

# Pacific Journal of Mathematics

**THE REVERSING RESULT FOR THE JONES POLYNOMIAL**

W. B. RAYMOND LICKORISH AND KENNETH MILLETT

## THE REVERSING RESULT FOR THE JONES POLYNOMIAL

W. B. R. LICKORISH AND K. C. MILLETT

**A short proof is given, using linear skein theory, of the theorem of V.F.R. Jones that the one variable "Jones" polynomial associated to an oriented link is independent of the choice of strand orientations, up to a multiple of the variable.**

The Jones polynomial of an oriented link  $K$  is the element  $V(K)$  of  $\mathbf{Z}[t^{\pm 1/2}]$  defined by

$$tV(K_+) - t^{-1}V(K_-) + (t^{1/2} - t^{-1/2})V(K_0) = 0$$

$$V(U) = 1,$$

where  $U$  is the unknot and  $K_+$ ,  $K_-$ , and  $K_0$  are oriented links identical except in a small ball where they have positive, negative and null crossings respectively. Details are to be found in [J], [F-Y-H-L-M-O] or [L-M]. This note gives a short proof of the following theorem of V. F. R. Jones which states, *inter alia*, that, up to multiplication by a unit of  $\mathbf{Z}[t^{\pm 1/2}]$ ,  $V(K)$  is independent of the orientation of  $K$ . The original proof used the theory of braids and plaits; the proof here is a simple induction together with a neat *illustration* of linear skein theory. The proof fails (as it must) for the general two-variable oriented link polynomial only at the start of the induction.

**THEOREM (V. F. R. Jones).** *Suppose that a component  $\gamma$  of an oriented link  $K$  has linking number  $\lambda$  with the union of the other components. Let  $\hat{K}$  be  $K$  with the direction of  $\gamma$  reversed. Then  $t^{3\lambda}V(K) = V(\hat{K})$ .*

*Proof.* The proof is in five sections.

(1) The theorem is true for the two links of Figure 1. This is an easy exercise in computation.

(2) It is well known that if the orientation of every component of  $K$  is reversed then  $V(K)$  is unchanged. Further,  $V(K_1\#K_2) = V(K_1)V(K_2)$  where  $K_1\#K_2$  is any connected sum of oriented links  $K_1$  and  $K_2$ , and also  $V(\bar{K}) = V(K)$  where  $\bar{K}$  is the obverse (reflection) of  $K$  and  $f(\bar{t}) = f(t^{-1})$ . Thus if the Theorem is true for  $K_1$  and  $K_2$  it is true for  $\bar{K}_1$  and for  $K_1\#K_2$ .



FIGURE 1

(3) Consider the self-crossings of the component  $\gamma$  in some presentation of  $K$ . Induction (as repeatedly used in section three of [L-M]) on the number of these crossings and on the number of them that have to be switched to unknot  $\gamma$  shows that  $\gamma$  may be assumed to be unknotted.

(4) Let the unknotted component  $\gamma$  bound a disc that meets the remainder of  $K$  in  $n$  points. Proceed by induction on  $n$ . The start of the induction will be given in (5); for the moment assume that  $n \geq 4$ . Figure 2 depicts a skein triple in which  $K$  is  $K_0$ . The disc bounded by  $\gamma$  is shown meeting the remainder of  $K$  in  $n$  points shown as crosses. In  $K_-$ ,  $\gamma$  has become two unlinked curves  $\gamma_1^-$  and  $\gamma_2^-$  that bound discs that meet the remainder of  $K_-$  in  $n_1$  and  $n_2$  points and that link the remainder of  $K_-$  with linking numbers  $\lambda_1$  and  $\lambda_2$  respectively. The situation of  $K_+$  is exactly similar except that  $\gamma_1^+$  and  $\gamma_2^+$  are linked as shown.

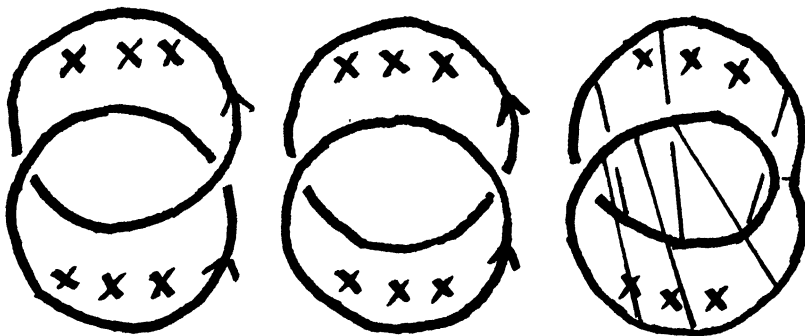


FIGURE 2

Thus  $n_1 + n_2 = n$  and  $\lambda_1 + \lambda_2 = \lambda$ . Choose  $n_1$  and  $n_2$  so that each is at most  $n - 2$  (recall  $n \geq 4$ ). Let  $\hat{K}_+$ ,  $\hat{K}_-$ , and  $\hat{K}_0$  be the same links but with the  $\gamma_i^\pm$  and  $\gamma$  all reversed. Then

$$tV(K_+) - t^{-1}V(K_-) + (t^{1/2} - t^{-1/2})V(K_0) = 0,$$

$$tV(\hat{K}_+) - t^{-1}V(\hat{K}_-) + (t^{1/2} - t^{-1/2})V(\hat{K}_0) = 0.$$

But, by the induction on  $n$ , reversing  $\gamma_1$  and then  $\gamma_2^-$  gives

$$t^{3\lambda_2}t^{3\lambda_1}V(K_-) = V(\hat{K}_1)$$

and reversing  $\gamma_1^+$  and then  $\gamma_2^+$  gives

$$t^{3(\lambda_2-1)}t^{3(\lambda_1+1)}V(K_+) = V(\hat{K}_+).$$

It follows immediately that  $t^{3\lambda}V(K) = V(\hat{K})$ .

This argument extends a little further when  $n = 3$ . If  $\lambda$  is also 3, choose  $n_1 = 1$  and  $n_2 = 2$ , then the above argument holds if the theorem is known for  $n = 3$  and  $\lambda = 1$  and for  $n \geq 2$ . Similarly when  $n = 3$ ,  $\lambda = -3$ .

(5) Suppose that  $n = 3$  and  $\lambda = \pm 1$ . It is required to show that whatever tangle is inserted into the room (the rectangle) of Figure 3 to give  $K$ , the Theorem holds true and  $t^3V(K) = V(\hat{K})$ . However, the module over  $Z[t^{\pm 1/2}]$  of the linearised skein of this room is generated by the six inhabitants shown in Figure 4.

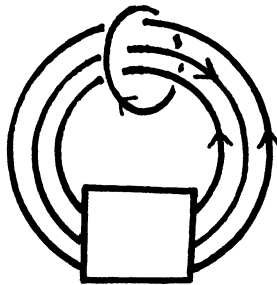


FIGURE 3

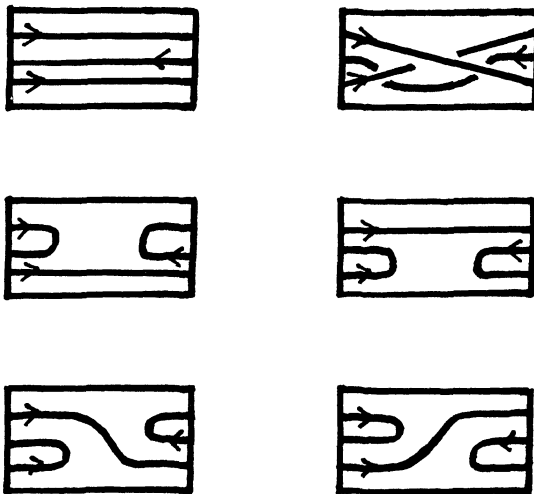


FIGURE 4

Thus all that is required is to check that whichever of these generators is inserted into the room to give  $K$  the theorem holds. This follows at once from (1) and (2). A simplified version of this proof works when  $n = 2$  there then being only two generators of the analogous rooms (see [L-M]). The case  $n = 1$  is immediate from (1) and (2) and  $n = 0$  is trivial.

This completes the proof; only step (1) fails to generalise to the general (two-variable) link polynomial.

#### REFERENCES

- [F-Y-H-L-M-O] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc., **12** (2) (1985), 239–246.
- [G] C. Giller, *A family of links and the Conway calculus*, Trans. Amer. Math. Soc., **270** (1982), 75–109.
- [J] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc., **12** (1) (1985), 103–111.
- [L-M] W. B. R. Lickorish and K. C. Millett, *A polynomial invariant of oriented links*, (to appear).

Received March 25, 1985.

DEPARTMENT OF PURE MATHEMATICS  
16, MILL LANE  
CAMBRIDGE, CB2 1SB  
ENGLAND

AND

UNIVERSITY OF CALIFORNIA  
SANTA BARBARA, CA 93106, U.S.A.

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

V. S. VARADARAJAN  
(Managing Editor)  
University of California  
Los Angeles, CA 90024  
HERBERT CLEMENS  
University of Utah  
Salt Lake City, UT 84112  
R. FINN  
Stanford University  
Stanford, CA 94305

HERMANN FLASCHKA  
University of Arizona  
Tucson, AZ 85721  
RAMESH A. GANGOLLI  
University of Washington  
Seattle, WA 98195  
VAUGHAN F. R. JONES  
University of California  
Berkeley, CA 94720  
ROBION KIRBY  
University of California  
Berkeley, CA 94720

C. C. MOORE  
University of California  
Berkeley, CA 94720  
H. SAMELSON  
Stanford University  
Stanford, CA 94305  
HAROLD STARK  
University of California, San Diego  
La Jolla, CA 92093

## ASSOCIATE EDITORS

R. ARENS      E. F. BECKENBACH      B. H. NEUMANN      F. WOLF      K. YOSHIDA  
(1906–1982)

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA	UNIVERSITY OF OREGON
UNIVERSITY OF BRITISH COLUMBIA	UNIVERSITY OF SOUTHERN CALIFORNIA
CALIFORNIA INSTITUTE OF TECHNOLOGY	STANFORD UNIVERSITY
UNIVERSITY OF CALIFORNIA	UNIVERSITY OF HAWAII
MONTANA STATE UNIVERSITY	UNIVERSITY OF TOKYO
UNIVERSITY OF NEVADA, RENO	UNIVERSITY OF UTAH
NEW MEXICO STATE UNIVERSITY	WASHINGTON STATE UNIVERSITY
OREGON STATE UNIVERSITY	UNIVERSITY OF WASHINGTON

<b>Kinetsu Abe and Martin Andrew Magid</b> , Relative nullity foliations and indefinite isometric immersions .....	1
<b>Erik P. van den Ban</b> , A convexity theorem for semisimple symmetric spaces .....	21
<b>Bo Berndtsson and Thomas Joseph Ransford</b> , Analytic multifunctions, the $\bar{\partial}$ -equation, and a proof of the corona theorem .....	57
<b>Brian Boe and David H. Collingwood</b> , Intertwining operators between holomorphically induced modules .....	73
<b>Giuseppe Ceresa and Alessandro Verra</b> , The Abel-Jacobi isomorphism for the sextic double solid .....	85
<b>Kun Soo Chang, Jae Moon Ahn and Joo Sup Chang</b> , An evaluation of the conditional Yeh-Wiener integral .....	107
<b>Charles Dale Frohman</b> , Minimal surfaces and Heegaard splittings of the three-torus .....	119
<b>Robert M. Guralnick</b> , Power cancellation of modules .....	131
<b>Kenneth Hardy and Kenneth S. Williams</b> , On the solvability of the Diophantine equation $dV^2 - 2eVW - dW^2 = 1$ .....	145
<b>Ray Alden Kunze and Stephen Scheinberg</b> , Alternative algebras having scalar involutions .....	159
<b>W. B. Raymond Lickorish and Kenneth Millett</b> , The reversing result for the Jones polynomial .....	173
<b>Guido Lupacchiolu</b> , A theorem on holomorphic extension of CR-functions .....	177
<b>William Schumacher Massey and Lorenzo Traldi</b> , On a conjecture of K. Murasugi .....	193
<b>Dinakar Ramakrishnan</b> , Spectral decomposition of $L^2(N \backslash GL(2), \eta)$ .....	215
<b>Steven L. Sperber</b> , On solutions of differential equations which satisfy certain algebraic relations .....	249