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Suppose $D \subset C^n$ is a smoothly bounded domain and u is bounded and pluriharmonic in D. Let u^* denote the boundary function of u, and let $\zeta_0 \in \partial D$. It is shown that if u^* has good averaging behavior on one curve in ∂D through ζ_0, then u^* has good averaging behavior on all curves in ∂D through ζ_0, provided the curves in question satisfy a certain directional condition. These results fail if the directional condition on the curve is violated.

I. Introduction. Let D be a domain in $\mathbf{C}^n$ with C^1-boundary, and for $\zeta \in \partial D$, let $T^C_{\partial D}(\zeta)$ denote the complex tangent space of ∂D at ζ. If f is a complex valued function defined in D, we denote by $f^*(\zeta)$ the nontangential limit of f at ζ, provided this limit exists.

Fix a point $\zeta_0 \in \partial D$. We will be interested in C^3-curves $\gamma\colon (-1,1) \to \partial D$ such that

$$\gamma(0) = \zeta_0, \quad \gamma'(0) \notin T^C_{\partial D}(\zeta_0). \tag{1}$$

Note that since $T^C_{\partial D}(\zeta_0)$ is of (real) codimension 1 within the full tangent space of ∂D at ζ_0, the "typical" smooth curve in ∂D through ζ_0 will satisfy the last condition in (1).

The results of this paper are concerned with averaging properties of pluriharmonic boundary values along such curves. The main thrust of these results is that if good averaging behavior occurs on one curve satisfying (1), then the same must be true of *every* curve satisfying (1). We first take up the case of H^∞-boundary values.

THEOREM 1. *If $f \in H^\infty(D)$, and if*

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h f^*(\gamma(x))\, dx = L \tag{2}$$

for one curve γ satisfying (1), *then* (2) *is true for every curve γ satisfying* (1).

Note that Theorem 1 also tells us that if (2) is true, then

$$\lim_{h \to 0^+} \frac{1}{h} \int_{-h}^0 f^*(\gamma(x))\, dx = L$$

as well, since the curve $\tilde{\gamma}$, defined by $\tilde{\gamma}(x) = \gamma(-x)$, satisfies (1) if and only if γ satisfies (1).

An easy corollary of Theorem 1 generalizes the familiar one variable fact that the boundary values of an H^∞-function cannot have a jump discontinuity.

COROLLARY. *If γ_1 and γ_2 are two curves satisfying* (1), *if $f \in H^\infty(D)$, and if*

$$\lim_{h \to 0^+} \left\| f^* \circ \gamma_j - L_j \right\|_{L^\infty(0,h)} = 0, \qquad j = 1, 2,$$

then $L_1 = L_2$.

1.2. The first thing one might wonder about in Theorem 1 is the existence of the boundary values $f^*(\gamma(x))$. It turns out that with γ as in (1), $f^*(\gamma(x))$ is guaranteed to exist for a.e. x in in a sufficiently small neighborhood of 0, by the following result of Nagel and Rudin [7]:

THEOREM A. *Suppose $\varepsilon > 0$, and that γ: $[-\varepsilon, \varepsilon] \to \partial D$ is a C^3-curve satisfying $\langle \gamma'(x), \nu(\gamma(x)) \rangle \neq 0$ for all $x \in [-\varepsilon, \varepsilon]$, where $\nu(\zeta)$ denotes the outward unit normal to ∂D at ζ, and $\langle\ ,\ \rangle$ is the usual complex inner product. If $f \in H^\infty(D)$, then $f^*(\gamma(x))$ exists for a.e. $x \in [-\varepsilon, \varepsilon]$.*

To see how Theorem A applies to our situation with γ as in (1), define $\alpha(x) = \langle \gamma'(x), \nu(\gamma(x)) \rangle$. Then $\alpha(0) \neq 0$, for this is precisely the last condition in (1). The continuity of α then shows $\alpha(x) \neq 0$ in $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, so that $f^*(\gamma(x))$ exists a.e. in $[-\varepsilon, \varepsilon]$ by Theorem A.

REMARK. Theorem A is true for curves with less than C^3-smoothness; see [8] for the strongest result. We are assuming γ is C^3 to keep the exposition as simple as possible. (Later, in Part IV, it will be convenient to assume γ is C^6.)

II. Proof of Theorem 1 and related results.

2.1. *One variable preliminaries*: For $\varepsilon > 0$, set $Q_\varepsilon = (-\varepsilon, \varepsilon) \times (0, \varepsilon)$. Theorem 1 hinges on the following one variable result:

THEOREM B. *If $f \in H^\infty(Q_\varepsilon)$, then the following statements are equivalent*:

(i) $\lim_{h \to 0^+} h^{-1} \int_0^h f^*(x)\, dx = L$

(ii) $\lim_{h \to 0^+} h^{-1} \int_{-h}^{0} f^*(x)\, dx = L$
(iii) $\lim_{y \to 0^+} f(iy) = L$.

Theorem B for the upper half plane follows from Corollaries 3.5, 3.6, and 3.7 in [**14**]; very similar results for the unit disc appear in [**1**]. Of course, conformal maps transfer Theorem B from one region to another, and in particular to our region Q_ε.

LEMMA 2.1. *Let* $\phi \in C^1 \cap L^\infty(Q_\varepsilon)$. *For* $z \in \mathbf{C}$ *set*

$$\psi(z) = \frac{1}{2\pi i} \int_{Q_\varepsilon} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

Then $\psi \in C^1(Q_\varepsilon)$, $\partial\psi/\partial\bar{z} = \phi$ *in* Q_ε, *and* ψ *is continuous on all of* $\mathbf{C}$.

Proof. We are using the notation $\partial/\partial\bar{z} = 1/2(\partial/\partial x + i\partial/\partial y)$. That $\psi \in C^1(Q_\varepsilon)$ and $\partial\psi/\partial\bar{z} = \phi$ in Q_ε is well known; see [**11**], p. 339. The continuity of ψ on all of $\mathbf{C}$ follows since ψ is the convolution of $\phi \in L^\infty(Q_\varepsilon)$ with $1/\zeta \in L^1_{\text{loc}}(\mathbf{C})$.

2.2. *Proof of Theorem* 1. The first part of this argument comes from [**7**], where we refer the reader for details. (The details are even easier here since we are dealing with C^3-curves.)

If γ satisfies (1), then one of the vectors $\pm i\gamma'(0)$ points transversally into D from ζ_0; suppose it is $i\gamma'(0)$ which does so. For $x + iy \in (-1, 1) \times \mathbf{R}$, define $\Gamma(x + iy) = \gamma(x) + iy\gamma'(x)$. Since $i\gamma'(0)$ points transversally into D and $\Gamma(x) = \gamma(x)$ for $x \in (-1, 1)$, there exists an $\varepsilon > 0$ such that $\Gamma(Q_\varepsilon) \subset D$, and such that $d(\Gamma(x + iy), \partial D)$ is on the order of $|y|$ for $x + iy \in Q_\varepsilon$. Here $d(w, \partial D)$ denotes the Euclidean distance from $w \in \mathbf{C}^n$ to ∂D.

For $f \in H^\infty(D)$, the chain rule gives

$$\frac{\partial(f \circ \Gamma)}{\partial\bar{z}}(x + iy) = \sum_{k=1}^{n} \frac{\partial f}{\partial w_k}(\Gamma(x + iy)) \frac{\partial \Gamma_k}{\partial\bar{z}}(x + iy) \tag{3}$$

($x + iy \in Q_\varepsilon$), where we have written $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$. Just as in [**7**], Cauchy's estimates show there is a constant C such that

$$|\partial f/\partial w_k(\Gamma(x + iy))| \leq C/|y| \quad \text{for } x + iy \in Q_\varepsilon,$$

and we easily compute $\partial\Gamma/\partial\bar{z}(x + iy) = 1/2iy\gamma''(x)$. Thus by (3), $\partial(f \circ \Gamma)/\partial\bar{z} \in C^1 \cap L^\infty(Q_\varepsilon)$.

Note that if instead it is $-i\gamma'(0)$ which points into D, everything said so far holds verbatim with $-Q_\varepsilon$ in place of Q_ε.

Now suppose γ_1, γ_2 satisfy (1), and that (2) holds for γ_1 and $f \in H^\infty(D)$. We assume that both $i\gamma_1'(0)$ and $i\gamma_2'(0)$ point transversally into D, the argument for the other cases being exactly the same. Associate Γ_j to γ_j on Q_{ε_j}, $j = 1, 2$, as described above. Since $\partial(f \circ \Gamma_j)/\partial\bar{z} \in C^1 \cap L^\infty(Q_{\varepsilon_j})$, we can define

$$\psi_j(z) = \frac{1}{2\pi i} \int_{Q_{\varepsilon_j}} \frac{\partial(f \circ \Gamma_j)/\partial\bar{\xi}}{\zeta - z}\, d\zeta \wedge d\bar{\xi} \qquad (x \in \mathbf{C})$$

as in Lemma 2.1. We then have $f \circ \Gamma_j - \psi_j \in H^\infty(Q_{\varepsilon_j})$. By the continuity of ψ_1 on $\mathbf{C}$,

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h \left[f^*(\gamma_1(x)) - \psi_1(x) \right] dx = L - \psi_1(0).$$

By Theorem B this implies

$$\lim_{y \to 0^+} (f \circ \Gamma_1 - \psi_1)(iy) = L - \psi_1(0),$$

which gives

$$\lim_{y \to 0^+} f \circ \Gamma_1(iy) = L.$$

But the curves $y \to \Gamma_j(iy)$ approach ζ_0 nontangentially as $y \to 0^+$. From Čirka's Lindelhöf theorem ([**11**], p. 168), we conclude

$$\lim_{y \to 0^+} f \circ \Gamma_2(iy) = L.$$

The above argument in reverse can now be applied to $f \circ \Gamma_2 - \psi_2$ to obtain

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h f^*(\gamma_2(x))\, dx = L.$$

REMARK. With some extra work, one can push the proof of Theorem 1 through to the case where γ is C^1 and $\gamma' \in \operatorname{Lip}\alpha$ for some $\alpha > 1/2$, but it seems substantially more difficult to treat the case where γ is merely C^1.

2.3. *Weakly admissible limits*. The notion of a *weakly admissible limit* was introduced in [**3**]; see also [**9**]. This coincides with the definition of *hypoadmissible limit* given in [**2**] and [**5**] and with the definition of *restricted K-limit* in [**11**]. Čirka's Lindelöf theorem ([**3**], and [**11**], p. 171), which we needed in the proof of Theorem 1, asserts that if $f \in H^\infty(D)$ and f has a limit L along a nontangential curve in D terminating at $\zeta_0 \in \partial D$, then f has a weakly admissible limit L at ζ_0.

THEOREM 2. *If $f \in H^\infty(D)$, and ∂D is C^3, then the following statements are equivalent*:

(i) $\lim_{h \to 0^+} h^{-1} \int_0^h f^*(\gamma(x))\, dx = L$ *for one curve* γ *satisfying* (1).

(ii) *f has a weakly admissible limit L at ζ_0.*

(iii) $\lim_{h \to 0^+} h^{-1} \int_0^h f^*(\gamma(x))\, dx = L$ *for every curve* γ *satisfying* (1).

Proof. Examining the proof of Theorem 1, we see that if γ satisfies (1), and Γ is associated with γ as before, then

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h f^*(\gamma(x))\, dx = L \quad \text{iff} \quad \lim_{y \to 0^+} f(\Gamma(iy)) = L. \tag{4}$$

(More precisely, (4) is true if $i\gamma'(0)$ points transversally into D from ζ_0; we assume without loss of generality that this is true of every γ under consideration here.) As noted earlier, the curve $y \to \Gamma(iy)$ tends to ζ_0 nontangentially as $y \to 0^+$. The implication (i) $\Rightarrow$ (ii) thus follows from Čirka's Lindelöf theorem.

If (ii) holds, then

$$\lim_{y \to 0^+} f(\Gamma(iy)) = L$$

for every γ as above, since a weakly admissible limit implies a nontangential limit. Thus (ii) $\Rightarrow$ (iii) by (4).

Since ∂D is C^3, there exist C^3-curves γ in ∂D satisfying (1), so that (iii) trivially implies (i).

2.4. *Other averaging properties.* Let B_n denote the open unit ball in $\mathbf{C}^n$, and let σ_n denote the usual rotation invariant Lebesgue measure on ∂B_n. If μ is a complex Borel measure on ∂B_n, we denote by $D\mu(\zeta_0)$ and $\mathscr{D}\mu(\zeta_0)$ the symmetric derivatives of μ at $\zeta_0 \in \partial B_n$ defined in [**10**].

Specializing to the case $D = B_n$, we obtain

THEOREM 3. *Let $f \in H^\infty(B_n)$, and put $d\mu = f^*\, d\sigma_n$. Then each of the statements $D\mu(\zeta_0) = L$, $\mathscr{D}\mu(\zeta_0) = L$ is equivalent to each statement of Theorem* 2.

Proof. As shown in [**10**], the statements $D\mu(\zeta_0) = L$ and $\mathscr{D}\mu(\zeta_0) = L$ are each equivalent to the assertion

$$\lim_{r \to 1} f(r\zeta_0) = L,$$

which is equivalent to (ii) of Theorem 2 for the case $D = B_n$ by Čirka's Lindelöf theorem.

III. Examples.

3.1. Theorems 1 and 2 can fail in a variety of ways if the directional condition in (1) is violated. In the first place, the boundary values $f^*(\gamma(x))$ may not even exist. Suppose for example that ∂D is C^2 and strictly pseudoconvex, and that $\gamma'(x) \in T^C_{\partial D}(\gamma(x))$ for every $x \in (-1,1)$. Then as is well known, $\gamma([-1/2,1/2])$ is a peak set for $A(D)$, the algebra of functions continuous on $\overline{D}$ and holomorphic in D (see, e.g., [**11**] p. 216). If $g \in A(D)$ peaks on $\gamma([-1/2,1/2])$, then $f = \exp[i \log(1-g)] \in H^\infty(D)$, and $f^*(\gamma(x))$ fails to exist for every $x \in [-1/2,1/2]$. (Any logarithms, powers, or arguments appearing in this paper will always denote the usual principal branches.)

But even if $f^*(\gamma(x))$ exists for every $x \in (-1,1)$, Theorems 1 and 2 can fail rather badly if (1) is not satisfied. For convenience, we work in the unit ball B in $\mathbf{C}^2$, with $\zeta_0 = e_1 = (1,0)$. Suppose γ: $(-1,1) \to \partial B$ is a C^2-curve such that

$$\gamma(0) = e_1, \quad \gamma'(0) \in T^C_{\partial B}(e_1), \quad \text{and} \quad \gamma'(0) \neq 0. \tag{5}$$

(Note that (5) can be true while at the same time there exists an $\varepsilon > 0$ such that $f^*(\gamma(x))$ exists a.e. in $(-\varepsilon,\varepsilon)$ for every $f \in H^\infty(D)$. This will happen if $\gamma'(x) \notin T^C_{\partial D}(\gamma(x))$ in a deleted neighborhood of 0, by Thm. A.) From (5) it follows that $\gamma(x) = (1 + ax^2 + o(x^2), bx + o(x))$ as $x \to 0$, where $a, b \in \mathbf{C}$ and $b \neq 0$. The fact that $|\gamma(x)| = 1$ for every $x \in (-1,1)$ implies $\operatorname{Re} a < 0$. Define $f(z,w) = w(1-z)^{-1/2}$. Since $|w(1-z)^{-1/2}|^2 \leq |w|^2(1-|z|)^{-1} \leq 2$ for $(z,w) \in B$, $f \in H^\infty(B) \in C(\overline{B} - \{e_1\})$. One easily computes

$$\lim_{x \to 0^+} f^*(\gamma(x)) = b(-a)^{-1/2}, \quad \lim_{x \to 0^-} f^*(\gamma(x)) = -b(-a)^{-1/2}.$$

Thus f^* has a jump discontinuity along every curve satisfying (5); i.e., the corollary to Theorem 1 fails for all of these curves. Also note that this jump discontinuity depends on a and b, i.e., the jump discontinuity varies from curve to curve. Finally, observe that f has a weakly admissible limit 0 at e_1.

3.2. Here we give an example of an $f \in H^\infty(B)$ with weakly admissible limit at e_1, but such that

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h f^*(\gamma(x))\, dx$$

fails to exist for every γ satisfying (5).

For $z \in U$, the open unit disc in $\mathbf{C}$, define $g(z) = \exp[i \log(1 - z)]$. Clearly $g \in H^\infty(U) \cap C(\overline{U} - \{1\})$. For $(z,w) \in B$, define $f(z,w) = w^2(1 - z)^{-1}g(z)$. As in 3.1 one verifes $f \in H^\infty(B) \cap C(\overline{B} - \{e_1\})$, and clearly f has a weakly admissible limit 0 at e_1. We claim that for every γ as in (5),

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h f^*(\gamma(x))\, dx \quad \text{fails to exist.} \tag{6}$$

Writing out the components of γ as in 3.1, and noting

$$\lim_{x \to 0} (bx + o(x))^2(-ax^2 + o(x^2))^{-1} = b^2(-a)^{-1} \neq 0,$$

it is clear that to prove (6) we need only look at

$$\frac{1}{h} \int_0^h g(1 + ax^2 + o(x^2))\, dx.$$

Since $g \in H^\infty(U)$, $|g'(z)|(1 - |z|)$ is bounded in U, hence

$$\lim_{x \to 0} [g(1 + ax^2 + o(x^2)) - g(1 + ax^2)] = 0.$$

(As $x \to 0$, $1 + ax^2$ approaches 1 nontangentially since $\operatorname{Re} a < 0$.) Thus it is enough to consider

$$\frac{1}{h} \int_0^h g(1 + ax^2)\, dx.$$

This latter integral equals

$$\frac{1}{h} \int_0^h e^{i \log(-ax^2)}\, dx = \int_0^1 e^{i \log(-ah^2x^2)}\, dx. \tag{7}$$

Since $\arg(-ah^2x^2)$ is constant, (7) is equal to

$$e^{-\arg(-a)} e^{i \log|ah^2|} \int_0^1 e^{i \log x^2}\, dx. \tag{8}$$

The integral in (8) is not zero since (7) clearly does not vanish for all h. Thus (8) has no limit as $h \to 0^+$, proving (6).

3.3. In the other direction, we give an example of an $f \in H^\infty(B)$ and a γ satisfying (5), such that f^* is constant a.e. on $\gamma(-1, 1)$, but such that $f^*(e_1)$ fails to exist. To do this we use the following theorem of Saerens [**13**]: if K is a peak set for $A(B)$, μ is a finite positive Borel measure on K, and $b \in L^\infty(\mu)$, then there exists $g \in H^\infty(B)$ such that

$$\|g\|_{H^\infty(B)} \leq \|b\|_{L^\infty(\mu)}$$

and $g^* = b$ μ-a.e. on K.

If γ: $(-1,1) \to \partial B$ is any C^1-curve such that $\gamma(0) = e_1$ and $0 \neq \gamma'(x) \in T^c_{\partial B}(\gamma(x))$ for every $x \in (-1,1)$ (the curve $\gamma(x) = (\cos x, \sin x)$ will do), then as was mentioned earlier, $\gamma([-1/2,1/2])$ is a peak set for $A(B)$. Define $\mu = s + \delta$, where s denotes arclength measure on $\gamma([-1/2,1/2])$, and δ is the unit point mass at e_1. Set $b(\gamma(x)) = 0$, $x \in [-1/2,1/2] - \{0\}$, and $b(e_1) = 1$. By Saerens's theorem, there exists a

$$g \in H^\infty(B), \quad \|g\|_{H^\infty(B)} \leq 1,$$

such that $g^* = b$ μ-a.e. on $\gamma([-1/2,1/2])$. Thus $g^*(\gamma(x)) = 0$ for a.e. $x \in [-1/2,1/2]$ and $g^*(e_1) = 1$. Setting $f = \exp[i \log(1-g)]$, we see $f^*(\gamma(x)) = 1$ a.e. in $[-1/2,1/2]$, while $\lim_{r \to 1^-} f(re_1)$ does not exist.

IV. The pluriharmonic case.

4.1. Theorems 1 and 2 fail for bounded pluriharmonic functions, and for a simple reason: Theorem B is false for bounded harmonic functions. (Example: $u(z) = \arg z$). There is, however, a substitute for Theorem B (Theorem C below), involving symmetric averages, which will enable us to prove the following theorem.

THEOREM 4. *If u is bounded and pluriharmonic in D, and ∂D is C^6, then the following statements are equivalent:*

(i) $\lim_{h \to 0^+} (1/2h) \int_{-h}^{h} u^*(\gamma(x))\, dx = L$ *for one C^6-curve γ satisfying* (1).

(ii) $\lim_{y \to 0^+} u(\zeta_0 - y\nu(\zeta_0)) = L$.

(iii) $\lim_{h \to 0^+} (1/2h) \int_{-h}^{h} u^*(\gamma(x))\, dx = L$ *for every C^6-curve γ satisfying* (1).

REMARKS. 1. The existence of the boundary values $u^*(\gamma(x))$ for a.e. x in a small enough interval about 0 follows easily from Theorem A. Fix a curve γ satisfying (1) and assume without loss of generality that u is real. Since there exists a neighborhood V of ζ_0 (open in $\mathbf{C}^n$) such that $V \cap D$ is simply connected, there exists a pluriharmonic v in $V \cap D$ such that $u + iv$ is holomorphic in $V \cap D$. The function $f = e^{u+iv}$ then belongs to $H^\infty(V \cap D)$, which implies $f^*(\gamma(x))$ exists a.e. in $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$ as in 1.2. Clearly $u^*(\gamma(x))$ exists wherever $f^*(\gamma(x))$ exists.

2. In proving Theorem 4 we will need to deal with the operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ rather than $\partial/\partial \bar{z}$. Thus twice as many derivatives will be involved here, accounting for the assumption $\gamma \in C^6$ rather than $\gamma \in C^3$.

4.2. Theorem C and Lemma 4.2 below will play the same roles in proving Theorem 4, respectively, as Theorem B and Lemma 2.1 did in proving Theorems 1 and 2.

THEOREM C. *If u is bounded and harmonic in Q_ε, then the following statements are equivalent:*

(i) $\lim_{h\to 0^+}(1/2h)\int_{-h}^{h} u^*(x)\,dx = L$

(ii) $\lim_{y\to 0^+} u(iy) = L$.

Theorem C for the upper half plane is due to Loomis [**6**]; again, a conformal map transfers this to Q_ε. (For generalizations of Theorem C to the upper half space of $\mathbf{R}^n$, see [**12**]. Versions of Theorem C for the unit ball in $\mathbf{C}^n$ were obtained in [**10**]. All of these results hold under the weaker hypothesis that u is positive.)

LEMMA 4.2. *Suppose $\phi \in C^2 \cap L^\infty(Q_\varepsilon)$. For $z \in \mathbf{C}$, define*

$$\psi(z) = \frac{1}{2\pi}\int_{Q_\varepsilon} \phi(\zeta)\log|z-\zeta|\,dx\,dy \qquad (\zeta = x+iy).$$

Then $\psi \in C^2(Q_\varepsilon)$, $\Delta\psi = \phi$ in Q_ε, and ψ is continuous on all of $\mathbf{C}$.

Proof. The first two assertions are well known; see [**5**], pp. 29–30. That ψ is continuous on all of $\mathbf{C}$ follows as in Lemma 2.1: ψ is the convolution of $\phi \in L^\infty(Q_\varepsilon)$ with $\log|\zeta| \in L^1_{\text{loc}}(\mathbf{C})$.

4.3. It will be convenient in the rest of the paper to suppose $\zeta_0 = 0$ and $\nu(\zeta_0) = (-i, 0, \ldots, 0)$; recall that $\nu(\zeta_0)$ is the outward unit normal to ∂D at ζ_0. We let $\mathbf{C}^n_+$ denote the half space $\{z = (z_1, \ldots, z_n) \in \mathbf{C}^n$: $\operatorname{Im} z_1 > 0\}$. Note that with these conventions, $\mathbf{C}^n_+$ can be thought of as the set of vectors pointing into D transversally from $\zeta_0 = 0$.

Define $Y = \{(iy, z')$: $y > 0$, $z' \in \mathbf{C}^{n-1}\}$. In Lemma 4.3, which is an analogue of Čirka's Lindelöf theorem, we consider C^1-curves η such that

$$\eta\colon [0,1] \to \overline{D}, \quad \eta(0) = 0, \quad \eta'(0) \in Y. \tag{9}$$

LEMMA 4.3. *If u is bounded and pluriharmonic in D, and if*

$$\lim_{y\to 0^+} u(\eta(y)) = L$$

for one curve η satisfying (9), *then*

$$\lim_{y\to 0^+} u(\eta(y)) = L$$

for every curve η satisfying (9).

Proof. The proof is very much like the proof of Čirka's Lindelöf theorem, and rests on two well known properties of ∇u. First, the boundedness of u implies the boundedness of $|\nabla u(z)|d(z, \partial D)$ for $z \in D$. Second, there is a better estimate on the derivatives of u in certain directions: if $\nabla_T u$ denotes the gradient of u with respect to the variables $x_2, y_2, \ldots, x_n, y_n$ (i.e., the gradient of u in the complex tangential directions), then $|\nabla_T u(z)|d(z, \partial D) \to 0$ as $z \to 0$ in D nontangentially. (See, e.g., inequality (20) of [9].)

From the first property it follows that

$$\lim_{y \to 0^+} u(\eta(y)) = L$$

if and only if

$$\lim_{y \to 0^+} u(y\eta'(0)) = L,$$

for any curve η as in (9). Thus we need only consider limits along rays contained in Y.

Suppose $v_1, v_2 \in Y$, and that $\lim_{y \to 0^+} u(yv_1) = L$; we wish to show $\lim_{y \to 0^+} u(yv_2) = L$. The definition of Y shows there exist $y_0 > 0$ and $z_0 \in \{0\} \times \mathbf{C}^{n-1}$ such that $v_2 = y_0 v_1 + z_0$, and we have the standard estimate

$$|u(yv_2) - u(yy_0v_1)| \leq \sup_{\substack{z \in \{0\} \times C^{n-1} \\ |z| \leq y|z_0|}} |\nabla_T u(yy_0v_1 + z)||yz_0|. \tag{10}$$

For small $y > 0$, the points $yy_0v_1 + z$ satisfying $|z| \leq y|z_0|$ lie within a nontangential approach region, and $d(yy_0v_1 + z, \partial D)$ is on the order of y. Thus the right hand side of (10) $\to 0$ as $y \to 0^+$ by the second property mentioned above, so that $\lim_{y \to 0^+} u(yv_2) = L$ as desired.

4.4. *Proof of Theorem* 4. If γ is any C^6-curve satisfying (1), we may assume that $i\gamma'(0)$ points transversally into D from 0, since the symmetric averages in (i) and (iii) are unaffected if $\gamma(x)$ is replaced by $\gamma(-x)$. For any such γ, define

$$\Gamma(x + iy) = \gamma(x) + iy\gamma'(x) - (y^2/2)\gamma''(x) \qquad (x + iy \in (-1, 1) \times \mathbf{R}).$$

Then for some $\varepsilon > 0$, Γ has the same properties on Q_ε as did the earlier Γ in the proof of Theorem 1, except this time $\partial\Gamma/\partial\bar{z}(x + iy) = -1/4\gamma'''(x)y^2$.

If u is bounded and pluriharmonic in D, we claim

$$\Delta(u \circ \Gamma) \in C^2 \cap L^\infty(Q_\varepsilon). \tag{11}$$

To calculate $\Delta(u \circ \Gamma)$ it is best to write $\Delta = 4\partial^2/\partial z \partial \bar{z}$, and use the $\partial/\partial \bar{z}$ version of the chain rule. This slightly tedious computation was carried out in the proof of Theorem 1 in [**9**], and we spare the reader the details here. Once $\Delta(u \circ \Gamma)$ has been computed, (11) follows from Cauchy-type estimates on the derivatives of u, the vanishing of $\partial^2 u/\partial z_i \partial \bar{z}_j$ for all i and j, and the second order vanishing of $\partial\Gamma/\partial\bar{z}$ on $(-\varepsilon, \varepsilon)$. (Note that second order derivatives of Γ appear in the expression for $\Delta(u \circ \Gamma)$. Since the definition of Γ already involves two derivatives of γ, we see that to assert $\Delta(u \circ \Gamma) \in C^2(Q_\varepsilon)$ requires that γ be C^6.)

Suppose now that (i) holds for u and γ. With Γ defined as above, we see by (11) that ψ can be defined as in Lemma 4.2, taking $\phi = \Delta(u \circ \Gamma)$. The function $u \circ \Gamma - \psi$ is then bounded and harmonic in Q_ε, and by Theorem C we conclude, since ψ is continuous in $\mathbf{C}$, that

$$(12) \qquad \lim_{y \to 0^+} u(\Gamma(iy)) = L \quad \text{iff} \quad \lim_{h \to 0^+} \frac{1}{2h} \int_{-h}^{h} u^*(\gamma(x))\, dx = L.$$

Setting $\eta(y) = \Gamma(iy)$, it is easily verified that η satisfies (9). Since the curve $y \to \zeta_0 - y\nu(\zeta_0) = (iy, 0, \ldots, 0)$ obviously satisfies (9), Lemma 4.3 shows that (ii) must hold.

A similar argument based on (12) and Lemma 4.3 shows that (ii) $\Rightarrow$ (iii), and the implication (iii) $\Rightarrow$ (i) is obvious since ∂D is C^6.

4.5. Specializing to the ball, we have the following analogue of Theorem 3.

THEOREM 5. *Suppose u is bounded and pluriharmonic in B_n, and put $d\mu = u^* d\sigma_n$. Then each of the statements $D\mu(\zeta_0) = L$, $\mathscr{D}\mu(\zeta_0) = L$ is equivalent to each statement of Theorem* 4.

Proof. Again, each of the statements $D\mu(\zeta_0) = L$, $\mathscr{D}\mu(\zeta_0) = L$ is shown in [**10**] to be equivalent to the statement $\lim_{r \to 1^-} u(r\zeta_0) = L$.

4.6. *Differentiability along γ and weakly admissible limits.* Unlike the case for an H^∞-function, the equivalent statements of Theorem 4 do not imply a weakly admissible limit for a bounded pluriharmonic function. The simple example $u(z_1, z_2) = \arg(1 - z_1)$ in B_2 (take $\zeta_0 = e_1$) shows this. What is needed is a stronger differentiability assumption on u^* along γ.

First we give some definitions. If γ is a curve satisfying (1), if ε is chosen as in 1.2, and if u is bounded and pluriharmonic in D, define

$$U_\gamma(t) = \int_{-\varepsilon}^{t} u^*(\gamma(x))\, dx \qquad (t \in (-\varepsilon, \varepsilon)).$$

For $z = (z_1, \ldots, z_n) \in \mathbf{C}_+^n$, define $a(z) = \arg z_1$; clearly $0 < a(z) < \pi$ for all $z \in \mathbf{C}_+^n$.

THEOREM 6. *If u is bounded and pluriharmonic in D, and ∂D is C^6, then the following statements are equivalent*:

(i) *there exists a C^6-curve γ satisfying* (1) *such that $U_\gamma'(0) = L$.*

(ii) *there exist $v_1, v_2 \in \mathbf{C}_+^n$, with $a(v_1) \neq a(v_2)$, such that $\lim_{r \to 0^+} u(rv_1) = L = \lim_{r \to 0^+} u(rv_2)$.*

(iii) *u has a weakly admissible limit L at 0.*

(iv) *$U_\gamma'(0) = L$ for every C^6-curve γ satisfying* (1).

As was the case earlier, the proof of Theorem 6 strongly depends on a one variable result:

THEOREM D. *If u is bounded and harmonic in Q_ε, put*

$$U(t) = \int_{-\varepsilon}^{t} u^*(x)\, dx \qquad (t \in (-\varepsilon, \varepsilon)).$$

Then the following statements are equivalent:

(i) $U'(0) = L$.

(ii) *There exist θ_1, θ_2, $0 < \theta_1 < \theta_2 < \pi$, such that $\lim_{r \to 0^+} u(re^{i\theta_j}) = L$, $j = 1, 2$.*

(iii) *u has a nontangential limit L at 0.*

Theorem D, proved for the upper half plane, is due to Loomis [**6**]. See also Gehring [**4**].

Proof of Theorem 6. As usual, we assume without loss of generality that if γ satisfies (1), then $i\gamma'(0)$ points transversally into D from 0. Since the reader by now is familiar with our strategy of proof, we will be brief about certain details.

Suppose (i) holds for u and γ. Associating Γ with γ as in the proof of Theorem 4, and using our usual argument together with Theorem D, we see that $u \circ \Gamma$ has a nontangential limit L at 0. So certainly there exist θ_1, θ_2, $0 < \theta_1 < \theta_2 < \pi$, such that

$$\lim_{r \to 0^+} u\big(\Gamma(re^{i\theta_j})\big) = L, \qquad j = 1, 2.$$

The fact that $\partial\Gamma/\partial\bar{z}(0) = 0$ implies the curves $r \to \Gamma(re^{i\theta_j})$ are tangent to the rays $re^{i\theta_j}\gamma'(0)$, $j = 1, 2$, respectively. Thus

$$\lim_{r \to 0^+} u\big(re^{i\theta_j}\gamma'(0)\big) = L, \qquad j = 1, 2.$$

Since

$$a\left(e^{i\theta_j}\gamma'(0)\right) = \theta_j,$$

we see that (ii) holds.

In proving (ii) $\Rightarrow$ (iii) we need some preliminaries. For $0 < \theta < \pi$, set $Y_\theta = \{(re^{i\theta}, z)\colon r > 0,\ z \in \mathbf{C}^{n-1}\}$. (The Y defined in §4.3 is $Y_{\pi/2}$ in the new notation.). If η_1 and η_2 are C^1-curves satisfying $\eta_j\colon [0,1] \to \overline{D}$, $\eta_j(0) = 0$, $\eta_j'(0) \in Y_\theta$, then

$$\lim_{r\to 0^+} u(\eta_1(r)) = L \quad \text{if and only if} \quad \lim_{r\to 0^+} u(\eta_2(r)) = L.$$

The proof of this is exactly the same as the proof of Lemma 4.3.

Now suppose (ii) holds, and put $\theta_j = a(v_j)$, $j = 1,2$. Let N be the complex line $\{(\lambda, 0, \ldots, 0)\colon \lambda \in \mathbf{C}\}$, and set $D_N = D \cap N$. The vectors $(e^{i\theta_1}, 0)$, v_1 are both in Y_{θ_1}, and similarly $(e^{i\theta_2}, 0)$, $v_2 \in Y_{\theta_2}$. We conclude

$$\lim_{r\to 0^+} u(re^{i\theta_j}, 0) = L, \qquad j = 1,2.$$

Since D_N can be thought of as a domain in $\mathbf{C}^1$, and since Theorem D is true for any domain with smooth boundary, we conclude that u has a nontangential limit within D_N at 0. Estimates on $\nabla_T u$ like the one mentioned in proving Lemma 4.3 now show that u has a weakly admissible limit L at 0. (We omit the proof of this last statement; the argument is essentially the same as in the proof of Lemma 4.3. Again, see [3] or [9] for the definition of a weakly admissible limit.)

If (iii) holds, let γ be a C^6-curve satisfying (1), and associate Γ with γ as before. If K is a nontangential approach region in Q_ε of small enough height, $\Gamma(K)$ will be contained in a nontangential approach region in D with vertex 0. Since a weakly admissible limit implies a nontangential limit, (iii) implies $u \circ \Gamma$ has a nontangential limit L at 0 within Q_ε. Our usual argument, together with Theorem D, then gives (iv).

Finally, since ∂D is C^6, (iv) implies (i).

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