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**DIRICHLET'S THEOREM FOR THE RING OF POLYNOMIALS  
OVER  $GF(2)$**

DOUGLAS AUSTIN HENSLEY

## DIRICHLET'S THEOREM FOR THE RING OF POLYNOMIALS OVER GF(2)

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Let  $G$  denote the ring  $\text{GF}(2)[x]$  of polynomials  $g(x)$  over the field of integers mod 2. Let

$$I(k) = \# \{ p \in G : \deg p = k \text{ and } p \text{ is irreducible in } G \}.$$

It is well known that  $I(k) = (1/k) \sum_{d|k} \mu(d) 2^{k/d}$ . Here we prove an analog to Dirichlet's Theorem on primes in arithmetic progressions. For any  $m \in G$  the  $p$  counted in  $I(k)$  are uniformly distributed among the congruence classes  $(b) \pmod m$  for which  $(b, m) = 1$ . The result is especially sharp when  $m$  is square-free.

**1. Introduction and notation.** As in the abstract,  $G = \text{GF}(2)[x]$ . We will suppress the variable and write, for instance, 1011 in place of  $x^3 + x + 1$ . We denote the set of irreducible  $p \in G$  by  $I$ . The only part of this work which does not seem to generalize easily to other  $\text{GF}(q)[x]$ ,  $q$  a prime, is the special role of square-free moduli. Defining  $\phi: G \rightarrow \mathbb{Z}$  in the natural way ( $\phi(m) = \# \{ a : \deg a = \deg m \text{ and } (a, m) = 1 \}$ ), we have that

$$(1.1) \quad \phi(m) \text{ is odd if and only if } m \text{ is square-free.}$$

Consequently, none of the "Dirichlet characters" on  $G/mG$  can have as their range  $\{-1, 0, 1\}$ . The absence of this kind of Dirichlet character permits sharper bounds. For fixed  $m \in G$ ,  $b \in G$  with  $(b, m) = 1$ , let  $I_b(n)$  denote the number of irreducible  $p \in G$  of degree  $n$  such that  $p \equiv b \pmod m$ .

**THEOREM.** *There exist positive effectively computable constants  $C_1$  and  $C_2$  such that for all integers  $M, N \geq 1$ , for all square-free polynomials  $m \in G$  of degree  $M$ , and for all congruence classes  $(b) \pmod m$  relatively prime to  $m$ ,*

$$\left| I_b(N) - \frac{2^N}{N\phi(m)} \right| \leq \frac{C_1 M 2^N}{N} \exp(-C_2 N M^{-9} (\log M)^{-3}).$$

That is,

$$I_b(N) = \frac{2^N}{N\phi(m)} \left( 1 + O\left( M\phi(m)e^{-C_2NM^{-9}(\log M)^{-3}} \right) \right)$$

uniformly in  $N$ ,  $M$ ,  $m$  and  $b$ .

The result, of course does not constitute any improvement on the trivial bounds  $0 \leq I_b(N) \leq I(N)$  unless  $N$  is larger, roughly, than  $M^9$ . It differs from results of Uchiyama and Carlitz [1, 3, 4] in its generality and uniformity with respect to the modulus, treating the ring  $G$  as fixed. Basically they kept  $G$  variable and constrained  $m$ .

When  $m$  is not square-free, characters of the second kind intrude, and we must settle for  $2^{-M}M^{-2}$  in place of  $M^{-9}(\log M)^{-3}$  in Theorem 1.

**2. Preliminaries.** For much of its length our proof follows the path of the classic proof of Dirichlet's theorem. There are analogs to Dirichlet characters, to  $L$ -functions, and product expansions valid in a half-plane. The difference is that in this case the  $L$ -functions are essentially polynomial functions on  $\mathbf{C}$ . This simplifies the analysis. We can dispense with contour integrations, and just compare coefficients in two expansions of

$$\sum_{\chi \bmod m} \frac{1}{\chi(b)} \frac{L'(s, \chi)}{L(s, \chi)},$$

as series in  $t = 2^{-s}$ . The reader who wants to see just what is *different* can skip this section.

Let  $Na = 2^{\deg a}$ , for  $a \in G$ . Let

$$(2.1) \quad \phi(m) = \#\{a: \deg a = \deg m \text{ and } (a, m) = 1\}.$$

Note that for  $p \in G$  irreducible,  $\phi(p) = Np - 1$  and is odd. Finally, the usual proof that

$$(2.2) \quad \phi(m) = (Nm) \prod_{p|m} \left( 1 - \frac{1}{Np} \right)$$

is valid in this setting too, so  $\phi(m)$  is multiplicative. Thus  $\phi(m)$  is odd if and only if  $m$  is square-free.

A *character mod  $m$*  is a function  $\chi: G \rightarrow \mathbf{C}$  such that

$$(2.3) \quad \begin{array}{ll} \text{(i)} & \chi(a)\chi(b) = \chi(ab) \quad \text{for } a, b \in G. \\ \text{(ii)} & \chi(a) = \chi(b) \quad \text{if } a \equiv b \pmod{m} \\ \text{(iii)} & \chi(a) = 0 \quad \text{for } (a, m) \neq 1. \end{array}$$

As with characters in the integers,  $\chi(1) = 1$ , and if  $(a, m) = 1$  then  $\chi(a)$  is a  $\phi(m)$ th root of 1. For every  $m$  except 1, 10, 11 and 110, there is a character other than the trivial character  $\chi_0$ , where

$$\chi_0(a) = 1 \quad \text{for } (a, m) = 1, \quad \chi_0(a) = 0 \quad \text{otherwise.}$$

Further, with the same exceptions,

$$(2.4) \quad \sum_{a \bmod m} \chi(a) = 0 \quad \text{for all } \chi \neq \chi_0$$

$$(2.5) \quad \sum_{\chi \bmod m} \chi(a) = 0 \quad \text{for all } a \not\equiv 1 \pmod m.$$

(All irreducibles except the factors of  $m$  are  $\equiv 1 \pmod m$  when  $m = 1, 10, 11$  or  $110$ , since only  $1 \pmod m$  is relatively prime to  $m$  in these cases. From now on, we assume  $m$  is not 1, 10, 11 or 110.)

$$(2.6) \quad \sum_{\chi \bmod m} \chi(1) = \phi(m)$$

and

$$(2.7) \quad \sum_{a \bmod m} \chi_0(a) = \phi(m).$$

*Proof.* The classical proofs go over word for word. See e.g. Landau [2].

We now define a power series  $f_\chi(t)$  corresponding to each  $\chi \bmod m$ . With the substitution  $t = 2^{-s}$  we get the analog of a Dirichlet  $L$ -series.

DEFINITION.

$$(2.8) \quad f_\chi(t) = \sum_{a \in G} \chi(a)t^{\deg a} = \sum_{j=0}^{\infty} \left\{ \sum_{\deg a=j} \chi(a) \right\} t^j,$$

and

$$L(s, \chi) = \sum_{\substack{a \in G \\ a \neq 0}} \chi(a)(Na)^{-s}.$$

Let  $C_j(\chi) = \sum_{\deg a=j} \chi(a)$ . Then by (2.4), for  $\chi = \chi_0$ ,  $C_j(\chi) = 0$  for  $j \geq \deg m$ .

Thus for  $\chi \neq \chi_0$ , and with  $M = \deg m$ ,

$$(2.9) \quad f_\chi(t) = \sum_{j=0}^{m-1} C_j(\chi)t^j$$

and is a polynomial over the complex numbers of degree  $\leq M - 1$ . We note here that

$$(2.10) \quad f_\chi(0) = 1, \quad f_\chi(1) = 0, \quad \text{and} \quad |C_j| \leq 2^j.$$

If we forget temporarily that  $f_\chi(t)$  is a polynomial, it is natural to ask for a product expansion. Formally,

$$(2.11) \quad f_\chi(t) = \prod_{p \in I} \left( 1 - \frac{\chi(p)}{(Np)^s} \right)^{-1} = \prod_{p \in I} (1 - \chi(p)t^{\deg p})^{-1},$$

and the product converges absolutely for  $|t| < \frac{1}{2}$  ( $\text{Re}(s) > 1$ ). The function corresponding to the Riemann zeta function here is

$$(2.12) \quad Z(t) := \sum_{a \neq 0} t^{\deg a} = \frac{1}{1 - 2t},$$

and this has the product expansion

$$(2.13) \quad Z(t) = \prod_{k=1}^{\infty} (1 - t^k)^{-I(k)}.$$

Finally, for  $\chi = \chi_0 \pmod{m}$ ,

$$(2.14) \quad f_{\chi_0}(t) = Z(t) \prod_{p|m} (1 - t^{\deg p}).$$

The well known identity

$$(2.15) \quad I(k) = \frac{1}{k} \sum_{d|k} \mu(d) 2^{k/d}$$

now follows from a (*much*) simplified reprise of the proof of the prime number theorem. We have  $Z'(t)/Z(t) = 2/(1 - 2t)$  on one hand, while from (2.13) it is  $\sum_{k=1}^{\infty} kI(k)t^{k-1}/(1 - t^k)$ . Expanding both sides as series about  $t = 0$  and equating coefficients gives

$$(2.16) \quad 2^k = \sum_{d|k} dI(d),$$

which is equivalent to (2.15).

The same ideas feature in the proof of Theorem 1: differentiate  $\log f_\chi(t)$ , use the product formula on one side, expand things as series in  $t$  and equate coefficients.

**3. Partial fractions.** For  $\chi = \chi_0 \pmod{m}$ ,

$$(3.1) \quad f_{\chi_0}(t) = \frac{1}{1 - 2t} \prod_{p|m} (1 - t^{\deg p})$$

for  $t \neq 1/2$ . With the notations  $I_m(k) = \#\{p \in I: p|m \text{ and } \deg p = k\}$ ,  $e(r) = e^{2\pi ir}$ , we have

$$(3.2) \quad \frac{f'_{\chi_0}(t)}{f_{\chi_0}(t)} = \frac{2}{1 - 2t} \sum_{k=1}^M \frac{1}{k} I_m(k) \sum_{j=0}^{k-1} \frac{1}{t - e(j/k)}$$

which has simple poles at  $t = 1/2$  and at various roots of unity. In all, there are  $1 + \sum_{k=1}^M kI_m(k)$  poles of  $(f'_{\chi_0}/f_{\chi_0})(t)$ , and for  $m$  square-free, this is just  $M + 1$ . Now for any polynomial  $f(t)$  over  $\mathbf{C}$  with zeros  $w_1, w_2, \dots, w_j$  to multiplicity  $N_1, N_2, \dots, N_j$ ,

$$(3.3) \quad \frac{f'(t)}{f(t)} = \sum_{i=1}^j \frac{N_i}{t - w_i}.$$

Thus for any character  $\chi \neq \chi_0 \pmod{m}$ ,

$$\frac{f'_\chi(t)}{f_\chi(t)} = \sum_{w \in \Omega_\chi} \frac{N(w)}{t - w},$$

where  $\Omega_\chi$  is the set of zeros of  $f_\chi(t)$  and  $N(w)$  the corresponding multiplicity, for  $w \in \Omega_\chi$ . By (2.11),  $f_\chi(t) \neq 0$  for  $|t| < 1/2$ , that is,  $|w| \geq 1/2$  if  $w \in \Omega_\chi$ . We now fix  $b \pmod{m}$ ,  $(b, m) = 1$ , and consider

$$(3.4) \quad \sum_{\chi \pmod{m}} \frac{1}{\chi(b)} \frac{f'_\chi(t)}{f_\chi(t)}.$$

On one hand, this is equal to

$$(3.5) \quad \sum_{\substack{\chi \pmod{m} \\ \chi \neq \chi_0}} \frac{1}{\chi(b)} \sum_{w \in W_\chi} \frac{N(w)}{t - w} + \frac{2}{1 - 2t} - \sum_{k=1}^M \frac{1}{k} I_m(k) \sum_{j=0}^{k-1} \frac{1}{t - e(j/k)}.$$

We anticipate that for small  $t$ , the series expansion of this about zero converges, and that the dominant contribution to the coefficient of  $t^n$  for large  $n$  comes from  $2/(1 - 2t)$ .

On the other hand, (3.4) equals

$$(3.6) \quad \sum_{\chi \pmod{m}} \frac{1}{\chi(b)} \sum_{p \in I} \frac{\chi(p)(\deg p)t^{\deg p-1}}{1 - \chi(p)t^{\deg p}} = \sum_{k=1}^{\infty} k \sum_{j=0}^{\infty} \sum_{p \in I} \sum_{\chi \pmod{m}} \frac{1}{\chi(b)} (\chi(p))^{j+1} t^{(j+1)k-1} = \sum_{n=1}^{\infty} nt^{n-1} \sum_{d|n} \frac{1}{d} \sum_{\substack{p \in I \\ \deg p = n/d}} \sum_{\chi \pmod{m}} \frac{1}{\chi(b)} (\chi(p))^d.$$

Thus the coefficient of  $t^{n-1}$  in the expansion of (3.6) about  $t = 0$  is

$$(3.7) \quad n \sum_{d|n} \frac{1}{d} \sum_{\substack{p \in I \\ \deg p = n/d}} \sum_{\chi \pmod{m}} \frac{1}{\chi(b)} \chi(p)^d.$$

In (3.7), the part due to  $d = 1$  is predominant, as we shall see. This part simplifies by (2.5) and (2.6) to

$$n\phi(m) \sum_{\substack{p \in I \\ \deg p = n}} 1 = n\phi(m)I_b(n).$$

The other terms may be estimated rather crudely. For any  $d$ ,

$$\left| \sum_{\chi \bmod m} \frac{1}{\chi(b)} \chi(p)^d \right| \leq \varphi(m),$$

and  $I(d) \leq 2^d/d$ . Thus in (3.7) the part of the sum due to a particular  $d$  has absolute value  $\leq (n/d)2^d\phi(m)$ .

This gives

$$(3.8) \quad \sum_{\chi \bmod m} \frac{1}{\chi(b)} \frac{f'_\chi(t)}{f_\chi(t)} = \sum_{n=1}^{\infty} n\phi(m) \left\{ I_b(n) + O\left(\frac{1}{n} 2^{n/2}\right) \right\} t^{n-1}.$$

The implicit constant is independent of  $b$ ,  $m$ , and  $n$ .

In (3.5) the expansion of  $2/(1 - 2t)$  is simple, and the coefficients of  $t^n$  arising from  $1/(t - e(j/k))$  are quite small by comparison. We just need a bound on  $|w|$  for  $w \in \Omega_\chi$ ,  $\chi \neq \chi_0$ . Here the distinction between characters of the *second kind* (real valued and taking  $-1$  as well as  $+1$ ) and *third kind* (not real) is important.

If  $\chi$  is a character of the second kind then following Landau's treatment in [2] one sees that  $f_\chi(1/2) \neq 0$ . But then

$$f_\chi(1/2) = \sum_{j=0}^{M-1} C_j(1/2)^j$$

and  $c_j = \sum_{\deg a=j} \chi(a)$  is an integer here, so  $|f_\chi(1/2)| \geq 2^{-M}$ . More sophisticated approaches led to no better an estimate. The estimate for  $I_b(n)$  when  $m$  is not square-free is done the same way as that for when  $m$  is square-free, except at this point. Since the main interest attaches to the uniformly good estimates to be had for square-free  $m$ , we shall not go into this any more.

Assume now that  $m$  is square-free. Then there are no real characters other than  $\chi_0$ .

**4. The zeros of  $f_\chi(t)$  for characters of the third kind.** By the familiar device based on the inequality  $3 + 4\cos\theta + \cos 2\theta \geq 0$  and the product expansion (2.11), we have

$$(4.1) \quad \left| f_{\chi_0}^3(t) f_\chi^4(t) f_{\chi^2}(t) \right| \geq 1 \quad \text{for } |t| < 1/2.$$

Since  $\chi$  takes on non-real values,  $\chi^2 \neq \chi_0$ , so  $|f_{\chi^2}(t)| \leq M$  for  $|t| \leq 1/2$ . The factor involving  $\chi_0$  is easily estimated:

$$|f_{\chi_0}(t)| \leq \left| \frac{1}{1-2t} \right| \prod_{p|m} \left( 1 + \frac{1}{Np} \right), \quad \text{for } |t| < \frac{1}{2}.$$

It is well known that for integer  $n \rightarrow \infty$ ,  $\phi(n) \gg n/\log \log n$ ; the worst case is when  $n$  is the product of the first  $k$  primes for some  $k$ .

Similarly here we have for  $\deg m = M$ ,  $M \rightarrow \infty$  that

$$(4.2) \quad \phi(m) \gg 2^M/\log M, \quad \text{uniformly in } m.$$

Since

$$\prod_{p|m} \left( 1 + \frac{1}{Np} \right) < \prod_{p|m} \left( 1 - \frac{1}{Np} \right)^{-1} = \frac{2^M}{\phi(m)},$$

$$\prod_{p|m} \left( 1 + \frac{1}{Np} \right) \ll \log M,$$

and so

$$(4.3) \quad |f_{\chi_0}(t)| \ll \left| \frac{\log M}{1-2t} \right|, \quad |t| < \frac{1}{2}.$$

Now from (4.1),

$$(4.4) \quad |f_{\chi}(t)| \gg M^{-1/4}(\log M)^{-3/4}|t-1/2|^{3/4} \quad \text{in } |t| < 1/2.$$

To estimate  $f'_{\chi}(t)/f_{\chi}(t)$  we also need an upper bound for  $f'_{\chi}(t) = \sum_{j=1}^{m-1} jC_j t^{j-1}$ , in  $|t| < 1/2$ .

Each  $|C_j| \leq 2^j$ , so  $|C_j t^{j-1}| \leq 2$ . Thus

$$(4.5) \quad |f'_{\chi}(t)| \leq M^2 \quad \text{for } |t| \leq 1/2.$$

Since no polynomial can have a zero of fractional order, for fixed  $\chi$ ,  $|f_{\chi}(t)| \gg 1$  in  $|t| < 1/2$ . But for variable  $M$ , we need a lemma.

LEMMA. *Uniformly in  $M \geq 1$ , in  $m$  with  $\deg m = M$ , in  $\chi \pmod m$  of the third kind, and in  $|t| \leq 1/2$ ,*

$$|f_{\chi}(t)| \gg M^{-7}(\log M)^{-3}.$$

*Proof.* By (4.4), there exists  $C > 0$  such that

$$|f_{\chi}(t)| \geq CM^{-1/4}(\log M)^{-3/4}|t-1/2|^{3/4}.$$

Let  $t_0$ ,  $0 < t_0 < 1/2$ , be the unique solution of

$$M^2 = \frac{3}{4}CM^{-1/4}(\log M)^{-3/4}\left|t-\frac{1}{2}\right|^{-1/4}; \quad t_0 = \frac{1}{2} - \left(\frac{3}{4}\right)^4 C^4 M^{-9}(\log M)^{-3}.$$



Then

$$|f_\chi(1/2)| \geq |f_\chi(t_0)| - M^2(1/2 - t_0)$$

from (4.5), and this is  $\geq (\frac{3}{4})^3 \frac{1}{4} C^4 M^{-7} (\log M)^{-3}$  from (4.4). Now for  $|\frac{1}{2} - t| < \frac{1}{10} |\frac{1}{2} - t_0|$ ,

$$|f_\chi(t)| \geq |f_\chi(t_0)| - \frac{1}{10} M^2 |\frac{1}{2} - t_0| \geq (\frac{3}{4})^3 (\frac{1}{4} - \frac{1}{10}) C^4 M^{-7} (\log M)^{-3}.$$

For  $|t - \frac{1}{2}| \geq \frac{1}{10} |t_0 - \frac{1}{2}|$ , though,

$$\begin{aligned} |f_\chi(t)| &\geq CM^{-1/4} (\log M)^{-3/4} |t - \frac{1}{2}|^{3/4} \quad \text{by (4.4),} \\ &\geq (\frac{3}{4})^3 (\frac{1}{10})^{3/4} C^4 M^{-7} (\log M)^{-3}. \end{aligned}$$

Thus uniformly in  $M$ ,  $m$ ,  $\chi \pmod{m}$  of the third kind, and for  $t$ ,  $|t| < 1/2$ ,

$$(4.6) \quad |f_\chi(t)| \geq C_1 M^{-7} (\log M)^{-3}.$$

The lemma follows by the continuity of the  $f_\chi(t)$ .

Now

$$f_\chi(t)^{(n)} = \sum_{j=n}^{M-1} C_j \frac{n!}{j!} t^{j-n},$$

so  $|f_\chi(t)^{(n)}| \leq (2M)^{n+1}$  in  $|t| \leq 1/2$ . Thus for  $|v| \leq M^{-9}$  and  $|T| = 1/2$  we have

$$\begin{aligned} (4.7) \quad f_\chi(T+v) &= f_\chi(T) + O\left(\sum_{j=1}^{M-1} \frac{1}{j!} |v|^j (2M)^{j+1}\right) \\ &\quad \text{(with the implicit constant = 1)} \\ &= f_\chi(T) + O(M^2|v|). \end{aligned}$$

Thus uniformly in  $M$ ,  $m$ , and  $\chi$ ,

$$(4.8) \quad f_\chi(t) \neq 0 \quad \text{in } |t| \leq 1/2 + C_2 (M^{-9} (\log M)^{-3})$$

for some  $C_2 > 0$ .

**5. Conclusions.** We now expand (3.5) as a series in  $t$ , and estimate the coefficient of  $t^{n-1}$ .

From  $\chi_0$ , we get

$$(5.1) \quad \sum_{n=1}^{\infty} 2^n t^{n-1} + \sum_{k=1}^M \frac{1}{k} I_m(k) \sum_{j=0}^{k-1} \sum_{n=1}^{\infty} t^{n-1} e\left(\frac{(n-1)j}{k}\right),$$

so the coefficient of  $t^{n-1}$  is

$$(5.2) \quad 2^n + \sum_{k=1}^M \frac{1}{k} I_m(k) \sum_{j=0}^{k-1} e\left(\frac{(n-1)j}{k}\right).$$

Now  $|\sum_{j=0}^{k-1} e((n-1)j/k)| \leq k$ , so the second term of (5.2) is  $O(\sum_{k=1}^M I_m(k))$ . Now trivially this latter is  $O(M)$ . (A little thought shows it to be  $O(M/\log M)$  but we have larger errors elsewhere.) Thus in (3.5) the coefficient of  $t^{n-1}$  due to  $\chi_0$  is

$$(5.3) \quad 2^n + O(M).$$

The expansion of the rest of (3.5) works out to  $\sum_{n=1}^\infty r_n t^{n-1}$ , where

$$(5.4) \quad \begin{aligned} r_n &= \sum_{\substack{\chi \pmod m \\ \chi \neq \chi_0}} \frac{1}{\chi(b)} \sum_{w \in \Omega_\chi} \frac{-N(w)}{w} \left(\frac{1}{w}\right)^{n-1} \\ &= - \sum_{\substack{\chi \pmod m \\ \chi \neq \chi_0}} \frac{N(w)}{\chi(b)} w^{-n}. \end{aligned}$$

Now  $|w| \geq 1/2 + C_2 M^{-9}(\log M)^{-3}$ . Thus

$$(5.5) \quad |r_n| \leq M\phi(m)2^n \exp(-C_3 n M^{-9}(\log M)^{-3}).$$

Now from (5.5), (5.3), and (3.8) we have

$$(5.6) \quad \begin{aligned} n\phi(m)(I_b(n) + O(\frac{1}{n}2^{n/2})) \\ = 2^n + O(M) + O(M\phi(m)2^n \exp(-C_3 n M^{-9}(\log M)^{-3})). \end{aligned}$$

The theorem follows upon renumbering the constants.

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