

Pacific Journal of Mathematics

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Yeh obtained the conditional Wiener integral of

$$\exp\left\{-\int_0^t V[x(u)] du\right\}$$

given $x(t)$ where x is in Wiener space $C[0, t]$ and V is a function on \mathbf{R}^1 satisfying certain conditions. In this paper we extend Yeh's result to the conditional Yeh-Wiener integral of $\exp\left\{-\int_0^t \int_0^s V[x(u, v)] du dv\right\}$ given $x(s, t)$ where x is in Yeh-Wiener space $C_2(Q)$ and V is a nonnegative continuous function on \mathbf{R}^1 satisfying the condition

$$\int_{\mathbf{R}^1} V(w) \cdot \exp\left\{-\frac{w^2}{2st}\right\} dm_L(w) < \infty.$$

1. Introduction. Yeh recently derived inversion formulae for conditional expectations [5] and for conditional Wiener integrals [6]. He also evaluated some conditional Wiener integrals using these inversion formulae. In [2] and [3], they introduced the conditional Yeh-Wiener integral and extended some of Yeh's results for the conditional Wiener integrals to the conditional Yeh-Wiener integrals.

Here the probability space is the Yeh-Wiener measure space on the Yeh-Wiener space $C_2(Q)$ of the real valued continuous functions x defined on $Q = [0, s] \times [0, t]$ for some fixed positive real numbers s and t such that $x(0, v) = x(u, 0) = 0$ for all $0 \leq u \leq s$ and $0 \leq v \leq t$. In this paper we shall always denote Q as a fixed above rectangle. Let $(C_2(Q), \mathcal{Y}, m_y)$ be the Yeh-Wiener measure space. For a complete discussion of Yeh-Wiener measure space, see [7].

A real valued functional F on $C_2(Q)$ is said to be Yeh-Wiener measurable if it is \mathcal{Y} -measurable. Its integral with respect to m_y if it exists, is called its Yeh-Wiener integral which is denoted by $E^y(F)$. In this case we write

$$(1.1) \quad E^y(F) = \int_{C_2(Q)} F(x) dm_y(x).$$

We say that F is Yeh-Wiener integrable or m_y -integrable when the Yeh-Wiener integral of F , $E^y(F)$, exists and is finite. The Yeh-Wiener measurability and Yeh-Wiener integrability of a complex valued functional on $C_2(Q)$ are defined in terms of its real and imaginary parts.

Let X and Y be the \mathbf{R}^n -valued and real valued Yeh-Wiener measurable functions on $C_2(Q)$, respectively, with $E^y(|Y|) < \infty$. Let P_X be the probability distribution determined by X . By the conditional Yeh-Wiener integral of Y given X we mean the conditional expectation $E^y(Y|X)$ which is given as a function on the value space of X . Throughout this paper we shall be exclusively concerned with X and Y given by $X(x) = x(s, t)$ and $Y(x) = \exp\{-\int_Q V[x(u, v)] du dv\}$ for $x \in C_2(Q)$ in which V is a nonnegative continuous function on \mathbf{R}^1 satisfying the condition

$$(1.2) \quad \int_{\mathbf{R}^1} V(w) \cdot \exp\left\{-\frac{w^2}{2st}\right\} dm_L(w) < \infty$$

where m_L is the Lebesgue measure on \mathbf{R}^1 .

The techniques of this paper are closely related to those of paper [6] of Yeh, but for the evaluation of the conditional Yeh-Wiener integral we use slightly different techniques. In Theorem 2.1 we evaluate the conditional Yeh-Wiener integral of $Y(x)$ given $X(x)$ which is the extension of Yeh's result [6; Theorem 5]. The proof of Theorem 2.1 is simpler than that of Yeh. To do this we will use the following Proposition which comes from [3; Theorem 3.5].

PROPOSITION 1.1. *Let X and Y be measurable transformations of $(C_2(Q), \mathcal{Y})$ into $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$ with $E^y(|Y|) < \infty$. Assume that P_X is absolutely continuous with respect to m_L on $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$ and $E^y(e^{iuX}Y)$ is a m_L -integrable function of u on \mathbf{R}^1 . Then there exists a version of $E^y[Y|X](dP_X/dm_L)$ such that for $\xi \in \mathbf{R}^1$,*

$$(1.3) \quad E^y[Y|X](\xi) \frac{dP_X}{dm_L}(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iu\xi} E^y(e^{iuX}Y) dm_L(u).$$

2. The conditional Yeh-Wiener integral of

$$\exp\left\{-\int_0^t \int_0^s V[x(u, v)] du dv\right\}$$

given $x(s, t)$.

THEOREM 2.1. *For some fixed positive real numbers s and t , let*

$$(2.1) \quad \begin{aligned} X_{(s,t)}(x) &= x(s, t) \quad \text{and} \\ Y_{(s,t)}(x) &= \exp\left\{-\int_0^t \int_0^s V[x(u, v)] du dv\right\} \end{aligned}$$

for $x \in C_2(Q)$ where V is a nonnegative continuous function on \mathbf{R}^1 satisfying the condition

$$(2.2) \quad \int_{\mathbf{R}^1} V(w) \exp\left\{-\frac{w^2}{2st}\right\} dm_L(w) < \infty$$

for every s and t in $(0, \infty)$. Then the conditional Yeh-Wiener integral of $Y_{(s,t)}$ given by $X_{(s,t)}$ is

$$\begin{aligned}
 (2.3) \quad & E^y [Y_{(s,t)} | X_{(s,t)}](\xi) \\
 &= 1 - \int_Q \left[\int_{\mathbf{R}^3} \sqrt{\frac{st}{(s-u)(t-v)}} \right. \\
 &\quad \times \left. \left\{ \exp \left(-\frac{(\xi - u_{21} - u_{12} + u_{11})^2}{2(s-u)(t-v)} + \frac{\xi^2}{2st} \right) \right\} \right. \\
 &\quad \times \left. \left\{ V(u_{11}) E^y [Y_{(u,v)} | X](u_{11}, u_{12}, u_{21}) \right. \right. \\
 &\quad \left. \left. - E^y [Z_{(u,v)} Y_{(u,v)} | X](u_{11}, u_{12}, u_{21}) \right\} dP_X(u_{11}, u_{12}, u_{21}) \right] dm_L(u, v)
 \end{aligned}$$

where $x(s, t) = \xi \in \mathbf{R}^1$,

$$(2.4) \quad Z_{(u,v)}(x) = \int_0^v V[x(u, r)] dr \int_0^u V[x(q, v)] dq,$$

$$(2.5) \quad X(x) = (X_{(u,v)}(x), X_{(u,t)}(x), X_{(s,v)}(x))$$

for $x \in C_2(Q)$, and

$$\begin{aligned}
 (2.6) \quad & dP_X(u_{11}, u_{12}, u_{21}) = \left\{ (2\pi)^3 u^2 v_1^2 (s-u)(t-v) \right\}^{-1/2} \\
 & \times \exp \left\{ -\frac{u_{11}^2}{2uv} - \frac{(u_{12} - u_{11})^2}{2u(t-v)} - \frac{(u_{21} - u_{11})^2}{2(s-u)v} \right\} dm_L(u_{11}, u_{12}, u_{21}).
 \end{aligned}$$

REMARK. The existence of V on (2.2) follows if V satisfies the order of growth condition

$$V(w) = O(\exp\{w^{2-\delta}\}) \quad \text{as } w \rightarrow \pm \infty$$

for some $\delta \in (0, 2)$. Under (2.2), if we define

$$(2.7) \quad \phi((s, t)) = \frac{1}{\sqrt{2\pi st}} \int_{\mathbf{R}^1} V(w) \exp\left\{-\frac{w^2}{2st}\right\} dm_L(w)$$

for $(s, t) \in (0, \infty)^2$, then ϕ is a nonnegative continuous function on $(0, \infty)^2$ and furthermore

$$\lim_{(s,t) \rightarrow (\sigma,\tau)} \phi(s, t) = V(o) \quad \text{for } (\sigma, \tau) \in [0, \infty)^2 - (0, \infty)^2.$$

Let us define

$$(2.8) \quad \phi(\sigma, \tau) = \lim_{(s,t) \rightarrow (\sigma,\tau)} \phi(s, t)$$

for $(\sigma, \tau) \in [0, \infty)^2 - (0, \infty)^2$ so that ϕ is continuous on $[0, \infty)^2$.

LEMMA 2.1. For $0 < u \leq s$ and $0 < v \leq t$,

$$(2.9) \quad E^y \left[e^{i w(x(s,t) - x(s,v) - x(u,t) + x(u,v))} \right] = \exp \left\{ - \frac{(s-u)(t-v)}{2} w^2 \right\}$$

for $x \in C_2(Q)$ and $w \in \mathbf{R}^1$.

The lemma can be followed from the fact that the left-hand side of (2.9) is the characteristic function of the random variable $x(s, t) - x(s, v) - x(u, t) + x(u, v)$ whose probability distribution is the normal distribution with mean 0 and variance $(s-u)(t-v)$.

Proof of Theorem 2.1. We can easily obtain that $X_{(s,t)}$ and $Y_{(s,t)}$ are measurable transformations of $(C_2(Q), \mathcal{Y})$ into $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$, with $E^y(|Y|) < \infty$. Now

$$\begin{aligned} & \frac{\partial^2}{\partial u \partial v} \left[\exp \left\{ - \int_0^v \int_0^u V[x(q, r)] dq dr \right\} \right] \\ &= \exp \left\{ - \int_0^v \int_0^u V[x(q, r)] dq dr \right\} \\ & \quad \times \left\{ \int_0^v V[x(u, r)] dr \cdot \int_0^u V[x(q, v)] dq - V[x(u, v)] \right\}, \end{aligned}$$

thus we have by (2.1)

$$(2.11) \quad Y_{(s,t)}(x) = 1 + \int_Q \left\{ \exp \left(- \int_0^v \int_0^u V[x(q, r)] dq dr \right) \right\} \\ \times \left\{ \int_0^v V[x(u, r)] dr \int_0^u V[x(q, v)] dq - V[x(u, v)] \right\} dm_L(u, v).$$

To show that $E^y[e^{i w X_{(s,t)}} Y_{(s,t)}]$ is a m_L -integrable function of w in \mathbf{R}^1 , let

$$(2.12) \quad E^y \left[e^{i w X_{(s,t)}} Y_{(s,t)} \right] = E^y \left[e^{i w x(s,t)} \right] - J_1(s, t) + J_2(s, t)$$

where

$$(2.13) \quad J_1(s, t) \\ = E^y \left[e^{i w x(s,t)} \int_Q V[x(u, v)] \right. \\ \left. \times \exp \left\{ - \int_0^v \int_0^u V[x(q, r)] dq dr \right\} dm_L(u, v) \right]$$

and

$$(2.14) \quad J_2(s, t) = E^y \left[e^{iwx(s,t)} \int_Q \left\{ \int_0^v V[x(u, r)] dr \int_0^u V[x(q, v)] dq \right\} \times \exp \left\{ - \int_0^v \int_0^u V[x(q, r)] dq dr \right\} dm_L(u, v) \right].$$

Then we can have

$$(2.15) \quad E^y [e^{iwx(s,t)}] = \frac{1}{\sqrt{2\pi st}} \int_{\mathbf{R}} e^{iwx\xi} e^{-\xi^2/2st} dm_L(\xi) = e^{-(st/2)w^2}$$

by the basic Yeh-Wiener integration formula and the formula

$$(2.16) \quad \int_{\mathbf{R}} \exp\{- (a\xi^2 + b\xi)\} dm_L(\xi) = \sqrt{\frac{\pi}{a}} \exp\left\{\frac{b^2}{4a}\right\}$$

for $a > 0$ and real or imaginary b . Thus

$$(2.17) \quad \int_{\mathbf{R}} |E^y [e^{iwx(s,t)}]| dm_L(w) = \sqrt{\frac{2\pi}{st}} < \infty.$$

To interchange the order of the Yeh-Wiener integral and the integral with respect to $dm_L(u, v)$ on Q in (2.13), note that

$$\left| e^{iwx(s,t)} V[x(u, v)] \exp \left\{ - \int_0^v \int_0^u V[x(q, r)] dq dr \right\} \right| \leq V[x(u, v)]$$

for $((u, v), x) \in Q \times C_2(Q)$ and note that by (2.7) and the continuity of ϕ on $[0, \infty)^2$,

$$\int_Q E^y [V(x(u, v))] dm_L(u, v) = \int_Q \phi(u, v) dm_L(u, v) < \infty.$$

By the Fubini Theorem we have

$$(2.18) \quad J_1(s, t) = \int_Q E^y \left[e^{iwx(s,t)} V[x(u, v)] \times \exp \left\{ - \int_0^v \int_0^u V[x(q, r)] dq dr \right\} \right] dm_L(u, v).$$

Since

$$\{x(s, t) - x(s, v) - x(u, t) + x(u, v), \{x(s, v), x(u, t), x(q, r)\}\}$$

is an independent system of random variables on $(C_2(Q), \mathcal{Y}, m_y)$ for every $(q, r) \in [0, u] \times [0, v]$, we can have by Lemma 2.1,

$$(2.19) \quad J_1(s, t) = \int_Q \exp\left\{-\frac{(s-u)(t-v)}{2}w^2\right\} \\ \times E^y \left[e^{iw(x(s,v)+x(u,t)-x(u,v))} V[x(u, v)] \right. \\ \left. \times \exp\left\{-\int_0^v \int_0^u V[x(q, r)] dq dr\right\} \right] dm_L(u, v).$$

Let X be a three dimensional random vector on $(C_2(Q), \mathcal{Y}, m_y)$ given by

$$(2.20) \quad X(x) = (X_1(x), X_2(x), X_3(x))$$

where $X_1 \equiv X_{(u,v)}$, $X_2 \equiv X_{(u,t)}$, and $X_3 \equiv X_{(s,v)}$. Let $Y_1 \equiv Y_{(u,v)}$. Then the regular conditional distribution of Y_1 given X , $P(Y_1 | X)$, exists since Y_1 is a real valued random variable. With fixed $w \in \mathbf{R}^1$ consider a complex valued function f on $\mathbf{R}^3 \times \mathbf{R}^1$ defined by

$$(2.21) \quad f((\xi_1, \xi_2, \xi_3), \eta) = e^{iw(\xi_3+\xi_2-\xi_1)} V(\xi_1) \eta$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$ and $\eta \in \mathbf{R}^1$. By Proposition 2 and Proposition 1 in [5], we have

$$(2.22) \quad E^y \left[e^{iw(x(s,v)+x(u,t)-x(u,v))} V[x(u, v)] \right. \\ \left. \times \exp\left\{-\int_0^v \int_0^u V[x(q, r)] dq dr\right\} \right] \\ = \int_{\mathbf{R}^3} \left\{ \int_{\mathbf{R}^1} e^{iw(\xi_3+\xi_2-\xi_1)} V(\xi_1) \eta P(Y_1 | X)(d\eta, \xi) \right\} dP_X(\xi) \\ = \int_{\mathbf{R}^3} e^{iw(\xi_3+\xi_2-\xi_1)} V(\xi_1) E^y(Y_1 | X)(\xi) dP_X(\xi) \\ = \int_{\mathbf{R}^3} e^{iw(u_{21}+u_{12}-u_{11})} V(u_{11}) E^y(Y_1 | X)(u_{11}, u_{12}, u_{21}) \\ dP_X(u_{11}, u_{12}, u_{21})$$

where

$$(2.23) \quad dP_X(u_{11}, u_{12}, u_{21}) = \{(2\pi)^3 u^2 v^2 (s-u)(t-v)\}^{-1/2} \\ \times \exp\left\{-\frac{u_{11}^2}{2uw} - \frac{(u_{12}-u_{11})^2}{2u(t-v)} - \frac{(u_{21}-u_{11})^2}{2(s-u)v}\right\} dm_L(u_{11}, u_{12}, u_{21}).$$

By (2.19) and (2.22) we can obtain

$$(2.24) \quad J_1(s, t) = \int_Q \exp\left\{-\frac{(s-u)(t-v)}{2}w^2\right\} \\ \times \left[\int_{\mathbf{R}^3} e^{iw(u_{21}+u_{12}-u_{11})}V(u_{11}) \right. \\ \left. \times E^y(Y_1|X)(u_{11}, u_{12}, u_{21}) dP_X(u_{11}, u_{12}, u_{21}) \right] dm_L(u, v).$$

To show that $J_1(s, t)$ is integrable, observe that

$$\left| V(u_{11})E^y(Y_1|X)(u_{11}, u_{12}, u_{21})e^{iw(u_{21}+u_{12}-u_{11})}e^{-((s-u)(t-v)/2)w^2} \right| \\ \leq V(u_{11})E^y(Y_1|X)(u_{11}, u_{12}, u_{21})e^{-((s-u)(t-v)/2)w^2}$$

for $((u_{11}, u_{12}, u_{21}), (u, v)) \in \mathbf{R}^3 \times Q$. Let π_1 be a function from \mathbf{R}^3 to \mathbf{R}^1 defined by $\pi_1(u_{11}, u_{12}, u_{21}) = u_{11}$. Then $X_1 = \pi_1 \circ X$. Thus by Proposition 3 in [5] and (2.7) we have

$$(2.25) \quad \int_{\mathbf{R}^3} V(u_{11})E^y(Y_1|X)(u_{11}, u_{12}, u_{21}) dP_X(u_{11}, u_{12}, u_{21}) \\ = E^y[(V \circ X_1)Y_1] \leq E^y[V(x(u, v))] = \phi(u, v).$$

Now

$$\int_Q \phi(u, v)e^{-((s-u)(t-v)/2)w^2} dm_L(u, v) = Ke^{-(st/2)w^2}$$

where

$$K \equiv \int_Q \phi(u, v)e^{((sv+ut-uv)/2)w^2} dm_L(u, v) < \infty$$

by the continuity of ϕ on $[0, \infty)^2$. Thus $J_1(s, t)$ is integrable since

$$(2.26) \quad \int_{\mathbf{R}^1} |J_1(s, t)| dm_L(w) \leq K \int_{\mathbf{R}^1} e^{-(st/2)w^2} dm_L(w) < \infty.$$

To interchange the order of the integrals in (2.14) note that

$$\left| e^{iwx(s,t)} \left\{ \int_0^v V[x(u, r)] dr \int_0^u V[x(q, v)] dq \right\} \right. \\ \left. \times \exp\left\{-\int_0^v \int_0^u V[x(q, r)] dq dr\right\} \right| \\ \leq \int_0^v V[x(u, r)] dr \cdot \int_0^u V[x(q, v)] dq$$

for $x \in C_2(Q)$ and

$$\int_Q E^y \left[\int_0^v V[x(u, r)] dr \cdot \int_0^u V[x(q, v)] dq \right] dm_L(u, v) \leq Mst < \infty$$

for some $M > 0$ since $V \circ x$ is a continuous function. Thus from (2.14) we obtain

$$(2.27) \quad J_2(s, t) = \int_Q \exp\left\{-\frac{(s-u)(t-v)}{2}w^2\right\} \\ \times E^y \left[e^{iw(x(s,v)+x(u,t)-x(u,v))} \right. \\ \left. \times \left\{ \int_0^v V[x(u, r)] dr \int_0^u V[x(q, v)] dq \right\} \right. \\ \left. \times \exp\left\{-\int_0^v \int_0^u V[x(q, r)] dq dr\right\} \right] dm_L(u, v)$$

by the Fubini Theorem and the same way as in (2.19). Let X be given as in (2.20) and let $Y_1 = Y_{(u,v)}$. Note that $E^y(|Y_1|) < \infty$. Let

$$Z_1(x) = Z_{(u,v)}(x) = \int_0^v V[x(u, r)] dr \int_0^u V[x(q, v)] dq$$

for $x \in C_2(Q)$. Then it is obvious that Z_1 is Yeh-Wiener measurable and Yeh-Wiener integrable on $C_2(Q)$. Put $F_1 \equiv Z_1 Y_1$. Then F_1 is a real valued random variable on $(C_2(Q), \mathcal{Y}, m_y)$ with $E^y(|F_1|) < \infty$. For fixed $w \in \mathbf{R}^1$ consider a complex valued function g on $\mathbf{R}^3 \times \mathbf{R}^1$ defined by

$$g((\xi_1, \xi_2, \xi_3), \eta) = e^{iw(\xi_3 + \xi_2 - \xi_1)} \eta$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$ and $\eta \in \mathbf{R}^1$. Applying Proposition 2 and Proposition 1 in [5] to the real and imaginary parts of g we have by (2.1),

$$(2.28) \quad E^y \left[e^{iw(x(s,v)+x(u,t)-x(u,v))} \left\{ \int_0^v V[x(u, r)] dr \int_0^u V[x(q, v)] dq \right\} \right. \\ \left. \times \exp\left\{-\int_0^v \int_0^u V[x(q, r)] dq dr\right\} \right] \\ = \int_{\mathbf{R}^3} \left\{ \int_{\mathbf{R}^1} e^{iw(\xi_3 + \xi_2 - \xi_1)} \eta P(F_1 | X)(d\eta, \xi) \right\} dP_X(\xi) \\ = \int_{\mathbf{R}^3} e^{iw(u_{21} + u_{12} - u_{11})} E^y(F_1 | X)(u_{11}, u_{12}, u_{21}) dP_X(u_{11}, u_{12}, u_{21})$$

where $dP_X(u_{11}, u_{12}, u_{21})$ is given as in (2.23). By (2.27) and (2.28) we have

$$(2.29) \quad J_2(s, t) = \int_Q \left\{ \exp\left(-\frac{(s-u)(t-v)}{2}w^2\right) \right\} \\ \times \left\{ \int_{\mathbf{R}^3} e^{iw(u_{21} + u_{12} - u_{11})} E^y(F_1 | X)(u_{11}, u_{12}, u_{21}) dP_X(u_{11}, u_{12}, u_{21}) \right\} \\ dm_L(u, v).$$

To show that $J_2(s, t)$ is integrable, observe that

$$\begin{aligned} & \left| E^y(F_1 | X)(u_{11}, u_{12}, u_{21}) e^{iw(u_{21} + u_{12} - u_{11})} e^{-((s-u)(t-v)/2)w^2} \right| \\ & \leq E^y(F_1 | X)(u_{11}, u_{12}, u_{21}) e^{-((s-u)(t-v)/2)w^2} \end{aligned}$$

for $((u_{11}, u_{12}, u_{21}), (u, v)) \in \mathbf{R}^3 \times Q$. Since the conditional Yeh-Wiener integral is P_X -integrable, we have

$$N \equiv \int_{\mathbf{R}^3} E^y(F_1 | X)(u_{11}, u_{12}, u_{21}) dP_X(u_{11}, u_{12}, u_{21}) < \infty.$$

Thus

$$\int_Q N e^{-((s-u)(t-v)/2)w^2} dm_L(u, v) = N L e^{-(st/2)w^2}$$

where

$$L \equiv \int_Q e^{(sv + ut - uv/2)w^2} dm_L(u, v) < \infty.$$

Hence $J_2(s, t)$ is integrable since

$$(2.30) \quad \int_{\mathbf{R}^1} |J_2(s, t)| dm_L(w) \leq NL \int_{\mathbf{R}^1} e^{-(st/2)w^2} dm_L(w) < \infty.$$

Therefore by (2.12), (2.17), (2.26) and (2.30), we have that $E^y[e^{iwX_{(s,t)}} Y_{(s,t)}]$ is a m_L -integrable function of w on \mathbf{R}^1 and thus, by Proposition 1.1, there exists a version of

$$E^y[Y_{(s,t)} | X_{(s,t)}] \frac{dP_{X_{(s,t)}}}{dm_L}$$

such that

$$\begin{aligned} (2.31) \quad & E^y[Y_{(s,t)} | X_{(s,t)}](\xi) \frac{dP_{X_{(s,t)}}}{dm_L}(\xi) \\ & = \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw\xi} E^y[e^{iwX_{(s,t)}} Y_{(s,t)}] dm_L(w) \\ & = \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw\xi} \{ E^y[e^{iwX_{(s,t)}}] - J_1(s, t) + J_2(s, t) \} dm_L(w) \end{aligned}$$

by (2.12). To evaluate the above integral first note that

$$(2.32) \quad \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw\xi} E^y[e^{iwX_{(s,t)}}] dm_L(w) = \frac{1}{\sqrt{2\pi st}} \exp\left\{-\frac{\xi^2}{2st}\right\}$$

by (2.15) and (2.16). And also note that

$$\begin{aligned}
 (2.33) \quad & \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw\xi} J_1(s, t) \, dm_L(w) \\
 &= \frac{1}{2\pi} \int_Q \left\{ \int_{\mathbf{R}^3} V(u_{11}) E^y(Y_1 | X)(u_{11}, u_{12}, u_{21}) \right. \\
 &\quad \times \left[\int_{\mathbf{R}^1} e^{-iw(\xi - u_{21} - u_{12} + u_{11})} \right. \\
 &\quad \left. \left. \times e^{-((s-u)(t-v)/2)w^2} \, dm_L(w) \right] dP_X(u_{11}, u_{12}, u_{21}) \right\} dm_L(u, v)
 \end{aligned}$$

by (2.24) and the Fubini Theorem. By (2.16) we can have

$$\begin{aligned}
 (2.34) \quad & \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw(\xi - u_{21} - u_{12} + u_{11}) - ((s-u)(t-v)/2)w^2} \, dm_L(w) \\
 &= \frac{1}{\sqrt{2\pi(s-u)(t-v)}} \exp\left\{ -\frac{(\xi - u_{21} - u_{12} + u_{11})^2}{2(s-u)(t-v)} \right\}.
 \end{aligned}$$

Substituting (2.34) in (2.33) we obtain

$$\begin{aligned}
 (2.35) \quad & \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw\xi} J_1(s, t) \, dm_L(w) \\
 &= \int_Q \left\{ \int_{\mathbf{R}^3} V(u_{11}) E^y(Y_1 | X)(u_{11}, u_{12}, u_{21}) \frac{1}{\sqrt{2\pi(s-u)(t-v)}} \right. \\
 &\quad \left. \times \exp\left\{ -\frac{(\xi - u_{21} - u_{12} + u_{11})^2}{2(s-u)(t-v)} \right\} dP_X(u_{11}, u_{12}, u_{21}) \right\} dm_L(u, v).
 \end{aligned}$$

Finally we can note that

$$\begin{aligned}
 (2.36) \quad & \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw\xi} J_2(s, t) \, dm_L(w) \\
 &= \int_Q \left\{ \int_{\mathbf{R}^3} E^y[F_1 | X](u_{11}, u_{12}, u_{21}) \frac{1}{\sqrt{2\pi(s-u)(t-v)}} \right. \\
 &\quad \left. \times \exp\left\{ -\frac{(\xi - u_{21} - u_{12} + u_{11})^2}{2(s-u)(t-v)} \right\} dP_X(u_{11}, u_{12}, u_{21}) \right\} dm_L(u, v)
 \end{aligned}$$

by (2.29), the Fubini Theorem and (2.34). Hence by (2.31), (2.32), (2.35)

and (2.36) we have

$$\begin{aligned}
 (2.37) \quad & E^y [Y_{(s,t)} | X_{(s,t)}](\zeta) \frac{dP_{X_{(s,t)}}}{dm_L}(\zeta) \\
 &= \frac{1}{\sqrt{2\pi st}} \exp\left\{-\frac{\zeta^2}{2st}\right\} \\
 &\quad - \int_Q \left[\int_{\mathbb{R}^3} \{V(u_{11}) E^y [Y_1 | X](u_{11}, u_{12}, u_{21}) \right. \\
 &\quad \quad \left. - E^y [F_1 | X](u_{11}, u_{12}, u_{21})\} \frac{1}{\sqrt{2\pi(s-u)(t-v)}} \right. \\
 &\quad \left. \times \exp\left\{-\frac{(\zeta - u_{21} - u_{12} + u_{11})^2}{2(s-u)(t-v)}\right\} dP_X(u_{11}, u_{12}, u_{21}) \right] dm_L(u, v).
 \end{aligned}$$

Since

$$\frac{dP_{X_{(s,t)}}}{dm_L}(\zeta) = \frac{1}{\sqrt{2\pi st}} \exp\left\{-\frac{\zeta^2}{2st}\right\},$$

we can have the desired result (2.3) from (2.37).

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Received March 19, 1985 and in revised form July 5, 1985. Research was partially supported by Korea Science and Engineering Foundation Grant.

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