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OPERATORS ON AMALGAMS**

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# POSITIVE DEFINITE FUNCTIONS AND $L^p$ CONVOLUTION OPERATORS ON AMALGAMS

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Let  $K_i$  be a countable collection of compact groups, and assume that  $H = \cap_i K_i$  is an open subgroup of  $K_i$  for every  $i$ . In this paper we consider positive definite functions and convolution operators on the amalgamated product  $G = *_H K_i$ , and we study their properties in relation with the notion of length of reduced words. In particular, if  $\sup_i k_i < \infty$ , we show that there exist unbounded approximate identities in  $A(G)$ , that the space of bounded convolution operators on  $L_p(G)$  is the dual space of the algebra  $A_p(G)$ , and, under the additional assumption that  $H$  be finite, that there exist unbounded approximate identities in  $A(G)$ .

**1. Introduction.** Considerable attention has been devoted, in the recent literature, to positive definite functions on groups acting isometrically on homogeneous trees. The Fourier-Stieltjes algebra of the free group  $F_r$  with  $r$  generators, which acts isometrically on the homogeneous tree of degree  $2r$ , has been studied in detail in [5, 14, 9, 6, 1, 13]. Other free products have been considered in [16, 17, 4]. The class of groups acting simply transitively on a homogeneous tree has been considered in [3]. Every locally compact group  $G$  acting isometrically on a homogeneous or semihomogeneous tree  $T$  is isomorphic to the amalgamated product  $K_1 *_H K_2$ , where  $K_1$  and  $K_2$  are the stability subgroups of two contiguous vertices and  $H = K_1 \cap K_2$  is the stability subgroup of the corresponding edge [18]. The subgroup  $H$  is open, and its indices in  $K_1$  and  $K_2$  are the homogeneity degrees of  $T$ . In particular, if the homogeneity degrees are finite,  $G$  is the amalgam of two compact groups. Some properties of positive definite functions on amalgams of two factors have been studied in [2, 11].

In this paper we consider amalgamated products  $G = *_H G_i$ , where  $\{G_i, i \in I\}$  is any collection of locally compact groups and  $H$  is a common open subgroup. These groups act isometrically on trees with periodical homogeneity degrees (and on "polygonal graphs": see [16]). The homogeneity degrees are finite if and only if the factors  $G_i$  are compact; they are bounded if and only if the indices  $k_i$  of  $H$  in  $G_i$  are bounded. For groups of this type, we consider several results originally obtained for free groups in [14, 8, 13]. Some of our arguments are adapted

from these references: we refer to Chapters 2 and 8 of [13] for a detailed account on free groups.

In §2, we prove that the amalgamated product of two-sided  $H$ -invariant functions  $f_i$  which are positive definite on the subgroups  $G_i$  is a positive definite function on  $G$ . This result was originally proved in [4] for free products of two factors. As a consequence, the length of  $G$  is a continuous negative definite function: this extends a result of [2] (see also [11]). In §3, we restrict attention to trees with bounded homogeneity degrees, that is, to amalgams  $G = *_H K_i$ , where the indices  $k_i = \#(K_i/H)$  are bounded. The result of the previous section yields a family of positive definite functions  $f_t$  on  $G$  which are radial, decay exponentially at infinity and converge to 1 uniformly on compact sets as  $t \searrow 0$ . By means of these functions, if  $H$  is finite we construct approximate identities for the Fourier algebra  $A(G)$  which are bounded in the multiplier norm (Theorem 3). In the terminology of [14], Theorem 3 asserts that the Fourier algebra  $A(G)$  has the metric approximation property; we observe that, by the same argument, the reduced  $C^*$ -algebra  $C_\lambda^*(G)$  has the metric approximation property (see [14, 11]). Theorem 3 is a consequence of the following result (due to U. Haagerup [13] in the special case of free groups; see also [16]): if  $f$  is a function supported on words of length  $n$ , then the norm of  $f$  as a convolution operators on  $L^2$  grows as  $O(n)$  (Theorem 2: this result holds without the assumption that  $H$  is finite). For the special case of amalgams of two finite groups, Theorems 2 and 3 have been obtained independently in [11].

We then consider functions on  $G$  which are multiplicative with respect to the length and, for every  $i$ , are constant on  $K_i - H$ . Using these functions, we prove a result which indicates that  $A(G)$  is “much smaller” than  $B(G)$  (Theorem 4; see also [9, 1, 4]). Even though Theorem 4 emphasizes the fact that  $G$  is nonamenable, nevertheless, in §4, we prove a result which restores some similarity with amenable groups. For every commutative group  $G$ , the space  $Cv_p$  of bounded convolution operators on  $L^p(G)$  is known to be isometrically isomorphic with the algebra  $A_p$  [12]. This result has been extended to all amenable groups in [15]. In the context of nonamenable groups a similar result is known for  $SL_2(\mathbf{R})$  and some other semisimple Lie groups [7], and for free groups [8, 13]. We show (Theorem 5) that  $A_p^* \simeq Cv_p$  for amalgams  $G = *_H K_i$  if the indices  $k_i$  are bounded. We do not know whether this assumption can be dropped.

Unexplained terminology and notations are as in [10, 13]. We are grateful to M. Bozejko, M. Enomoto and Y. Watatani who communicated to us their results of [4, 5, 11] prior to publication.

**2. Amalgamated products and length.** Let  $I$  be any set of indices and let  $\{G_i, i \in I\}$  be a family of locally compact groups containing a common open subgroup  $H = \bigcap_i G_i$ . Then the amalgamated product  $G = *_H G_i$ , endowed with the strongest topology such that the natural embeddings  $G_i \rightarrow G$  are continuous, is a locally compact group [2, 17]. Choose and fix a set  $S_i$  of representatives of the right  $H$ -cosets in  $G_i$ . Then every element  $s$  of  $G$  can be uniquely written as a reduced word  $s = s_{i_1} \cdots s_{i_n} h$  with  $h \in H$ ,  $s_{i_j} \in S_{i_j} - H$  and  $i_j \neq i_{j+1}$  for every  $j$ . The number  $n$  is called the *length* of  $s$ , and is denoted by  $|s|$ . In this section we prove that the length is a continuous negative definite function on  $G$ .

**DEFINITION.** Let  $f_i$  be right  $H$ -invariant functions on  $G_i$ ,  $i \in I$ , with  $f_i|_H = 1$ .

The function  $f = *_H f_i$ , defined by the rule  $f(g) = \prod_{m=1}^n f_{i_m}(s_{i_m})$  if  $g = s_{i_1} \cdots s_{i_n} h$ , is called the amalgamated product of the  $f_i$ 's over  $H$ .  $\square$

The result about the length function is a consequence of the following theorem, whose argument is essentially due to M. Bozejko [4, Thm. 1].

**THEOREM 1.** *For  $i \in I$ , let  $f_i$  be a positive definite bi- $H$ -invariant function on  $G_i$  with  $f_i|_H = 1$ . Suppose that the amalgamated product  $f = *_H f_i$  is bi- $H$ -invariant. Then  $f$  is positive definite on  $G$ .*

*Proof.* First, we claim that the assumption that  $f$  is bi- $H$ -invariant is equivalent to the property

$$(1) \quad f(xy) = f(x)f(y) \quad \text{if } |xy| = |x| + |y|.$$

Indeed, we show that (1) holds if  $f$  is bi- $H$ -invariant (the converse, not needed here, is clear; see also the definition and remarks following Thm. 3). Write, as before,  $x = s_{i_1} \cdots s_{i_n} h$ , and set  $x_0 = s_{i_1} \cdots s_{i_n}$ . Write also  $hy = s_{j_1} \cdots s_{j_m} h_0$ , and observe that  $s_{j_1} \neq s_{i_n}^{-1} \bmod H$ , since  $|xy| = |x| + |y|$ . Hence, by the definition of  $f$ , it follows  $f(xy) = f(x_0)f(hy)$ . On the other hand, since  $f$  is bi- $H$ -invariant,  $f(x_0) = f(x_0 h) = f(x)$ , and  $f(hy) = f(y)$ . Thus  $f(xy) = f(x_0)f(y) = f(x)f(y)$ , and the claim is proved. For  $k = 1, \dots, n$ , choose  $c_k$  in  $C$ ,  $x_k$  in  $G$ . To prove that  $f$  is positive definite, we must prove that, for each such choice,

$$(2) \quad \sum_{m,k} c_m \bar{c}_k f(x_k^{-1} x_m) \geq 0.$$

The left hand side in this inequality is a pre-norm for the function  $h = \sum c_k \delta_k$ ; we denote it by  $\|h\|_f$ . Observe that  $\sum c_k f(x_k) = \langle h, \delta_e \rangle_f$  and

$\|\delta_e\|_f = f(e) = 1$ . Thus the Cauchy-Schwarz inequality shows that (2) is equivalent to the apparently stronger inequality

$$(3) \quad \sum_{m,k} c_m \bar{c}_k f(x_k^{-1} x_m) \geq \left| \sum_k c_k f(x_k) \right|^2.$$

For  $i \in I$ , denote by  $s_{i_j}$  the representatives of the cosets in  $G_i/H$ , and let  $S = \{s_{i_j}\}$ . Define a function  $i: G \rightarrow S$  by  $i(x) = i(s_{i_1 j_1} \cdots s_{i_n j_n} h) = s_{i_1 j_1}$ , and let  $\tilde{x} = s_{i_2 j_2} \cdots s_{i_n j_n} h$ . For some  $j \in I$ , say  $j = 1$ , the subset  $A$  of  $U \equiv \{x_1, \dots, x_n\}$  given by  $A = U \cap i^{-1}(G_j/H)$  is non-empty. We now let  $B = U - A$ , and proceed by induction on  $n = \#U$ . We first show that (2) holds in the case  $U = A$ . Let  $A_r = A \cap i^{-1}(s_{1r})$ . In order to compute the left hand side of (2), it is enough to assume that  $A_r \neq A$  for every  $r$ : if this is not so for some  $r$ , it suffices to replace  $U = A$  by  $s_{1r}^{-1} \cdot A$ . Now (1) and the definition of  $f$  yield

$$\begin{aligned} \sum_{m,k} c_m \bar{c}_k f(x_k^{-1} x_m) &= \sum_r \sum_{A_r \times A_r} c_m \bar{c}_k f(\tilde{x}_k^{-1} \tilde{x}_m) \\ &\quad + \sum_{r \neq j} \sum_{A_r \times A_j} c_m \bar{c}_k f(\tilde{x}_k^{-1}) f(\tilde{x}_m) f_1(i(x_k)^{-1} i(x_m)). \end{aligned}$$

Set  $b_r = \sum_{x_k \in A_r} c_k f(\tilde{x}_k)$ . As  $\#A_r < \#U$ , by the induction hypothesis the right hand side of this inequality is greater than or equal to  $\sum_r |b_r|^2 + \sum_{r \neq j} b_r \bar{b}_j f_1(s_{1j}^{-1} s_{1r})$ , which is non-negative because  $f_1$  is positive definite on  $G_1$ . Thus (2), and therefore (3), hold in this case.

The general case  $U = A \cup B$  is handled by the same token. Indeed, we have just proved that (3) holds for the subset  $A$  of  $U$ , and, as  $B$  is properly contained in  $U$ , we can assume, by induction on  $\#U$ , that (3) holds for  $B$ . Moreover, if  $x \in A$  and  $y \in B$ , then  $|xy| = |x| + |y|$ , and we can apply (1). Now we decompose the sum in the left hand side of (3) into three parts, corresponding to the conditions that  $x_k$  and  $x_m$  both belong to  $A$ , or both to  $B$ , or one to  $A$  and the other to  $B$ . By the induction hypothesis, it follows that

$$\begin{aligned} \sum_{m,k} c_m \bar{c}_k f(x_k^{-1} x_m) &\geq \left| \sum_A c_k f(x_k) \right|^2 + \left| \sum_B c_m f(x_m) \right|^2 \\ &\quad + 2 \operatorname{Re} \sum_{A \times B} c_m \bar{c}_k \bar{f}(x_k) f(x_m) \geq \left| \sum_U c_k f(x_k) \right|^2. \quad \square \end{aligned}$$

**REMARK 1.** As a consequence of Theorem 1, for  $t \geq 0$ , the function  $x \rightarrow \exp(-t|x|)$  is a (continuous) positive definite function on  $G$ . This result is due to U. Haagerup [14] for the special case of free groups (see also [1], [11]).

**3. Convolution operators and positive definite functions on amalgams.** From now on, we assume that  $G = *_H K_i$ , where the  $K_i$ ,  $i \in I$ , are a family of compact groups, and the indices  $k_i$  of  $H = \bigcap_i K_i$  in  $K_i$  are bounded. We shall refer to the groups  $K_i$  as the *factors* of  $G$ . We normalize the Haar measure  $m$  of  $G$  so that  $m(H) = 1$ .

Following [16], we say that two words  $x = s_{i_1} \cdots s_{i_n} h$ ,  $y = s_{j_1} \cdots s_{j_m} h'$  give rise to a *simple cancellation* if  $|xy| = |x| + |y| - 2$ , that is, if  $s_{i_n} h s_{j_1} \in H$ . In order that a cancellation occurs,  $s_{i_n}$  and  $s_{j_1}$  must belong to the same set  $S_i$ , i.e., to the same factor  $K_i$ , and the right  $H$ -cosets determined by  $h s_{j_1}$  and  $s_{i_n}^{-1}$  must coincide.

We say that  $x$  and  $y$  give rise to a *reduction* if  $|xy| = |x| + |y| - 1$ , that is, if  $s_{i_n}$  and  $s_{j_1}$  belong to the same factor  $K_i$  but do not give rise to a cancellation.

Denote by  $W_n$  the set of words of length  $n$ , and by  $B_{n,m}(x)$  the set  $\{(y, z) \in W_n \times W_m : x = yz\}$ . Obviously  $B_{n,m}(x) = \emptyset$  unless  $|n - m| \leq |x| \leq n + m$ . In particular, if  $\text{supp } f \subset W_n$ ,  $\text{supp } g \subset W_m$ , then  $f * g(x) = 0$  unless  $|n - m| \leq |x| \leq n + m$ .

**LEMMA 1.** *Let  $p = n + m - t$ ,  $0 \leq t \leq n + m$ , and  $|x| = p$ ; write  $x = s_{i_1} \cdots s_{i_p} h$  with  $h \in H$ . Let  $q = [(t + 1)/2]$ ,  $q' = [(t - 1)/2]$ ,  $y_0 = s_{i_1} \cdots s_{i_{n-q}}$ ,  $z_0 = s_{i_{n-q'+1}} \cdots s_{i_p} h$ . Then:*

- (i) *if  $t$  is even,  $t = 2j$ , then  $y_0 = s_{i_1} \cdots s_{i_{n-j}}$ ,  $z_0 = s_{i_{n+j+1}} \cdots s_{i_p} h$ , and  $B_{n,m}(x) = \{(y, z) \in W_n \times W_m : \text{there exist } w \in W_j \text{ such that } y = y_0 w, z = w^{-1} z_0\}$ .*
- (ii) *if  $t$  is odd,  $t = 2j + 1$ , then  $y_0 = s_{i_1} \cdots s_{i_{n-j-1}}$ ,  $z_0 = s_{i_{n+j+1}} \cdots s_{i_p} h$  and  $B_{n,m}(x) = \{(y, z) \in W_n \times W_m : \text{there exist } p_1, p_2 \in W_1, w \in W_j \text{ such that } y = y_0 p_1 w, z = w^{-1} p_2 z_0 \text{ and } p_1 p_2 = s_{i_n}\}$ .*
- (iii) *If  $u \in K_i - H$ , and  $\pi_2$  denotes the projection of  $G \times G$  onto its second factor, then  $\int_{\pi_2(B_{1,1}(u))} dm = k_i - 2$ .*

*Proof.* Let  $t = 2j$ . Then  $(y_0, z_0) \in B_{n-j, m-j}(x)$ . By uniqueness of the reduced word expression in  $G$  [18], for every  $(y, z) \in B_{n,m}(x)$  there exists  $w \in W_j$  such that  $y = y_0 w$ . Then  $y_0 z_0 = x = y_0 w z$ , hence  $z = w^{-1} z_0$ . To prove (ii), let  $t = 2j + 1$ : then  $x = y_0 s_{i_n} z_0$ . If  $(y, z) \in B_{n,m}(x)$ , then, again,  $y = y_0 u$ , for some  $u \in W_{j+1}$ . Then  $y_0 s_{i_n} z_0 = x = yz = y_0 u z$  and  $z = u^{-1} s_{i_n} z_0$ . As  $|z| = m$ ,  $|z_0| = m - j - 1$  and  $|u| = j + 1$ , a reduction occurs in the product  $u^{-1} s_{i_n} z_0$ . By definition of  $z_0$ , no reduction occurs in the product  $s_{i_n} z_0$ . Hence a reduction occurs in the product  $u^{-1} s_{i_n}$ . If  $s_{i_n} \in K_{i_n}$  and  $u^{-1} = s'_{r_1} \cdots s'_{r_{j+1}}$  this means that  $e \neq s'_{r_{j+1}} s_{i_n} \in K_{i_n}$ . Let  $p_1 = s'_{r_{j+1}}^{-1}$ ,  $p_2 = s'_{r_{j+1}} s_{i_n}$ ,  $w = u^{-1} p_1$ . Then  $|w| = j$ ,  $y = y_0 p_1 w$ ,  $z = w^{-1} p_2 z_0$ .

To prove (iii), pick any representative  $s \in S_i$ . Then there is exactly one representative  $s' \in S_i$  such that  $u = ss' \bmod H$ . Since  $u$  is nontrivial,  $s'$  is nontrivial if  $s$  is trivial, i.e., if  $s \in H$ . Thus there are exactly  $k_i - 2$  choices of  $s$  such that both  $s$  and  $s'$  are nontrivial and  $u = ss'h$  for some  $h \in H$ . By part (ii),

$$\int \int_{\pi_2(B_{1,1}(u))} dm = (k_i - 2)m(H) = k_i - 2. \quad \square$$

Denote by  $\chi_n$  the characteristic function of  $W_n$ ; in particular,  $\chi_0 = \chi_H$ . We shall denote by  $H_l$  the expectation defined by left average over  $H$  on  $L^1(G) \cap L^\infty(G)$ :  $H_l f(x) = \int_H f(hx) dh$ . Similarly, the right average is denoted by  $H_r$ . Observe that  $H_l$  and  $H_r$  have norm 1 on  $L^p(G)$ ,  $1 \leq p \leq \infty$ .

Let  $k = \max_i k_i = \max_i \#(K_i/H) < \infty$ .

**LEMMA 2.** *Let  $f, g$  be square-integrable functions with support in  $W_n, W_m$  respectively. Then:*

- (i)  $\|(f * g)\chi_p\|_2 \leq \|f\|_2 \|g\|_2$  if  $p = n + m - 2j$ , with  $0 \leq j \leq \min(n, m)$
- (ii)  $\|(f * g)\chi_p\|_2 \leq \sqrt{k-2} \|f\|_2 \|g\|_2$  if  $p = n + m - 2j - 1$  with  $0 \leq j \leq \min(n, m) - 1$ .

*Proof.* The argument follows the lines of [14, Lemma 1.3], [16, Lemma 2]. By density, we can assume  $g \in L^1 \cap L^\infty$ . Let us first handle the case  $p = n + m$ .

Let us write  $B(x) = B_{n,m}(x)$ . It follows from Lemma 1.i that  $m(\pi_2(B(x))) = m(H) = 1$ . By the remarks preceding the statement and by Lemma 1.i, one has

$$\begin{aligned} \|(f * g)\chi_p\|_2^2 &= \int_{W_p} dx \left| \int_{\pi_2(B(x))} f(xz) g(z^{-1}) dm(z) \right|^2 \\ &= \sum_{|s_{i_1} \cdots s_{i_n} s_{i_{n+1}} \cdots s_{i_p}|=p} |H_r f(s_{i_1} \cdots s_{i_n})|^2 |H_l g(s_{i_{n+1}} \cdots s_{i_p})|^2 \\ &\leq \|H_r f\|_2^2 \|H_l g\|_2^2 \leq \|f\|_2^2 \|g\|_2^2. \end{aligned}$$

To complete the proof of part (i), denote by  $\lambda_y$  the left translation by  $y^{-1}$ , and by  $\rho_z$  the right translation by  $z$ . Let  $p = n + m - 2j$ , and define two auxiliary functions  $f', g'$ , with support in  $W_{n-j}, W_{m-j}$  respectively, by

$f'(y) = \|\lambda_{y^{-1}} f\|_{L^2(W_j, m)}$  for  $|y| = n - j$ , and  $g'(z) = \|\rho_z f\|_{L^2(W_j, m)}$  for  $|z| = m - j$ . It follows again from Lemma 1.ii that  $\|f'\|_2 = \|f\|_2$ ,  $\|g'\|_2 = \|g\|_2$ . If  $|x| = p = n + m - 2j$  and  $y_0, z_0$  are as in Lemma 1.i., one has

$$\begin{aligned} |f * g(x)| &= \left| \int_{\pi_2(B(x))} f(xz) g(z^{-1}) dm(z) \right| \\ &= f'(y_0) g'(z_0) = f' * g'(x), \end{aligned}$$

by Lemma 1.i. This yields (i).

Let now  $p = n + m - 1$ , and  $|x| = p$ ,  $x = s_{i_1} \cdots s_{i_p} h$ . By Lemma 1.iii,  $m(\pi_2(B_{1,1}(s_{i_n}))) = k_{i_n} - 2 \leq k - 2$ . Write  $y = xz^{-1}$ . Then Lemma 1.ii, yields

$$\begin{aligned} \|(f * g)\chi_p\|_2^2 &= \int_{W_p} dx \left| \int_{\pi_2(B_{n,m}(x))} f(y) g(z) dm(z) \right|^2 \\ &= \int_{W_p} dx \left| \int \int_{\pi_2(B_{1,1}(s_{i_n}))} \int_H f(y_0 p_1 h) g(h^{-1} p_2 z_0) dh dm(p_2) \right|^2 \\ &\leq \|H_r f\|_2^2 \|H_l g\|_2^2 m(\pi_2(B_{1,1}(s_{i_n}))) \leq (k - 2) \|f\|_2 \|g\|_2. \end{aligned}$$

Finally, to complete the proof of (ii), we reduce to the case  $p = n + m - 1$  by using the auxiliary functions  $f', g'$  as in the proof of (i).  $\square$

As in [14, 16], the previous lemma provides estimates for the norm of a function supported on  $W_n$  as a convolution operator on  $L^2(G)$ . Indeed, the argument of [14, Lemma 2] now yields:

**THEOREM 2.** *Let  $f$  be a function on  $G = *_H K_i$ , with support in  $W_n$ , and let  $k = \max_i k_i < \infty$ . Then there exists a constant  $C(n, k) = O(n)$  such that the norm of  $f$  as a left convolution operator on  $L^2$  satisfies the inequality  $\|f\|_{C_\lambda^*} \leq C(n, k) \|f\|_2$ .*

Denote by  $\|f\|_{C_{v_p}}$  the norm of a function  $f$  as a left convolution operator on  $L^p(G)$  (the norm of convolution operators on  $L^2$  will be denoted by  $\|\cdot\|_{C_\lambda^*}$ ). By Theorem 2,  $\|f\|_{C_\lambda^*} \leq C \cdot (n + 1) \|f\|_2$  for some constant  $C$  independent of  $f$ . Moreover,  $\|f\|_{C_{v_1}} \leq \|f\|_1$ . Hence bilinear interpolation yields:

**COROLLARY 1.** *There exists a constant  $C$  such that, for every function  $f$  with support in  $W_n$ ,  $\|f\|_{C_{v_p}} \leq C \cdot (n + 1)^{1/q} \|f\|_p$ , where  $1 \leq p \leq 2$  and  $1/q = 1 - 1/p$ .*



If  $X, Y$  are normed spaces of functions on  $G$ , we denote by  $\mathcal{M}(X, Y)$  the space of pointwise multipliers from  $X$  to  $Y$ , endowed with the multiplier norm  $\|u\|_{\mathcal{M}(X, Y)} = \sup\{\|uv\|_Y: \|v\|_X \leq 1\}$ . Recall that  $\chi_n$  denotes the characteristic function of  $W_n$ .

COROLLARY 2.  $\|\chi_n\|_{\mathcal{M}(A, L^2)} \leq C(n, k)$ .

*Proof.* It is enough to observe that, by the duality between  $A(G)$  and  $C_\lambda^*$  [10],

$$\begin{aligned} \|u\chi_n\|_2 &= \sup\left\{\left|\int uv \, dm\right|: \|v\|_2 \leq 1, \text{supp } v \subset W_n\right\} \\ &\leq C(n, k)\|u\|_A. \end{aligned} \quad \square$$

Denote by  $B_\lambda$  the closed ideal of  $B(G)$  consisting of the functions which can be uniformly approximated on compact sets by bounded sequences in  $A(G)$ . We can now characterize  $B_\lambda$  as follows:

PROPOSITION 1. *Let  $X$  be the largest closed subspace of  $B(G)$  such that  $\|\chi_n\|_{\mathcal{M}(X, L^2)} = O(n)$ . Then  $X = B_\lambda$ .*

*Proof.* If  $u \in B_\lambda$ , there exists a sequence  $u_j \in A$ , such that  $\|u_j\|_A \leq C$  and  $u_j(x) \rightarrow u(x)$  on compact subsets of  $G$ . Then  $\|u_j\|_\infty \leq C$ , and Corollary 2 yields

$$\begin{aligned} \|u\chi_n\|_2 &= \sup\{\|u\chi_n\chi_K\|_2: K \subset G, K \text{ compact}\} \\ &= \sup_K \lim_j \|u_j\chi_n\chi_K\|_2 \leq C \cdot C(n, k) = O(n). \end{aligned}$$

Conversely, let  $u \in B$ , and suppose that  $\|u\chi_n\|_2 = O(n)$ . Then

$$\|e^{-t|x|}u(x)\|_2^2 = \sum_{n=0}^{\infty} \|e^{-t|x|}u\chi_n\|_2^2 = \sum_{n=0}^{\infty} e^{-tn} \|u\chi_n\|_2^2 < \infty.$$

Therefore  $e^{-t|x|}u(x) \in L^2$ . Moreover,  $e^{-t|x|}u \in B$ , by Theorem 1. Hence  $e^{-t|x|}u \in A$ , and  $\|e^{-t|x|}u\|_A \leq \|e^{-t|x|}\|_B \|u\|_B = \|u\|_B$ , by Theorem 1 again.

Finally,  $\lim_{t \rightarrow 0} e^{-t|x|}u(x) = u(x)$  for each  $x$ . Thus  $u \in B_\lambda$ .  $\square$

Denote by  $\mathcal{M}(A)$  the space of pointwise multipliers of  $A(G)$ . It is obvious that  $B(G) \subset \mathcal{M}A$ , with norm-decreasing inclusion. Suppose  $H$  is finite (i.e.,  $G$  is discrete), so that  $L^2(G) \subset A(G)$ . Then we show that  $\mathcal{M}A$  contains  $B$  properly.

**PROPOSITION 2.** *Let  $M = \sum_{n=0}^{\infty} C(n, k)^2 \|u\chi_n\|_{\infty}^2 < \infty$ . If  $H$  is finite, then  $u \in \mathcal{MA}$  and  $\|u\|_{\mathcal{MA}} \leq \sqrt{M}$ .*

*Proof.* For  $h \in A(G)$ , Corollary 2 yields  $\|uh\|_2^2 \leq M\|h\|_A^2$ . Then  $\|uh\|_A \leq \|uh\|_2 \leq \sqrt{M}\|h\|_A$ .  $\square$

**THEOREM 3.** *If  $H$  is finite, then, for every  $\varepsilon > 0$ , there exists a sequence  $u_n \in A(G)$  such that  $\lim_n \|u_n v - v\|_A = 0$  for every  $v \in A(G)$ , and  $\|u_n\|_{\mathcal{MA}} \leq 1 + \varepsilon$ .*

*Proof.* As  $C(n, k) = O(n)$ , for every integer  $j$  there exists a smallest integer  $n_j$  such that  $\sum_{n=n_j}^{\infty} e^{-2n/j} C(n, k)^2 < \varepsilon^2$ . Let  $u_j(x) = e^{-|x|/j}$  if  $|x| < n_j$ ,  $u_j(x) = 0$  otherwise. Then

$$\begin{aligned} \|u_j\|_{\mathcal{MA}} &\leq \|u_j - e^{-|x|/j}\|_{\mathcal{MA}} + \|e^{-|x|/j}\|_{\mathcal{MA}} \\ &\leq 1 + \left( \sum_{n=n_j}^{\infty} e^{-2n/j} C(n, k)^2 \right)^{1/2} \leq 1 + \varepsilon, \end{aligned}$$

by Theorem 1 and Proposition 2. Moreover,  $\lim_j u_j = 1$  uniformly on compact sets. Since all norms are equivalent on finite-dimensional vector spaces,  $\|u_j v - v\|_A \xrightarrow{j \rightarrow \infty} 0$  for every compactly supported  $v \in A$ . As the norms  $\|u_j\|_{\mathcal{MA}}$  are uniformly bounded, the same is true for every  $v \in A$ .  $\square$

In the remainder, we shall not need to restrict attention to the case of  $G$  discrete. We conclude this section by showing that the Fourier-Stieltjes algebra of  $G$  is much larger than its ideal  $B_{\lambda}$ . For this, we introduce a family of positive definite functions as follows:

**DEFINITION.** A function  $u$  on  $G = *_H K_i$  is *multiplicative with respect to the length* if  $0 < u \leq 1$ ,  $u(xy) = u(x)u(y)$  if  $|xy| = |x| + |y|$  and, for every  $i \in I$ ,  $u_i = u|_{K_i - H}$  is a constant (depending on  $i$ ).  $\square$

It is obvious that every such function  $u$  is bi- $H$ -invariant (for the converse, see the proof of Thm.1). It is also immediate that, for every  $i$ ,  $u_i$  is positive definite on  $K_i$ . Indeed, denote by  $c_i$  the constant value attained by  $u_i$  on  $K_i - H$ . Then  $u_i|_{K_i} = c_i\chi_{K_i} + (1 - c_i)\chi_H$ , because  $u_i \equiv 1$  on  $H$ . Thus  $u_i|_{K_i}$  is the sum of two positive definite functions. The multiplicative property yields  $u = *_H u_i$ . Therefore, by Theorem 1,  $u$  is positive definite on  $G$ .

With notations as in [10, 13], let now  $P$  be the central projection, in the von Neumann algebra of the universal representation of  $G$ , defined by

$PB(G) = B_\lambda$ . If  $u$  is a function in  $B(G)$ , denote by  $u_\lambda$  its projection on  $B_\lambda$ :  $u_\lambda = Pu$ . We say that  $u \in B_\lambda^\perp$  if  $u_\lambda = 0$ .

**THEOREM 4.** *Let  $u$  be a multiplicative function with respect to the length. If  $\|u\chi_1\|_2 < 1$ , then  $u \in A(G)$ . If  $\|u\chi_1\|_2 = \infty$ , then  $u \in B_\lambda^\perp$ .*

*Proof.* Observe that, by the multiplicative property,

$$\|u\chi_n\|_2^2 = \int_{W_n} |u|^2 \leq \left( \int_{W_1} |u|^2 \right)^n = \|u\chi_1\|_2^{2n}.$$

Thus, if  $\|u\chi_1\|_2 < 1$ , then

$$\|u\|_2^2 = \sum_{n=0}^{\infty} \|u\chi_n\|_2^2 \leq \sum_{n=0}^{\infty} \|u\chi_1\|_2^{2n} < \infty.$$

Hence  $u \in B \cap L^2 \subset A$ . On the other hand, if  $\|u\chi_1\|_2 = \infty$ , then the set of indices  $I$  is infinite (because  $u$  is constant on the compact factors  $K_i$ ). Let us denote by  $u_i$  the constant  $u|_{K_i}$ . Then there exists a function  $v$  on  $I$ , and a partition of  $I$  into disjoint subsets  $I_n$  such that: (i)  $\sum_{i \in I_n} u_i v_i = 1$ , (ii)  $\sum_{i \in I} |v_i|^2 < \infty$ . For each  $i \in I$ , choose a nontrivial representative  $s_i$  in  $S_i$ , and let  $E = \bigcup_i s_i H$ ,  $E_n = \bigcup_{i \in I_n} s_i H$ .

The function  $f = \sum_{i \in I_n} v_i \chi_{s_i H}$  is supported in  $E_n \subset W_1$ . Therefore, by Theorem 2, there exists a constant  $C$  such that the norm of  $f_n$  as a left convolution operator on  $L^2$  satisfies the inequality

$$\|f_n\|_{C_\lambda^*} \leq C \|f_n\|_2 < C \left( \sum_{i \in I} |v_i|^2 \right)^{1/2} \rightarrow 0.$$

If  $g$  is a finitely supported function on  $G$ , define

$$\|g\|_u^2 = \int_G g^* * g(x) u(x) dx.$$

We claim that

$$\lim_n \|f_n - \delta_e\|_u = 0.$$

Indeed,

$$\begin{aligned} \|f_n - \delta_e\|_u^2 &= 1 + \iint u(y^{-1}x) f_n(x) \overline{f_n}(y) dx dy \\ &\quad - \int u(y^{-1}) \overline{f_n}(y) dy - \int u(x) f_n(x) dx \end{aligned}$$

For every  $x, y \in E$ , either  $x$  and  $y$  belong to the same right  $H$ -coset or  $|x^{-1}y| = |x| + |y|$  and  $u(x^{-1}y) = u(x)u(y)$ . Thus, by (i),

$$\begin{aligned} \|f_n - \delta_e\|_u^2 &= -1 + \int |f_n|^2 - \int u^2 |f_n|^2 \\ &\quad + \int \int u(x)u(y)f_n(x)\overline{f_n(y)} dx dy \\ &= \int_E (1 - u^2(x))|f_n(x)|^2 dx. \end{aligned}$$

The right hand side tends to zero, because of (ii), and the claim is proved. Denote by  $\lambda(f)$  the left convolution operator on  $L^2$  determined by  $f$ . Let  $F$  be the positive functional on  $C^*(G)$  associated with the positive definite function  $u$ , and define  $F_\lambda$  on  $C_\lambda^*(G)$  by  $F_\lambda = F \circ P$ . By the Cauchy-Schwarz inequality,

$$\begin{aligned} F_\lambda(\mathbf{1} - \lambda(f_n)) &\leq F(P)F((\mathbf{1} - \lambda(f_n))^*(\mathbf{1} - \lambda(f_n))) \\ &= F(P)\|f_n - \delta_e\|_u^2, \end{aligned}$$

which converges to zero as  $n \rightarrow \infty$ . Thus  $F(P\lambda(f_n)) = F_\lambda(\lambda(f_n)) \rightarrow F_\lambda(\mathbf{1}) = F(P)$ . On the other hand, as  $u_\lambda(x) = Pu(x) = F_\lambda(\delta_x)$ , one has

$$|F_\lambda(\lambda(f_n))| = \left| \int u_\lambda f_n dx \right| \leq \|u_\lambda\|_B \|f_n\|_{C_\lambda^*} \rightarrow 0.$$

It follows that  $F(P) = 0$ . Since  $F(P) = u_\lambda(e)$ , this implies  $u_\lambda = 0$ .  $\square$

**4. Convolution operators on  $L^p(G)$  and  $A_p$ -algebras.** In this section, we characterize the predual of the Banach space  $Cv_p$  of left translation invariant operators on  $L^p(G)$ ,  $G = *_H K_i$ , with  $\sup_i \#(K_i/H) < \infty$ .

**DEFINITION.** For  $1 < p < \infty$ , let  $1/q = 1 - 1/p$ , and  $A_p = \{h = \sum_{i=1}^\infty f_i * g_i : f_i \in L^p(G), g_i \in L^q(G), \sum_{i=1}^\infty \|f_i\|_p \|g_i\|_q < \infty\}$ . We endow  $A_p$  with the norm  $\|h\|_{A_p} = \inf\{\sum_{i=1}^\infty \|f_i\|_p \|g_i\|_q : h = \sum_{i=1}^\infty f_i * g_i\}$ .

We now show that, if  $G = *_H K_i$ , with  $\#(K_i/H)$  bounded, then  $A_p^* = Cv_p$ .

**LEMMA 3.** *For  $1 < p < \infty$ , every bounded operator  $T$  on  $L^p(G)$  which commutes with right translations can be approximated, in the weak operator topology, by left convolution operators by functions with compact support.*

*Proof.* Denote by  $v_\alpha$  an approximate identity in  $L^1(G)$ . As  $T$  commutes with right translations, for every  $u \in L^p(G)$  one has  $\lim_\alpha T(v_\alpha * u) = \lim_\alpha T \cdot v_\alpha * u = Tu$ . Thus we can assume that  $T$  is a left convolution operator by a function  $f \in L^p(G)$ ,  $T = \lambda(f)$ .

We first handle the case  $1 < p \leq 2$ . For  $t > 0$ , let  $f_t(x) = \exp(-t|x|)f(x)$ . Now  $\lambda(f_t) \xrightarrow{t \rightarrow 0} \lambda(f)$  in the strong operator topology, since  $B(G) \subset B_p(G) \subset \mathcal{M}(Cv_p)$  [15]. Therefore, it suffices to show that there exists a sequence of functions  $h_n$  with compact support such that  $\lambda(h_n) \xrightarrow{n} \lambda(f_t)$  in norm.

We construct this sequence as follows:  $h_n = \sum_{m=0}^n f_t \chi_m$ . Indeed, by Corollary 1,

$$\begin{aligned} \|\lambda(h_n) - \lambda(f_t)\|_{Cv_p} &\leq \sum_{m=n+1}^{\infty} \|f_t \chi_m\|_{Cv_p} \\ &\leq C \sum_{m=n+1}^{\infty} (m+1)^{1/q} \|f_t \chi_m\|_p \leq C \sum_{m=n+1}^{\infty} (m+1)^{1/q} e^{-tm} \|f\|_p. \end{aligned}$$

This completes the proof if  $1 < p \leq 2$ . By the same argument, every right convolution operator on  $L^p$  can be approximated, in the strong operator topology, by right convolution operators  $\rho(h)$ , where  $h$  has compact support. Let now  $J$  be the involution of  $L^p$  given by  $Ju(x) = u(x^{-1})$ . Then, for  $p > 1$ , the adjoint of  $\lambda(f)$  on  $L^p$  is the operator  $J\rho(f)J$  on  $L^q$ . If  $p > 2$ , the first part of the proof yields a sequence of operators  $\rho(h_n)$  on  $L^q$ , with  $h_n$  of compact support, which approximate  $\rho(f)$  in the weak operator topology. The result now follows from the fact that  $J\rho(h_n)J$  converges to  $J\rho(f)J$  in the weak operator topology.  $\square$

**THEOREM 5.** *Let  $G = *_H K_i$ , with  $\#(K_i/H)$  bounded. Then the dual space of  $A_p$  is isometrically isomorphic with  $Cv_p$ .*

*Proof.* Let  $T \in Cv_p$ ,  $h = \sum_{i=1}^{\infty} f_i * g_i$ ,  $f_i \in L^p(G)$ ,  $g_i \in L^q(G)$ , and  $\langle T, h \rangle = \sum_{i=1}^{\infty} T f_i * g_i(e)$ . The inclusion  $Cv_p \subset A_p^*$  is immediate if we prove that the map  $h \rightarrow \langle T, h \rangle$  is well defined on  $A_p$ . To prove this, let  $\sum_i f_i * g_i(x) = 0$  for every  $x \in G$ . Then  $\sum_i \lambda(h) f_i * g_i(e) = 0$  for every function  $h$  with compact support. By Lemma 3,  $\sum_i T f_i * g_i(e) = 0$ .

Conversely, let  $\Phi \in A_p^*$ , and fix  $f$  in  $L^p(G)$ . Then  $\Phi$  determines a continuous linear functional on  $L^q(G)$  by the rule  $g \rightarrow \Phi(f * g)$ . This functional is associated with a function  $F \in L^p(G)$ . Define an operator  $T$  on  $L^p(G)$  by  $Tf = F$ . It is easily seen that  $T$  commutes with right translations, that is,  $T \in Cv_p$ , and  $\|T\| = \|\Phi\|$ . Thus  $A_p^* \simeq Cv_p$ .  $\square$

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