^k g \in L^p$, the result follows. □

In particular, this result shows that $L^p_\alpha \subset L^p_{\alpha,\text{loc}}$. Let $L^p_{\alpha,\text{comp}}$ denote the $f \in L^p_\alpha$ with compact support. The g with $\hat{g} = (1 + k^2)^{\alpha/2}\hat{f}$ will not have compact support if $\alpha \neq 2l$, but since the Fourier transform of $(1 + k^2)^{\alpha/2}$ is a distribution given away from zero by a function with exponential decay, $|g(x)| \leq Ce^{-D|x|}$ for $x < p$. Thus:

THEOREM A.2. *If $1 \leq q < p \leq \infty$, $L^p_{\alpha,\text{comp}} \subset L^q_{\alpha,\text{comp}}$ and $L^p_{\alpha,\text{loc}} \subset L^q_{\alpha,\text{loc}}$.*

The next few results require a basic fact about L^p_α proven, for example, in Stein **[13]**, p. 135ff.

THEOREM A.3. *Let $1 < p < \infty$. Then $f \in L^p_1$ if and only if f and $\vec{\nabla} f$ (distributional derivative) lie in L^p and $\|f\|_{L^p_\alpha}$ is equivalent to $\|f\|_p + \|\vec{\nabla}\|_p$.*

There is a more general L^p_α-Hölder inequality: $L^p_\alpha \cdot L^q_\alpha$ lies in L^r_α $(r^{-1} = p^{-1} + q^{-1})$. We only require the result that follows:

THEOREM A.4. *If $f \in L^p_1$ and $g \in L^q_\beta$ with $0 \leq \beta \leq 1$ and $p(p - 1)^{-1} < q < \infty$, then $fg \in L^r_\beta$ with $r^{-1} = p^{-1} + q^{-1}$.*

Proof. By interpolation, we need only prove the result for $\beta = 0, 1$. $\beta = 0$ is the ordinary Hölder inequality. $\beta = 1$ follows from Theorem A.3 and the ordinary Hölder inequality if we note $\nabla(fg) = (\nabla f)g + f(\nabla g)$.□

THEOREM A.5. *If f is a continuous function on R^ν everywhere strictly positive, and $f \in L^p_{1,\mathrm{loc}}$, then $f^{-1} \in L^p_{1,\mathrm{loc}}$.*

Proof. By mollifying f, it is easy to check that, in distributional sense, $\nabla(f^{-1}) = -f^{-2}\nabla f$. Thus ∇f in L^p_{loc} and f^{-1} in L^∞_{loc} implies that $\nabla f^{-1} \in L^p_{\mathrm{loc}}$. □

The following result is a special case of the Gagliardo-Nirenberg inequalities (see e.g. [**9**]).

THEOREM A.6. *If $f \in L^\infty \cap L^2_{2,\mathrm{loc}}$, then $\nabla f \in L^4_{\mathrm{loc}}$ and so $f \in L^4_{1,\mathrm{loc}}$.*

Next we will construct $f \in L^\infty$ with compact support so that f lies in no L^1_α ($\alpha > 0$) (and so by Theorem A.2 in no L^p_α ($\alpha > 0$)).

THEOREM A.7. *There exists f, a continuous function, with compact support, so $f \notin \bigcup_{\alpha > 0} L^1_\alpha$.*

Proof. Let $k_n = (2^n, 0, \ldots, 0) \in R^\nu$, let $g \in C_0^\infty(R^\nu)$ with $\hat{g}(0) = 1$ and let $f(x) = \sum_n n^{-2} e^{ik_n \cdot x} g(x)$. The sum converges uniformly, so f is continuous. Moreover, since $\hat{g}(k)$ decays faster than k^{-1}, it is easy to see that $n^2 \hat{f}(k_n) \to 1$. In particular, $(1 + |k_n|^2)^{\alpha/2} \hat{f}(k_n) \to \infty$ for any $\alpha > 0$. It follows that $f \notin L^1_\alpha$ for any $\alpha > 0$. □

As a final result, we need

THEOREM A.8. *If $f, \Delta f \in L^p_{\alpha,\mathrm{loc}}$, then $f \in L^p_{\alpha+2,\mathrm{loc}}$.*

Proof. The proof is only somewhat involved since we have the loc's. For $\varphi \in C_0^\infty$,

$$(1 - \Delta)(\varphi f) = g - 2\nabla \cdot (\nabla \varphi f)$$

where $g = \varphi(1 - \Delta)f + (\Delta\varphi)f \in L^p_\alpha$ by hypothesis. Thus

$$(1) \qquad (1 - \Delta)^{1/3}(\varphi f) = h - \sum_i A_i[(\nabla_i \varphi) f]$$

where $h = (1-\Delta)^{-2/3} g \in L^p_{\alpha+4/3}$ and $A_i = (1-\Delta)^{-3/4}\nabla_i$ is convolution with a function in L^1 and so a bounded map on each L^p_α. By 91) and $f \in L^p_{\alpha,\mathrm{loc}}$, we conclude that $(1-\Delta)^{1/3}(\varphi f) \in L^p_\alpha$, so $f \in L^p_{\alpha+2/3,\mathrm{loc}}$. Iterating this argument twice, we find $f \in L^p_{\mathrm{loc}}$. The iteration stops because h is only in $L^p_{\alpha+4/3}$. □

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