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**ON A PROBLEM OF GAUSS-KUZMIN TYPE FOR CONTINUED
FRACTION WITH ODD PARTIAL QUOTIENTS**

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Let x be a number of the unit interval. Then x may be written in a unique way as a continued fraction

$$x = 1/(\alpha_1(x) + \varepsilon_1(x)/(\alpha_2(x) + \varepsilon_2(x)/(\alpha_3(x) + \cdots)))$$

where $\varepsilon_n \in \{-1, 1\}$, $\alpha_n \geq 1$, $\alpha_n \equiv 1 \pmod{2}$ and $\alpha_n + \varepsilon_n > 1$. Using the ergodic behaviour of a certain homogeneous random system with complete connections we solve a variant of Gauss-Kuzmin problem for the above expansion.

1. Introduction. We define continued fraction with odd partial quotients as follows. Let us partition the unit interval into

$$\left(\frac{1}{2k}, \frac{1}{2k-1}\right], \quad k = 1, 2, \dots, \quad \text{and} \quad \left(\frac{1}{2k-1}, \frac{1}{2k-2}\right], \quad k = 2, 3, \dots$$

and define the transformation $T: [0, 1] \rightarrow [0, 1]$ by

$$Tx = e \left(\frac{1}{x} - (2k - 1) \right)$$

where

$$e = 1 \quad \text{if } x \in \left(\frac{1}{2k}, \frac{1}{2k-1}\right],$$

and

$$e = -1 \quad \text{if } x \in \left(\frac{1}{2k-1}, \frac{1}{2k-2}\right].$$

We arrive at

$$x = \frac{1}{2k - 1 + e(Tx)}$$

and therefore the map T generates a continued fraction

$$(1.1) \quad x = \frac{1}{\alpha_1(x) + \varepsilon_1(x)/(\alpha_2(x) + \varepsilon_2(x)/(\alpha_3(x) + \cdots))} \\ = \left[\begin{array}{c} 1, \varepsilon_1(x), \varepsilon_2(x), \dots \\ \alpha_1(x), \alpha_2(x), \alpha_3(x), \dots \end{array} \right]$$

where $\varepsilon_n \in \{-1, 1\}$, $\alpha_n \geq 1$, $\alpha_n \equiv 1 \pmod{2}$ and $\alpha_n + \varepsilon_n > 1$. The expression (1.1) is called the *continued fraction with odd partial quotients expansion* of x .

Let us denote

$$r_n = \alpha_n + \left[\begin{smallmatrix} \varepsilon_{n+1}, \dots \\ \alpha_{n+1}, \dots \end{smallmatrix} \right], \quad n = 1, 2, \dots$$

The purpose of this paper is to find the limit

$$\lim_{n \rightarrow \infty} \mu(r_n > t) = l$$

for a given nonatomic measure μ on the σ -algebra of the Borel sets of $[0, 1]$ and to estimate the error $\mu(r_n > t) - l$. This is the variant of Gauss-Kuzmin problem for the continued fraction with odd partial quotients expansion. For solving of the above problem we shall use the approach of the random system with complete connections.

NOTATION.

$N^* = \{1, 2, 3, \dots\}$,

$N = \{0, 1, 2, \dots\}$,

R = the set of real numbers,

$[a]$ = the integral part of $a \in R$,

I_A = the characteristic function of A ,

$G = (\sqrt{5} + 1)/2$,

$\mathcal{B}_{[0,1]}$ = the σ -algebra of the Borel sets of $[0, 1]$,

$\mathcal{P}(X)$ = the power set of X ,

(X, \mathcal{X}^n) = the n -fold product measurable space of (X, \mathcal{X}) .

2. Preliminaries.

DEFINITION 2.1. A quadruple $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ is named a homogeneous random system with complete connections (RSCC) if

- (i) (W, \mathcal{W}) and (X, \mathcal{X}) are arbitrary measurable spaces;
- (ii) $u: W \times X \rightarrow W$ is a $(\mathcal{W} \otimes \mathcal{X}, \mathcal{W})$ -measurable function;
- (iii) P is a transition probability function from (W, \mathcal{W}) to (X, \mathcal{X}) .

Next, denote the element $(x_1, \dots, x_n) \in X^n$ by $x^{(n)}$.

DEFINITION 2.2. The functions $u^{(n)}: W \times X^n \rightarrow W$, $n \in N^*$, are defined as follows:

$$u^{(n+1)}(w, x^{(n+1)}) = \begin{cases} u(w, x), & \text{if } n = 0 \\ u(u^{(n)}(w, x^{(n)}), x_{n+1}), & \text{if } n \geq 1. \end{cases}$$

Convention. We shall write $wx^{(n)}$ instead of $u^{(n)}(w, x^{(n)})$.

DEFINITION 2.3. The transition probability functions P_r , $r \in N^*$, are defined by

$$P_r(w, A) = \begin{cases} P(w, A), & \text{if } r = 1 \\ \sum_{x_1 \in X} P(w, x_1) \sum_{x_2 \in X} P(wx_1, x_2) \cdots \sum_{x_r \in X} P(wx^{(r-1)}, x_r) I_A(x^{(r)}), & \text{if } r > 1, \end{cases}$$

for any $w \in W$, $r \in N^*$ and $A \in \mathcal{X}^r$.

DEFINITION 2.4. Assume that $X^0 \times A = A$. Then we define

$$P_r^n(w, A) = P_{n+r-1}(w, X^{n-1} \times A),$$

for any $w \in W$, $n, r \in N^*$ and $A \in \mathcal{X}^r$.

THEOREM 2.5. (*Existence theorem.*) Let $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ be a homogeneous RSCC and let $w_0 \in W$. Then there exist a probability space $(\Omega, \mathcal{X}, P_{w_0})$ and two chains of random variables $(\xi_n)_{n \in N^*}$ and $(\zeta_n)_{n \in N}$ defined on Ω with values in X and W respectively, such that

$$(i)(a) \quad P_{w_0}((\xi_n, \dots, \xi_{n+r-1}) \in A) = P_r^n(w_0, A),$$

$$(b) \quad P_{w_0}((\xi_{n+m}, \dots, \xi_{n+m+r-1}) \in A | \xi^{(n)}) = P_r^m(w_0 \xi^{(n)}, A), \quad P_{w_0}\text{-a.e.}$$

$$(c) \quad P_{w_0}((\xi_{n+m}, \dots, \xi_{n+m+r-1}) \in A | \xi^{(n)}, \zeta^{(n)}) = P_r^m(\zeta_n, A), \quad P_{w_0}\text{-a.e.}$$

for any $n, m, r \in N^*$ and $A \in \mathcal{X}^r$, where $\xi^{(n)}$, $\zeta^{(n)}$ denote the random vectors (ξ_1, \dots, ξ_n) and $(\zeta_1, \dots, \zeta_n)$ respectively.

(ii) $(\zeta_n)_{n \in N}$ is a homogeneous Markov chain with initial distribution concentrated in w_0 and with the transition operator U defined by

$$(2.1) \quad Uf(w) = \sum_{x \in X} P(w, x) f(wx),$$

for any f real W -measurable and bounded function.

This theorem is proved by Iosifescu [2].

REMARK. (i) Letting $m = r = 1$ in (i)b we obtain

$$P_{w_0}(\xi_{n+1} \in A | \xi^{(n)}) = P(w_0 \xi^{(n)}, A), \quad P_{w_0}\text{-a.e.}$$

that is the conditioned distribution of ξ_{n+1} by the past depends actually by this, through $u^{(n)}$. This fact justifies the name of *chain of infinite order* or *chain with complete connections* used for $(\xi_n)_{n \in N^*}$.

(ii) On account of (2.1) we have

$$(2.2) \quad U^n f(w) = \sum_{x^{(n)} \in X^n} P_n(w, x^{(n)}) f(wx^{(n)}), \quad n \in N^*$$

for any f real \mathcal{W} -measurable and bounded function.

(iii) The transition probability function of the Markov chain $(\xi_n)_{n \in N^*}$ is

$$Q(w, A) = \sum_{x \in X} P(w, x) I_A(wx) = P(w, A_w),$$

where $A_w = \{x \in X: wx \in A\}$, $w \in W$. It follows that the transition probability after n paths of the Markov chain $(\xi_n)_{n \in N}$ is

$$Q^n(w, A) = P_n(w, A_w^{(n)}),$$

where $A_w^{(n)} = \{x^{(n)}: wx^{(n)} \in A\}$.

2.6. Let Q_n be the transition probability function defined by

$$Q_n(w, A) = n^{-1} \sum_{k=1}^n Q^k(w, A)$$

and let U_n be the Markov operator associated with Q_n . Next, denote $L(W)$ the space of all real Lipschitz functions defined on W and assume that $(L(W), \|\cdot\|)$ is a Banach space with respect to a norm $\|\cdot\|$.

(i) If there exists a linear bounded operator U^∞ from $L(W)$ to $L(W)$ such that

$$\lim_{n \rightarrow \infty} \|U_n f - U^\infty f\| = 0,$$

for any $f \in L(W)$ with $\|f\| = 1$, we say U *ordered*.

(ii) If

$$\lim_{n \rightarrow \infty} \|U^n f - U^\infty f\| = 0,$$

for any $f \in L(W)$ with $\|f\| = 1$, we say U *aperiodic*.

(iii) If U is ordered and $U^\infty(L(W))$ is one-dimensional space, it is named *ergodic* with respect to $L(W)$.

(iv) If U is ergodic and aperiodic, it is named *regular* with respect to $L(W)$ and the corresponding Markov chain has the same name.

DEFINITION 2.7. If $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ is a RSCC which satisfies the properties

(i) (W, d) is a metric separable space;

(ii) $r_1 < \infty$, where

$$r_k = \sup_{w' \neq w''} \sum_{X^k} P_k(w, x^{(k)}) \frac{d(w'x^{(k)}, w''x^{(n)})}{d(w', w'')}, \quad k \in N^*;$$

(iii) $R_1 < \infty$, where

$$R_1 = \sup_{A \in \mathcal{X}} \sup_{w' \neq w''} \frac{|P(w', A) - P(w'', A)|}{d(w', w'')};$$

(iv) there exists $k \in N^*$ such that $r_k < 1$, it is named RSCC with contraction.

This definition is due to M. F. Norman [3].

THEOREM 2.8. *Let (W, d) be a compact space and $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ be a RSCC with contraction.*

The Markov chain associated to the RSCC is regular, if and only if, there exists a point $\tilde{w} \in W$ such that

$$\lim_{n \rightarrow \infty} d(\sigma_n(\tilde{w}), w) = 0,$$

for any $w \in W$, where $\sigma_n(w) = \text{supp } Q^n(w, \cdot)$ ($\text{supp } \mu$ denotes the support of the measure μ).

LEMMA 2.9. *We have*

$$\sigma_{m+n}(w) = \overline{\bigcup_{w' \in \sigma_m(w)} \sigma_n(w')},$$

for any $m, n \in N$, $w \in W$ (the line designates the topological adherence).

Theorem 2.8 and Lemma 2.9 are due to Iosifescu [1].

DEFINITION 2.10. Let $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ be a RSCC. The RSCC is called uniformly ergodic if for any $r \in N^*$ there exists a probability P_r^∞ on \mathcal{X}^r such that $\lim \varepsilon_n = 0$, as $n \rightarrow \infty$, where

$$\varepsilon_n = \sup_{\substack{w \in W, r \subseteq N^* \\ A \in \mathcal{X}^r}} |P_r^n(w, A) - P_r^\infty(A)|.$$

THEOREM 2.11. *Let (W, d) be a compact space. If the RSCC $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ with contraction has regular associated Markov chain, then it is uniform ergodic.*

This result one can find in [1].

3. The Gauss-Kuzmin type equation. Let μ be a nonatomic measure on $\mathcal{B}_{[0,1]}$ and define

$$F_n(w) = F_n(w, \mu) = \mu(r_{n+1}^{-1} < w), \quad n \in N, w \in [0, 1].$$

Clearly $F_0(w) = \mu([0, w])$.

PROPOSITION 3.2. (*The Gauss-Kuzmin type equation*) F_n , $n \in N$, satisfy the relation

$$F_{n+1}(w) = \sum_{\substack{(k, \epsilon) \\ k \equiv 1 \pmod{2} \\ |\epsilon|=1, k+\epsilon > 1}} \epsilon \left(F_n\left(\frac{1}{k}\right) - F_n\left(\frac{1}{k + \epsilon w}\right) \right), \quad w \in [0, 1].$$

Proof. We start from the relation

$$r_{n+1} = \alpha_{n+1} + \frac{\epsilon_{n+1}}{r_{n+2}}.$$

Thus

$$\begin{aligned} F_{n+1}(w) &= \mu(r_{n+2}^{-1} < w, \epsilon_{n+1} = 1) + \mu(r_{n+2}^{-1} < w, \epsilon_{n+1} = -1) \\ &= \sum_{k \equiv 1 \pmod{2}} \mu((k + w)^{-1} < r_{n+1}^{-1} < k^{-1}) \\ &\quad + \sum_{\substack{k \equiv 1 \pmod{2} \\ k \neq 1}} \mu(k^{-1} < r_{n+1}^{-1} < (k - w)^{-1}) \\ &= \sum_{(k, \epsilon)} \epsilon \left(F_n\left(\frac{1}{k}\right) - F_n\left(\frac{1}{k + \epsilon w}\right) \right). \end{aligned}$$

and this completes the proof.

Further, suppose that F'_0 exists and it is bounded (μ has bounded density). By induction we obtain that F'_n exists and it is bounded too for any $n \in N^*$. Deriving the Gauss-Kuzmin type equation we arrive at

$$(3.1) \quad F'_{n+1}(w) = \sum_{(k, \epsilon)} \frac{1}{(k + \epsilon w)^2} F'_n\left(\frac{1}{k + \epsilon w}\right).$$

Let us denote for $\rho(w) = (G - 1 + w)^{-1} - (-G - 1 + w)^{-1}$, $w \in [0, 1]$ and $n \in N$

$$f_n(w) = F'_n(w)/\rho(w).$$

Then (3.1) becomes

$$f_{n+1}(w) = (G^2 - (1 - w)^2) \times \sum_{(k, \varepsilon)} \frac{1}{((G - 1)(k + \varepsilon w) + 1)((G + 1)(k + \varepsilon w) - 1)} f_n\left(\frac{1}{k + \varepsilon w}\right).$$

Now, we prove

PROPOSITION 3.3. *The function*

$$P(w, (k, \varepsilon)) = \frac{G^2 - (1 - w)^2}{((G - 1)(k + \varepsilon w) + 1)((G + 1)(k + \varepsilon w) - 1)}$$

defines a transition probability function from $([0, 1], \mathcal{B}_{[0,1]})$ to $(X, \mathcal{P}(X))$ where

$$X = \{(k, \varepsilon): k \geq 1, k \equiv 1 \pmod{2}, |\varepsilon| = 1, k + \varepsilon > 1\}.$$

Proof. We must verify that

$$\sum_{(k, \varepsilon)} P(w, (k, \varepsilon)) = 1.$$

Indeed, noting that $(G - 1)^{-1} = G$ and $(G + 1)^{-1} = -G + 2$, we have

$$\begin{aligned} & \sum_{(k, \varepsilon)} \frac{G^2 - (1 - w)^2}{((G - 1)(k + \varepsilon w) + 1)((G + 1)(k + \varepsilon w) - 1)} \\ &= \frac{G^2 - (1 - w)^2}{G^2 - 1} \left(\sum_{k=1,3,\dots} \frac{1}{(k + w + G)(k + w + G - 2)} \right. \\ & \quad \left. + \sum_{k=3,5,\dots} \frac{1}{(k - w + G)(k - w + G - w)} \right) \\ &= \frac{G^2 - (1 - w)^2}{2G} \left(\sum_{k=1,3,\dots} \left(\frac{1}{k - 2 + w + G} - \frac{1}{k + w + G} \right) \right. \\ & \quad \left. + \sum_{k=3,5,\dots} \left(\frac{1}{k - 2 - w + G} - \frac{1}{k - w + G} \right) \right) \\ &= \frac{G^2 - (1 - w)^2}{2G} \left(\frac{1}{G - (1 - w)} + \frac{1}{G + (1 - w)} \right) = 1 \end{aligned}$$

that is the desired result.

Now, we can define a random system with complete connections as follows.

$$(3.2) \quad \{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$$

where

$$W = [0, 1], \quad \mathcal{W} = \mathcal{B},$$

$$X = \{(k, \varepsilon) : k \geq 1, k \equiv 1 \pmod{2}, |\varepsilon| = 1, k + \varepsilon > 1\},$$

$$\mathcal{X} = \mathcal{P}(X), \quad u(w, (k, \varepsilon)) = \frac{1}{k + \varepsilon w},$$

$$P(w, (k, \varepsilon)) = \frac{G^2 - (1 - w)^2}{((G - 1)(k + \varepsilon w) + 1)((G + 1)(k + \varepsilon w) - 1)}.$$

4. The ergodic behaviour of the RSCC. In this section we study the ergodic behaviour of RSCC (3.2) in order to solve a Gauss-Kuzmin type theorem.

In what follows we shall introduce the norm $\|\cdot\|_L$ defined by

$$\|f\|_L = \sup_{w \in W} |f(w)| + \sup_{w' \neq w''} \frac{|f(w') - f(w'')|}{|w' - w''|}, \quad f \in L(W).$$

Then $(L(W), \|\cdot\|_L)$ is a Banach algebra.

PROPOSITION 4.1. *RSCC (3.2) is a RSCC with contraction and its associated Markov operator U is regular with respect to $L(W)$.*

Proof. We have

$$\begin{aligned} \frac{dP}{dw} &= \frac{2(1 - w)((G - 1)(x + \varepsilon w) + 1)((G + 1)(x + \varepsilon w) - 1)}{((G - 1)(k + \varepsilon w) + 1)^2((G + 1)(k + \varepsilon w) - 1)^2} \\ &\quad - \frac{2\varepsilon(G^2 - (1 - w)^2)((G^2 - 1)(x + \varepsilon w) + 1)}{((G - 1)(k + \varepsilon w) + 1)^2((G + 1)(k + \varepsilon w) - 1)^2}, \\ \frac{du}{dw} &= \frac{\varepsilon}{(k + \varepsilon w)^2}. \end{aligned}$$

Therefore

$$\begin{aligned} \sup_{w \in W, (x, \varepsilon) \in X} \left| \frac{d}{dw} P(w, (x, \varepsilon)) \right| &< \infty, \\ \sup_{w \in W} \left| \frac{d}{dw} u(w, (x, \varepsilon)) \right| &< \frac{1}{(x - 1)^2}, \quad k = 3, 5, \dots \end{aligned}$$

It follows that $R_1 < \infty$ and $r_1 < 1$, that is, (3.2) is a SALC with contraction.

To prove the regularity of U with respect to $L(W)$, define the recurrence relation $w_{n+1} = (w_n + 1)^{-1}$, $n \in N$, with $w_0 = w$. Clearly $w_{n+1} \in \sigma(w_n)$. Then using Lemma 2.9 and by induction we obtain $w_n \in \sigma_n(w)$, $n \in N^*$. Because w_n tends to $G - 1$ as $n \rightarrow \infty$, for every $w \in [0, 1]$, then

$$|\sigma_n(w), G - 1| \leq |w_n - G + 1| \rightarrow 0$$

as $n \rightarrow \infty$. The regularity of U with respect to $L(W)$ follows from Theorem 2.8. and the proof is completed.

Now, by virtue of Theorem 2.11, RSCC (3.2) is uniformly ergodic. Moreover, Theorem 2.1.57 of Iosifescu-Theodorescu [2], implies that $Q^n(\cdot, \cdot)$ converges uniformly to a probability Q^∞ and there exist two positive constants $q < 1$ and c such that

$$(4.1) \quad \|U^n f - U^\infty f\|_L \leq cq^n$$

for all $n \in N^*$, $f \in L(W)$, where

$$(4.2) \quad U^\infty f = \int_W f(w) Q^\infty(dw).$$

Further, by virtue of Lemma 2.1.58 of Iosifescu-Theodorescu [2], U has no eigenvalues of modulus 1 other than 1. Then, taking into account Proposition 2.1.6 of [2] the adjoint of the operator U with the transition probability function Q has the only eigenvector the measure Q^∞ , that is

$$(4.3) \quad \int_0^1 Q(w, B) Q^\infty(dw) = Q^\infty(B),$$

for all the Borel sets B of $[0, 1]$.

Generally, the form of Q^∞ cannot be identified but in our case this is possible as we shall show below

PROPOSITION 4.2. *The probability Q^∞ has the density*

$$\rho(w) = \frac{1}{w + G - 1} - \frac{1}{w - G - 1}, \quad w \in [0, 1]$$

and the normalizing constant $1/(3 \log G)$.

Proof. By virtue of uniqueness of Q^∞ we have to prove the equality (4.3) where

$$Q(w, B) = \sum_{\substack{(x, \varepsilon) \in X \\ (x + \varepsilon w)^{-1} \in B}} P(w, (x, \varepsilon)), \quad w \in [0, 1], \quad B \in \mathcal{B}_{[0, 1]}.$$

Since the intervals $[0, u] \subset [0, 1]$ generate $\mathcal{B}_{[0, 1]}$, it suffices to verify the equality (4.3) only for $B = [0, u]$, $0 < u \leq 1$. First, we consider that $[u^{-1}]$ is even. Then

$$\begin{aligned} & \int_0^1 Q(w, [0, u]) \rho(w) dw \\ &= \int_0^1 \left(\sum_{\substack{k=1,3,\dots \\ k > [u^{-1}-w]}} P(w, (k, 1)) + \sum_{\substack{k=3,5,\dots \\ k > [u^{-1}]}} P(w, (k, -1)) \right) \rho(w) dw \\ &= \int_0^1 \frac{G^2 - (1-w)^2}{2(G^2 - 1)} \\ & \quad \times \left(\sum_{k=[u^{-1}]+1}^{\infty} \left(\frac{1}{k-2+w+G} - \frac{1}{k+w+G} \right) \right) \rho(w) dw \\ & \quad + \int_0^{1+[u^{-1}]-u^{-1}} \frac{G^2 - (1-w)^2}{2G} \\ & \quad \times \left(\sum_{k=[u^{-1}]+1}^{\infty} \left(\frac{1}{k-2+w+G} - \frac{1}{k+w+G} \right) \right) \rho(w) dw \\ & \quad + \int_{1+[u^{-1}]-u^{-1}}^1 \frac{G^2 - (1-w)^2}{2G} \\ & \quad \times \left(\sum_{k=[u^{-1}]+3}^{\infty} \left(\frac{1}{k-2-w+G} - \frac{1}{k-w+G} \right) \right) \rho(w) dw \\ &= \log \frac{G + [u^{-1}]}{G + [u^{-1}] - 1} \cdot \frac{G + [u^{-1}] - 1}{G + u^{-1} - 2} \cdot \frac{G + u^{-1}}{G + [u^{-1}]} \\ &= \log \frac{(G+1)(G-1+u)}{(G-1)(G+1-u)} = \int_0^u \rho(w) dw. \end{aligned}$$

Analogously if $[u^{-1}]$ is odd we have

$$\begin{aligned}
 & \int_0^1 Q(w, [0, u)) \rho(w) dw \\
 &= \int_0^{u^{-1}-[u^{-1}]} \frac{G^2 - (1-w)^2}{2G} \cdot \frac{\rho(w)}{[u^{-1}] + w + G} dw \\
 &+ \int_{u^{-1}-[u^{-1}]}^1 \frac{G^2 - (1-w)^2}{2G} \cdot \frac{\rho(w)}{[u^{-1}] - 2 + w + G} dw \\
 &+ \int_0^1 \frac{G^2 - (1-w)^2}{2G} \cdot \frac{\rho(w)}{[u^{-1}] - w + G} dw \\
 &= \log \frac{u^{-1} + G}{[u^{-1}] + G} \cdot \frac{[u^{-1}] + G - 1}{u^{-1} + G - 1} \cdot \frac{[u^{-1}] + G}{[u^{-1}] + G - 1} \\
 &= \log \frac{(G+1)(G-1+u)}{(G-1)(G+1-u)} = \int_0^u \rho(w) dw.
 \end{aligned}$$

5. The Gauss-Kuzmin type theorem. Now, we may determine where $\mu(r_n > t)$ tends as $n \rightarrow \infty$ and give the rate of this convergence.

PROPOSITION 5.1. (*The solution of Gauss-Kuzmin type problem.*) *If the density F'_0 of μ is a Riemann integrable function, then*

$$\lim_{n \rightarrow \infty} \mu(r_n > t) = \frac{1}{3 \log G} \cdot \log \frac{(G+1)(t(G-1)+1)}{(G-1)(t(G+1)-1)}, \quad t \geq 1.$$

If the density F'_0 of μ is a Lipschitz function, then there exist two positive constants c and $q < 1$ such that for all $t \geq 1$, $n \in N^$*

$$\mu(r_n > t) = \frac{1}{3 \log G} (1 + \theta q^n) \log \frac{(G+1)(t(G-1)+1)}{(G-1)(t(G+1)-1)}$$

where $\theta = \theta(\mu, n, t)$ with $|\theta| \leq c$.

Proof. Let F'_0 be a Lipschitz function. Then $f_0 \in L(W)$ and by virtue of (4.2)

$$U^\infty f_0 = \int_0^1 f_0(w) Q^\infty(dw) = \frac{1}{3 \log G} \int_0^1 F'_0(w) dw = \frac{1}{3 \log G}.$$

According to (4.1) there exist two constants c and $q < 1$ such that

$$U^n f_0 = U^\infty f_0 + T^n f_0, \quad n \in N^*,$$

with $\|T^n f_0\|_L \leq cq^n$.

Further, consider $C[0, 1]$ the metric space of real continuous functions defined on $[0, 1]$ with the norm $|\cdot| = \sup|\cdot|$. Since $L([0, 1])$ is a dense subset of $C([0, 1])$ we have

$$(5.1) \quad \lim_{n \rightarrow \infty} |T^n f_0| = 0$$

for $f_0 \in C([0, 1])$. Therefore (5.1) is valid for measurable f_0 which is Q^∞ -almost surely continuous, that is for Riemann integrable f_0 . Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(r_n > t) &= \lim_{n \rightarrow \infty} F_{n-1}\left(\frac{1}{t}\right) \\ &= \lim_{n \rightarrow \infty} \int_0^{1/t} U^{n-1} f_0(u) \rho(u) dw \\ &= \frac{1}{3 \log G} \int_0^{1/t} \rho(w) dw \end{aligned}$$

and the desired result follows.

REFERENCES

- [1] S. Grigorescu and M. Iosifescu, *Dependence with Complete Connections and Applications*, Ed. St. Encicl., Bucharest, 1982.
- [2] M. Iosifescu and R. Theodorescu, *Random Processes and Learning*, Springer-Verlag, Berlin, 1969.
- [3] M. F. Norman, *Markov Processes and Learning Models*, Academic Press, New York, 1972.
- [4] F. Schweiger, *Continued fractions with odd and even partial quotients*, Arbeitsbericht Math. Institut. der Univ. Salzburg, 4 (1982), 59–70.

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