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The Kato-Rosenblum Theorem has no straightforward generalization to operators with non-absolutely continuous spectra. For example, if A is a bounded selfadjoint operator such that the singular continuous parts of H and $H + A$ are unitarily equivalent for every selfadjoint operator H, then $A = 0$.

1. Introduction. The classical theorem of Kato and Rosenblum (1957) asserts the invariance of absolutely continuous parts under tracse class perturbations. [**5**, p. 540; **6**, p. 26]

THEOREM (*Kato-Rosenblum*). *If H and A are selfadjoint, and A is trace class, then the absolutely continuous parts of H and $H + A$ are unitarily equivalent.*

It is notable that the theorem gives a unitarily invariant condition on the perturbation A alone, and that Lebesgue measure plays a distinguished role.

That the trace condition cannot be radically improved, follows from the Weyl-von Neumann theorem [**5**, p. 523], which states that given any selfadjoint operator H, there is a selfadjoint perturbation A of arbitrarily small Hilbert-Schmidt norm, such that $H + A$ has pure point spectrum—a phenomenon often termed *curdling*. Moreover, according to Kuroda, the Hilbert-Schmidt norm may be replaced by any cross-norm *except* the trace norm. [**5**, p. 525]

For singular measures, there are a few, largely negative, results. Donoghue [**2**], following earlier work of Aronszajn, gave examples in which a purely singular continuous spectrum is curdled by a perturbation of rank one. He also obtained the following result, which we shall use [**2**, p. 565; **4**, Cor. 1].

THEOREM. (*Donoghue*). *Let H be selfadjoint and $A = c\langle \cdot , \phi\rangle\phi$ where ϕ if cyclic for H and c is real and non-zero. Then the singular parts of H and $H + A$ are supported on disjoint sets (i.e. are mutually singular).*

A generalization was proved in [**4**].

Following Donoghue's approach, Carey and Pincus [**1**] proved that the spectrum of any operator with *purely singular* spectrum can be curdled by a perturbation of arbitrarily small *trace* norm. A proof of this fact following the Weyl-von Neumann construction has recently been given by Eugene Wayne [**6**].

These results leave it difficult to imagine a unitarily invariant condition on A alone which might guarantee that A preserves singular continuous parts. Indeed, as we shall prove, there is no such condition: if H and $H + A$ have unitarily equivalent singular parts for every H, then $A = 0$.

We shall, in fact, prove that it is impossible to generalize the Kato-Rosenblum theorem to other measures in the following sense. Let μ be a non-zero Borel measure, and $A \neq 0$ a bounded operator. If the parts of H and $H + A$ which are absolutely continuous with respect to μ are unitarily equivalent for all selfadjoint H, then μ is absolutely continuous with respect to Lebesgue measure, and, moreover, the entire absolutely continuous parts of $H$and $H + A$ are unitarily equivalent.

We shall also prove that A is necessarily compact. The Weyl-von Neumann-Kuroda result strongly suggests that A is trace class, but we know of no proof.

The author wishes to thank Ira Herbst and Eugene Wayne for valuable conversations. He has, however, resisted Professor Herbst's rather gratuitous suggestion that this paper be entitled "A New Characterization of Lebesgue Measure."

2. Preservation of measures. Let $\mathscr{H}$ be a separable Hilbert space, and $H = \int \lambda\, E(d\lambda)$ a selfadjoint operator on $\mathscr{H}$. We shall assume throughout that *all operators are bounded*. For Borel measures m and μ on $\mathbf{R}$, write $u \ll \mu$ if m is absolutely continuous with respect to μ. For $x \in \mathscr{H}$, let m_x be the Borel measure

$$m_x(d\lambda) = \langle E(d\lambda)x, x\rangle.$$

The set

$$\mathscr{H}_\mu(H) = \{x \in \mathscr{H}\colon m_x \ll \mu\}$$

is a closed reducing subspace of H, called the *absolutely continuous subspace* of H with respect to μ. Its orthogonal complement is

$$\mathscr{H}_\mu^s(H) = \{x \in \mathscr{H}\colon m_x \text{ and } \mu \text{ are mutually singular}\}.$$

(See [**5**, p. 516.] The proof is given for Lebesgue measure, but holds in general without change.)

For any Borel measure μ, define H_μ to be the *restriction* of H to $\mathscr{H}_\mu(H)$. If $\nu \ll \mu$, then

$$(H_\mu)_\nu = H_\nu. \tag{2.1}$$

For real t, define the translated measure

$$\mu_t(S) = \mu(S - t). \tag{2.2}$$

Then

$$(H + t)_{\mu_t} = H_\mu + t.$$

Write $A \cong B$ to mean that A and B are unitarily equivalent.

2.1 DEFINITION. *Let μ be a Borel measure on $\mathbf{R}$. A selfadjoint operator A preserves μ iff*

$$(H + A)_\mu \cong H_\mu$$

for every selfadjoint operator H.

The trivial zero measure is preserved by every A, because the space $\mathscr{H}_\mu$ is then always zero-dimensional. The Kato-Rosenblum theorem says that trace class operators preserve Lebesgue measure.

2.2 PROPOSITION. *Let A and B preserve μ. Then*:

(a) *$A + B$ and cA also preserves μ, if c is real*:

(b) *if $\nu \ll \mu$, then A preserves ν*;

(c) *A preserves μ_t for all t*;

(d) *if $W \cong A$, then W preserves μ*; *and*

(e) *If P is an orthogonal reducing projection for A, then AP preserves μ on $P\mathscr{H}$*;

Proof.

(a) We have

$$(H + A + B)_\mu \cong (H + A)_\mu \cong H_\mu.$$

and similarly

$$(H + cA)_\mu = c(c^{-1}H + A)_\mu \cong c(c^{-1}H)_\mu = H_\mu.$$

(b) By (2.1),

$$(H + A)_\nu = [(H + A)_\mu]_\nu \cong (H_\mu)_\nu = H_\nu.$$

(c) By (2.2),

$$(H + A)_{\mu_t} = (H + t + A)_\mu - t \cong (H + t)_\mu - t = H_{\mu_t}.$$

(d) If $W = UAU^*$, with U unitary, then

$$(H + W)_\mu = (H + UAU^*)_\mu = \left[U(U^*HU + A)U^*\right]_\mu$$
$$\cong (U^*HAU + A)_\mu \cong (U^*HU)_\mu \cong H_\mu$$

(e) Let A be the restriction of A of $P\mathscr{H}$. Writing operator matices for the decomposition $\mathscr{H} = P\mathscr{H} \oplus (I - P)\mathscr{H}$ gives

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

If H is defined on $\mathscr{H}$ by

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix}$$

then $(H + A)_\mu \cong H_\mu$ says that

$$\begin{pmatrix} (H_1 + A_1)_\mu & 0 \\ 0 & (A_2)_\mu \end{pmatrix} \cong \begin{pmatrix} (H_1)_\mu & 0 \\ 0 & 0_\mu \end{pmatrix}$$

which gives $(H_1 + A_1)_\mu \cong (H_1)_\mu$, by equating the first components.

THEOREM 1. *If A preserves a non-zero measure μ, then A is compact.*

Proof. If A is not compact, then, possibly replacing A by $-A$, there is an infinite dimensional reducing projection P of A such that $A_1 = AP \geq \delta P$ for some $\delta > 0$. By restriction and translation (2.2(b) and (c)), we can assume that $[0, 1]$ supports μ, and that $\mu[0, \varepsilon] > 0$ for every $\varepsilon > 0$. Choosing H_1 to be an operator on $P\mathscr{H}$ unitarily equivalent to multiplication by λ on $\mathscr{L}^2([0, 1], d\mu(\lambda))$, we see that $H_1 \geq 0$ and that the spectrum of $H_1 = (H_1)_\mu$ contains 0. By (e),

$$(H_1 + A_1)_\mu = (H_1)_\mu.$$

But $H_1 + A_1 \geq \delta P > 0$, so 0 is not in the spectrum of $H_1 + A_1$, a contradiction. Hence, A is compact.

THEOREM 2. *If A preserves μ and $A \neq 0$, then μ is absolutely continuous with respect to Lebesgue measure.*

Proof. Choose a vector ϕ of norm one, which is *not* an eigenvector of A, but for which $A\phi \neq 0$. The operator

$$U = 1 - 2 \cdot \langle \cdot , \phi \rangle \phi$$

is unitary, and we compute that

$$B = A - UAU^* = 2\langle \cdot , A\phi \rangle \phi + 2\langle \cdot , \phi \rangle A - 4\langle A\phi, \phi \rangle \langle \cdot , \phi \rangle \phi.$$

Since ϕ and $A\phi$ are independent, B has rank exactly two. By 2.2(a) and (d), B also preserves μ.

Let ϕ_1 and ϕ_2 be the two eigenvectors of B with non-zero eigenvalues. Let $\mathscr{H}_1$ be the orthogonal complement of ϕ_1, which reduces B. By 2.1(e), the restriction B_1 of B to $\mathscr{H}_1$ must preserve μ on $\mathscr{H}_1$. But $B_1 = c\langle\,\cdot\,,\phi_2\rangle\phi_2$ has rank one. Since H_μ depends only on a class of mutually absolutely continuous measures, we can assume that μ is finite, with total mass one. Identify $\mathscr{H}_1$ with $L^2(\mu)$ and ϕ_2 with the constant function 1, and let H_1 be multiplication by λ. Since ϕ_2 is cyclic for H_1, Donoghue's theorem says that the singular parts of H_1 and $H_1 + B_1 = H_1 + c\langle\,\cdot\,,\phi_2\rangle\phi_2$ are disjointly supported. Thus, if μ had a singular part, B_1 could not preserve μ, so μ must be absolutely continuous.

THEOREM 3. *If A preserves a non-zero absolutely continuous measure, then A preserves Lebesgue measure.*

Denote by $\chi_S(\lambda)$ the characteristic function of the Borel set S, by $|S|$ its Lebesgue measure, and by μ_S, the measure

$$\mu_S(d\lambda) = \chi_S(\lambda)\,d\lambda$$

Write H_S for H_{μ_S}. Let

$$\mathscr{B} = \{S\colon A \text{ preserves } \mu_S,\ S \text{ Borel}\}.$$

2.3 LEMMA. (i) *$\mathscr{B}$ contains every set of measure zero.*
(ii) *$\mathscr{B}$ contains a set of positive measure.*
(iii) *If $S \in \mathscr{B}$, then $S + t \in \mathscr{B}$ for every t*
(iv) *If $S \in \mathscr{B}$ and $F \subset S$, then $F \in \mathscr{B}$. Hence, $\mathscr{B}$ is closed under intersection and difference.*
(v) *$\mathscr{B}$ is closed under countable unions.*

Proof. (i) If $|S| = 0$, μ_S is the zero measure, which is always preserved.
(ii) If A preserves the measure $f(\lambda)\,d\lambda$, with density $f(\lambda)$, then $S = \{\lambda\colon f(\lambda) > 0\}$ is in $\mathscr{B}$ and has positive measure.
(iii) follows from 2.2(c), and (iv) from 2.2(b).
(v) Let $S = S_1 \cup S_1 \cup \cdots$, with $S_j \in \mathscr{B}$.
Writing

$$S = S_1 \cup (S_2 \cup S_1) \cup (S_2 \sim [S_1 \cup S_2]) \cup \cdots$$

and noting (iv) permits us to assume that S_1, $S_2 \dots$ are *disjoint*. In that case

$$(H+A)_S \cong \bigoplus \sum_{j\geq 1} (H+A)_{S_j} \cong \bigoplus \sum_{j\geq 1} H_{S_j} \cong H_S$$

so that $S \in \mathscr{B}$.

Proof of Theorem 3. We wish to show that $\mathscr{B}$ contains the whole line **R**. By (iii) and (v), it suffices that $\mathscr{B}$ contain [0, 1]. Let

$$\mathscr{B}_0 = \{S \in \mathscr{B}\colon S \subset [0,1]\}.$$

If we can prove that

$$\sup\{|S|\colon S \in \mathscr{B}_0\} = 1 \tag{2.3}$$

then (cf. [**3**, p. 75]) the union F of a sequence of sets $F_n \in \mathscr{B}_0$ with $|F_\mu| \to 1$, is in $\mathscr{B}_0$ and has measure 1. Hence [0, 1], which is the union of F with a null set, is also in $\mathscr{B}_0$.

It remains to prove (2.3). Let $\varepsilon > 0$ and $0 < \alpha < 1$ be arbitrary. Use (i), (ii) and (iii) to find an $S \in \mathscr{B}_0$ with $0 < |S| < \varepsilon$, and then an interval I with

$$|I \cap S| > \alpha |I|$$

[**3**, p. 68]. Note that $|I| < \varepsilon/\alpha$.

Lay off on [0, 1] consecutive intervals $I_1, I_2, \dots$ of the same length as I, starting at 0 and continuing until $I_1, \dots, I_{n+1}$ just cover [0, 1]. Each I_j is a translate $I_j = I + t_j$ of I. If $F_j = (I \cap S) + t_j$, and $F = F_1 \cup \cdots \cup F_n$, then $F, F_j \in \mathscr{B}_0$ and

$$|F| = |F_1 \cup \cdots \cup F_n| = N|I \cap S| > N\alpha|I|$$

$$= \alpha|I_1 \cup \cdots \cup I_n| > \left(1 - |I_{n+1}|\right) > \alpha(1 - \varepsilon/\alpha).$$

The right side can be made arbitrarily close to 1 by choosing ε small and α close to 1.

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