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We prove the holomorphic extendabilty on a domain $D \Subset \mathbf{C}^n$, $n \geq 2$, of the continuous CR-functions on a relatively open connected subset of ∂D, provided the complementary subset of ∂D is $\mathcal{O}(\overline{D})$-convex.

Introduction. Let D be a relatively compact open domain in $\mathbf{C}^n$, $n \geq 2$, with boundary ∂D, and K a compact subset of ∂D. We require D and K to be such that $\partial D \setminus K$ is a real hypersurface of class C^1 in $\mathbf{C}^n \setminus K$.

The purpose of this paper is to give a sufficient condition on D and K guaranteeing the holomorphic extendability on all of D of the CR-functions on $\partial D \setminus K$. Our theorem, which states the condition, improves and generalizes previous results in this direction obtained in Lupacciolu-Tomassini [**6**] and in Tomassini [**10**].[1]

Let $\mathcal{O}(\overline{D})$ be the algebra of complex-valued functions on $\overline{D}$ each of which is holomorphic on an open neighborhood of $\overline{D}$, and $\hat{K}_{\overline{D}}$ the $\mathcal{O}(\overline{D})$-hull of K. i.e.,

$$\hat{K}_{\overline{D}} = \bigcap_{\varphi \in \mathcal{O}(\overline{D})} \left\{ z \in \overline{D}; |\varphi(z)| \leq \max_K |\varphi| \right\}.$$

Our main result is the following theorem on holomorphic extension of CR-functions.

THEOREM 1. *Assume that K is $\mathcal{O}(\overline{D})$-convex, i.e., $\hat{K}_{\overline{D}} = K$, and $\partial D \setminus K$ is connected. Then every continuous CR-function f on $\partial D \setminus K$ has a unique extension F continuous on $\overline{D} \setminus K$ and holomorphic on D.*

A seemingly more general theorem is the following one.

THEOREM 2. *Assume that $\partial D \setminus \hat{K}_{\overline{D}}$ is a connected real hypersurface of class C^1 in $\mathbf{C}^n \setminus \hat{K}_{\overline{D}}$. Then every continuous CR-function f on $\partial D \setminus \hat{K}_{\overline{D}}$ has a unique extension F continuous on $\overline{D} \setminus \hat{K}_{\overline{D}}$ and holomorphic on $D \setminus \hat{K}_{\overline{D}}$.*

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[1] *Added in proof*. Recently Edgar Lee Stout kindly informed me of his paper [**12**], where the same condition is already recognized to be sufficient, when D is a domain of holomorphy, for a parallel extendability's property in the setting of holomorphic functions.

However, if we set $D' = D \setminus \hat{K}_{\overline{D}}$ and $K' = \overline{D}' \cap \hat{K}_{\overline{D}}$, it is an easy matter to see that Theorem 2 is equivalent to Theorem 1 with D' and K' in place of D and K.

Before going into the proof of Theorem 1, let us discuss a nontrivial situation where it applies.

Observe that, since plainly

$$\hat{K}_{\overline{D}} = \bigcap_{U \supset \overline{D}} \hat{K}_U,$$

where U ranges over the open neighbourhoods of $\overline{D}$, it suffices, in order that $\hat{K}_{\overline{D}} = K$, that, for some U, $\hat{K}_U \cap \overline{D} = K$, i.e. $\hat{K}_U$ does not meet $\overline{D} \setminus K$. Suppose, then, that the following holds: *there is an upper semicontinuous plurisubharmonic function ρ on a Stein open neighbourhood U of $\overline{D}$, so that $K \subset \{\rho = 0\}$ and $\overline{D} \setminus K \subset \{\rho > 0\}$*. Since $\hat{K}_U$ coincides with $\hat{K}_U^P$, the hull of K with respect to the plurisubharmonic functions on U (cf. Hörmander [5], p. 91), it follows that $\hat{K}_U$ is contained in $\{\rho \leq 0\}$, and hence $\hat{K}_U \cap \overline{D} = K$. In the case ρ is *pluriharmonic*, U may be required to be *simply connected*, instead that Stein; for ρ has then a unique pluriharmonic extension $\tilde{\rho}$ to the envelope of holomorphy $\tilde{U}$ of U, and hence $\hat{K}_U \subset \hat{K}_{\tilde{U}} = \hat{K}_{\tilde{U}}^P \subset \{\tilde{\rho} \leq 0\}$.

1. Preliminary facts. (a) We denote by $\omega(\zeta)$ the Martinelli form relative to a point $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$, that is

$$\omega(\zeta) = C_n \frac{dz_1 \wedge \cdots \wedge dz_n}{|z - \zeta|^{2n}} \wedge \sum_{\alpha=1}^{n} (-1)^{\alpha-1} (\bar{z}_\alpha - \bar{\zeta}_\alpha)\, d\bar{z}_1 \wedge \cdots \hat{\alpha} \cdots \wedge d\bar{z}_n$$

(where $C_n = (-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n$).

Given a holomorphic function φ on an open set $U \subset \mathbf{C}^n$ and a point $\zeta \in U$, we denote by $L_\zeta(\varphi)$ the level set of φ through ζ, that is

$$L_\zeta(\varphi) = \{z \in U; \varphi(z) = \varphi(\zeta)\}.$$

It is known that for any $\varphi \in \mathcal{O}(U)$ there exist holomorphic maps $h = (h_1, \ldots, h_n) \in \mathcal{O}^n(U \times U)$ such that, for each $(z, \zeta) \in U \times U$,

$$(*) \qquad \varphi(z) - \varphi(\zeta) = \sum_{\alpha=1}^{n} h_\alpha(z, \zeta)(z_\alpha - \zeta_\alpha)$$

(cf. Harvey [3], Lemma 2.3). Then we set:

$$(1.1) \qquad \mathcal{O}_\varphi^n(U \times U) = \{h \in \mathcal{O}^n(U \times U); (*) \text{ holds}\}.$$

Any $h \in \mathcal{O}_\varphi^n(U \times U)$ allows one to define canonically, for $\zeta \in U$, a $\bar{\partial}$-primitive of $\omega(\zeta)$ on $U \setminus L_\zeta(\varphi)$, that is $(n, n-2)$-form $\Phi_h(\zeta)$ on

$U \setminus L_\zeta(\varphi)$ such that

$$\omega(\zeta) = \bar\partial\Phi_h(\zeta) = d\Phi_h(\zeta).$$

As a matter of fact, consider, for every $\alpha = 1, \ldots, n$, the following $(n, n-2)$-form on $\mathbf{C}^n \setminus \{z_\alpha = \zeta_\alpha\} = \mathbf{C}^n \setminus L_\zeta(z_\alpha)$:

$$\Omega_\alpha(\zeta) = \frac{(-1)^{n+\alpha}}{n-1} C_n \frac{dz_1 \wedge \cdots \wedge dz_n}{(z_\alpha - \zeta_\alpha)|z-\zeta|^{2n-2}}$$
$$\wedge \left[\sum_{\beta=1}^{\alpha-1} (-1)^\beta \left(\bar z_\beta - \bar\zeta_\beta\right) d\bar z_1 \wedge \cdots \hat\beta \cdots \hat\alpha \cdots \wedge d\bar z_n \right.$$
$$\left. + \sum_{\beta=\alpha+1}^{n} (-1)^{\beta-1} \left(\bar z_\beta - \bar\zeta_\beta\right) d\bar z_1 \wedge \cdots \hat\alpha \cdots \hat\beta \cdots \wedge d\bar z_n \right].$$

One verifies that, on $\mathbf{C}^n \setminus L_\zeta(z_\alpha)$, $\omega(\zeta) = \bar\partial\Omega_\alpha(\zeta)$.[2] Then set

$$\Phi_h(\zeta) = \frac{1}{\varphi(z) - \varphi(\zeta)} \sum_{\alpha=1}^{n} h_\alpha(z,\zeta)(z_\alpha - \zeta_\alpha)\Omega_\alpha(\zeta). \tag{1.2}$$

It is plain that $\Phi_h(\zeta)$ is indeed a real analytic $\bar\partial$-primitive of $\omega(\zeta)$ on $U \setminus L_\zeta(\varphi)$.

Such $\bar\partial$-primitives of the Martinelli form will play a fundamental role in the proof of our extension theorem. Now we derive the properties of them that will be needed.

Let there be given open sets $U, U' \subset \mathbf{C}^n$ such that $U \cap U' \neq \varnothing$, functions $\varphi \in \mathcal{O}(U)$, $\varphi' \in \mathcal{O}(U')$ and maps $h \in \mathcal{O}^n_\varphi(U \times U)$, $h' \in \mathcal{O}^n_{\varphi'}(U' \times U')$, and let ζ be a point in $U \cap U'$. Suppose first that $n \geq 3$, and consider, for every $\alpha, \beta = 1, \ldots, n$ with $\alpha \neq \beta$, the $(n, n-3)$-form $\Lambda_{\alpha,\beta}(\zeta)$ on $\mathbf{C}^n \setminus (L_\zeta(z_\alpha) \cup L_\zeta(z_\beta))$ defined as follows: for $\alpha < \beta$

$$\Lambda_{\alpha,\beta}(\zeta) = \frac{(-1)^{n+\alpha+\beta}}{(n-1)(n-2)} C_n \frac{dz_1 \wedge \cdots \wedge dz_n}{(z_\alpha - \zeta_\alpha)(z_\beta - \zeta_\beta)|z-\zeta|^{2n-4}}$$
$$\wedge \left[\sum_{\gamma=1}^{\alpha-1} (-1)^\gamma \left(\bar z_\gamma - \bar\zeta_\gamma\right) d\bar z_1 \wedge \cdots \hat\gamma \cdots \hat\alpha \cdots \hat\beta \cdots \wedge d\bar z_n \right.$$
$$+ \sum_{\gamma=\alpha+1}^{\beta-1} (-1)^{\gamma-1} \left(\bar z_\gamma - \bar\zeta_\gamma\right) d\bar z_1 \wedge \cdots \hat\alpha \cdots \hat\gamma \cdots \hat\beta \cdots \wedge d\bar z_n$$
$$\left. + \sum_{\gamma=\beta+1}^{n} (-1)^{\gamma} \left(\bar z_\gamma - \bar\zeta_\gamma\right) d\bar z_1 \wedge \cdots \hat\alpha \cdots \hat\beta \cdots \hat\gamma \cdots \wedge d\bar z_n \right],$$

[2] The forms $\Omega_\alpha(\zeta)$ were considered first by Martinelli [7], to give a proof of Hartogs' theorem.

and for $\alpha > \beta$ $\Lambda_{\alpha,\beta}(\zeta) = -\Lambda_{\beta,\alpha}(\zeta)$. One can verify that $\Omega_\alpha(\zeta) - \Omega_\beta(\zeta) = \bar{\partial}\Lambda_{\alpha,\beta}(\zeta)$. Then, consider the following $(n, n-3)$-form on $(U \setminus L_\zeta(\varphi)) \cap (U' \setminus L_\zeta(\varphi'))$:

$$X_{h,h'}(\zeta) = \frac{1}{(\varphi(z) - \varphi(\zeta))(\varphi'(z) - \varphi'(\zeta))} \cdot \sum_{1 \le \alpha < \beta \le n} \left(h_\alpha h'_\beta - h_\beta h'_\alpha\right)(z_\alpha - \zeta_\alpha)(z_\beta - \zeta_\beta)\Lambda_{\alpha,\beta}(\zeta).$$

It is easily seen that, on $(U \setminus L_\zeta(\varphi)) \cap (U' \setminus L_\zeta(\varphi'))$,

$$\Phi_h(\zeta) - \Phi_{h'}(\zeta) = \bar{\partial}X_{h,h'}(\zeta). \tag{1.3}$$

In case $n = 2$ we simply have:

$$\Omega_1(\zeta) - \Omega_2(\zeta) = -\frac{1}{(2\pi i)^2}\frac{dz_1 \wedge dz_2}{(z_1 - \zeta_1)(z_2 - \zeta_2)},$$

and hence we find, on $(U \setminus L_\zeta(\varphi)) \cap (U' \setminus L_\zeta(\varphi'))$:

$$\Phi_h(\zeta) - \Phi_{h'}(\zeta) = -\frac{1}{(2\pi i)^2}\frac{\left(h_1 h'_2 - h_2 h'_1\right) dz_1 \wedge dz_2}{(\varphi(z) - \varphi(\zeta))(\varphi'(z) - \varphi'(\zeta))}. \tag{1.4}$$

Next, we observe that all the above differential forms depend in a real analytic fashion also on the point ζ, so that we may perform any derivative of these with respect to the parameters $\operatorname{Re}\zeta_\alpha$, $\operatorname{Im}\zeta_\alpha$, $\alpha = 1, \dots, n$ (by taking the derivative of each coefficient). In particular we may consider the forms $\partial\omega/\partial\bar{\zeta}_\alpha$, $\partial\Omega_\beta/\partial\bar{\zeta}_\alpha$, etc., obtained by applying the Wirtinger operator $\partial \cdot /\partial\bar{\zeta}_\alpha$. We first note that, for every $\alpha = 1, \dots, n$, the $(n, n-2)$-form $\partial\Omega_\alpha/\partial\bar{\zeta}_\alpha$ satisfies

$$\frac{\partial\Omega_\alpha}{\partial\bar{\zeta}_\alpha}(\zeta) = (n-1)\frac{z_\alpha - \zeta_\alpha}{|z - \zeta|^2}\Omega_\alpha(\zeta),$$

and hence is defined (and real analytic) on $\mathbf{C}^n \setminus \zeta$, instead that only on $\mathbf{C}^n \setminus L_\zeta(z_\alpha)$ as $\Omega_\alpha(\zeta)$. It follows that, on $\mathbf{C}^n \setminus \zeta$,

$$\frac{\partial\omega}{\partial\bar{\zeta}_\alpha}(\zeta) = \bar{\partial}\left[\frac{\partial\Omega_\alpha}{\partial\bar{\zeta}_\alpha}(\zeta)\right] \qquad (\alpha = 1, \dots, n). \tag{1.5}$$

Similarly, if $n \ge 3$, for every $\alpha, \beta = 1, \dots, n$ with $\alpha \ne \beta$, the $(n, n-3)$-form $\partial\Lambda_{\alpha,\beta}/\partial\bar{\zeta}_\alpha$ satisfies

$$\frac{\partial\Lambda_{\alpha,\beta}}{\partial\bar{\zeta}_\alpha}(\zeta) = (n-2)\frac{z_\alpha - \zeta_\alpha}{|z - \zeta|^2}\Lambda_{\alpha,\beta}(\zeta),$$

and hence is defined on $\mathbf{C}^n \setminus L_\zeta(z_\beta)$, instead that only on $\mathbf{C}^n \setminus (L_\zeta(z_\alpha) \cup L_\zeta(z_\beta))$ as $\Lambda_{\alpha,\beta}(\zeta)$. It follows that, on $\mathbf{C}^n \setminus L_\zeta(z_\beta)$,

$$\frac{\partial \Omega_\alpha}{\partial \bar{\zeta}_\alpha}(\zeta) - \frac{\partial \Omega_\beta}{\partial \bar{\zeta}_\alpha}(\zeta) = \bar{\partial}\left[\frac{\partial \Lambda_{\alpha,\beta}}{\partial \bar{\zeta}_\alpha}(\zeta)\right].$$

If $n = 2$ we simply have, for $\alpha = 1, 2$:

$$\frac{\partial \Omega_1}{\partial \bar{\zeta}_\alpha}(\zeta) - \frac{\partial \Omega_2}{\partial \bar{\zeta}_\alpha}(\zeta) = 0.$$

Now, let there be given an open set $U \subset \mathbf{C}^n$, a function $\varphi \in \mathcal{O}(U)$ and a map $h \in \mathcal{O}^n_\varphi(U \times U)$, and let ζ be a point in U. In case $n \geq 3$ consider, for every $\alpha = 1, \ldots, n$, the following $(n, n-3)$-form on $U \setminus L_\zeta(\varphi)$:

$$\Psi^\alpha_h(\zeta) = \frac{1}{\varphi(z) - \varphi(\zeta)} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n} h_\beta(z_\beta - \zeta_\beta) \frac{\partial \Lambda_{\alpha,\beta}}{\partial \bar{\zeta}_\alpha}(\zeta).$$

Then we find, on $U \setminus L_\zeta(\varphi)$:

$$\frac{\partial \Phi_h}{\partial \bar{\zeta}_\alpha}(\zeta) = \frac{\partial \Omega_\alpha}{\partial \bar{\zeta}_\alpha}(\zeta) - \bar{\partial}\Psi^\alpha_h(\zeta) \qquad (\alpha = 1, \ldots, n). \tag{1.6}$$

On the other hand, if $n = 2$, we have:

$$\frac{\partial \Phi_h}{\partial \bar{\zeta}_\alpha}(\zeta) = \frac{\partial \Omega_\alpha}{\partial \bar{\zeta}_\alpha}(\zeta) \qquad (\alpha = 1, 2). \tag{1.7}$$

(b) It is well known that, given an oriented real hypersurface Σ of class C^1 in $\mathbf{C}^n$ (without boundary, not necessarily closed) and a complex-valued function f in $L^1_{\text{loc}}(\Sigma)$, one may say that f is a CR-function on Σ in case it satisfies the tangential Cauchy-Riemann equation in the weak form, that is

$$\int_\Sigma f \bar{\partial}\lambda = 0, \tag{1.8}$$

for every $(n, n-2)$-form λ of class C^1 on an open neighbourhood of Σ, such that $\Sigma \cap \operatorname{Supp}(\lambda)$ is compact. However we need for our purposes a sharper characterization of continuous CR-functions on Σ than (1.8) is. This is provided by the following proposition.

Proposition 1.9. *Let f be a complex-valued continuous function on Σ. Then f is a CR-function if and only if it satisfies*

$$\int_{c_{n+q}} f \bar{\partial}\mu = \int_{\partial c_{n+q}} f\mu, \tag{1.10}$$

for every singular $(n+q)$-chain c_{n+q} of Σ of class C^1 and every $(n, q-1)$-form μ of class C^1 on an open neighbourhood of Σ $(1 \leq q \leq n-1)$.[3]

Proof. This proposition asserts that (1.8) and (1.10) are equivalent for a continuous f (which would be quite immediate if f were of class C^1). We shall prove only that (1.8) implies (1.10), the converse being trivial.

For every differential form μ of class C^1 on an open neighbourhood V of Σ, we denote by $\mu|_\Sigma$ the restriction of μ to Σ (i.e. the pull-back of μ by the inclusion map $\Sigma \hookrightarrow V$). Then $\mu|_\Sigma$ is a continuous regular form on Σ.[4]

Consider the continuous n-form on Σ

$$u = f(dz_1 \wedge \cdots \wedge dz_n)|_\Sigma.$$

We claim that (1.10) is equivalent to the following assertion:

$(*)$ *u is regular on Σ and $du = 0$.*

As a matter of fact, taking in particular $q = 1$ and $\mu = dz_1 \wedge \cdots \wedge dz_n$, (1.10) gives:

$$0 = \int_{\partial c_{n+1}} f\, dz_1 \wedge \cdots \wedge dz_n = \int_{\partial c_{n+1}} u,$$

for every singular $(n+1)$-chain c_{n+1} of Σ of class C^1; and this is just as to say that $(*)$ holds. Conversely, assume that $(*)$ holds. Any $(n, q-1)$-form μ as in the statement can be written as $\mu = dz_1 \wedge \cdots \wedge dz_n \wedge \tilde{\mu}$, where $\tilde{\mu}$ is a $(0, q-1)$-form of class C^1 on an open neighbourhood of Σ. Then $u \wedge \tilde{\mu}|_\Sigma$ is a continuous regular $(n+q-1)$-form on Σ and, since $du = 0$, $d(\tilde{\mu}|_\Sigma) = (d\tilde{\mu})|_\Sigma$, we have:

$$d(u \wedge \tilde{\mu}|_\Sigma) = (-1)^n u \wedge (d\tilde{\mu})|_\Sigma = f(d\mu)|_\Sigma = f(\bar{\partial}\mu)|_\Sigma.$$

It follows that

$$\int_{c_{n+q}} f\bar{\partial}\mu = \int_{\partial c_{n+q}} u \wedge \tilde{\mu}|_\Sigma = \int_{\partial c_{n+q}} f\mu,$$

that is, (1.1) holds. Next, we claim that $(*)$ is equivalent to:

$(**)$ *u is weakly closed on Σ, that is $\int_\Sigma u \wedge dv = 0$*

for every $(n-2)$-form v on Σ of class C^1 and with compact support.

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[3] The same result is proved in Lupacciolu-Tomassini [**6**] under the additional assumption that f is locally Lipschitz, but the argument used there does not work without that assumption.

[4] For the definition and basic properties of continuous regular forms we refer to Whitney [**11**] pp. 103–108. We denote, as usual, by d the differential acting on such forms (defined by means of Stokes' formula), as the ordinary exterior differential.

This latter equivalence is a straightforward consequence of the following general facts about continuous differential forms on a manifold of class C^1:

(i) The differential acting on continuous regular forms may be understood in the strong sense. This means that, if η, θ are continuous forms, then η, θ are regular and $d\eta = \theta$ in the sense of regular forms if and only if there exists a sequence $\{\eta_s\}_{s=1}^{\infty}$ of forms of class C^1 such that $\eta_s \to \eta$ and $d\eta_s \to \theta$ as $s \to \infty$, both uniformly on compact sets (cf. Whitney [**11**]);

(ii) The differential in the strong sense coincides with the differential in the weak sense. This means that, if η, θ are continuous forms, then $d\eta = \theta$ in the strong sense if and only if $\int \eta \wedge d\xi = (-1)^{\deg \eta + 1} \int \theta \wedge \xi$, for every form ξ of class C^1 and with compact support (cf. Friedrichs [**2**], or Fichera [**1**]).[5]

Now we show that (1.8) implies (**), which will conclude the proof. We shall use the following fact: there exists an open neighbourhood W of Σ in $\mathbf{C}^n$ and a retraction $r\colon W \to \Sigma$ of class C^1 (which means that $r(z) = z$ for each $z \in \Sigma$). This is a special case of a standard theorem in Differential Topology (cf. Munkres [**8**], p. 51, or Whitney [**11**], p. 121).[6] If v is any $(n-2)$-form on Σ of class C^1 and with compact support, consider its pull-back r^*v to W. r^*v is a continuous regular $(n-2)$-form on W, and hence we can find a sequence $\{\eta_s\}_{s=1}^{\infty}$ of $(n-2)$-forms of class C^1 on W such that

$$\lim_{s\to\infty} \eta_s = r^*v, \qquad \lim_{s\to\infty} d\eta_s = r^*dv,$$

both uniformly on compact subsets of W. Moreover, since $\Sigma \cap \operatorname{Supp}(r^*v) = \operatorname{Supp}(v)$ is compact, we can arrange that so too is $\Sigma \cap \operatorname{Supp}(\eta_s)$, for every s. It follows that

$$\begin{aligned}\int_\Sigma u \wedge dv &= \lim_{s\to\infty} \int_\Sigma u \wedge (d\eta_s)|_\Sigma \\ &= \lim_{s\to\infty} \int_\Sigma f\,dz_1 \wedge \cdots \wedge dz_n \wedge d\eta_s \\ &= (-1)^n \lim_{s\to\infty} \int_\Sigma f\,\bar\partial(dz_1 \wedge \cdots \wedge dz_n \wedge \eta_s),\end{aligned}$$

and hence (1.8) implies $\int_\Sigma u \wedge dv = 0$.

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[5] Clearly, the interest of this fact is in the "if", the "only if" being trivial.

[6] If Σ were of class C^2, we could use the more elementary "tubular neighbourhood theorem".

2. Proof of Theorem 1. Let V be an open neighbourhood of K in $\mathbf{C}^n$ and $\sigma\colon \mathbf{C}^n \to \mathbf{R}$ a C^∞ function such that $0 \le \sigma(z) \le 1$ for all z, $\sigma(z) = 1$ for $z \in K$, $\operatorname{Supp}(\sigma)$ is compact and contained in V. For a generic small $\varepsilon > 0$, set $D_\varepsilon = D \cap \{1 - \sigma > \varepsilon\}$, $\Gamma_\varepsilon = \partial D \cap \{1 - \sigma \ge \varepsilon\}$ and $K_\varepsilon = \overline{D} \cap \{1 - \sigma = \varepsilon\}$. Then D_ε is a subdomain of D, $\partial D_\varepsilon = \Gamma_\varepsilon \cup K_\varepsilon$, Γ_ε and K_ε are compact real hypersurfaces with boundary, of class C^1, such that $\Gamma_\varepsilon \cap K_\varepsilon = \partial\Gamma_\varepsilon = \partial K_\varepsilon$, and Γ_ε is connected. Clearly, D is exhaustible by an increasing sequence of subdomains of this sort, $\{D_s\}_{s=1}^\infty$, say, so that

$$\partial D_s = \Gamma_s \cup K_s \qquad (s = 1, 2, \ldots),$$

with obvious meaning of Γ_s, K_s, and

$$D = \bigcup_{s=1}^\infty D_s, \qquad \partial D \setminus K = \bigcup_{s=1}^\infty \Gamma_s.$$

We assume that the sequence $\{D_s\}_{s=1}^\infty$ has been chosen once for all.

Now, let U be an open neighbourhood of $\overline{D}$ and let $\varphi \in \mathcal{O}(U)$. For every positive integer s we set:

$$U_s(\varphi) = \left\{\zeta \in U;\, |\varphi(\zeta)| > \max_{\overline{D\setminus D_s}} |\varphi|\right\}.$$

Then $U_s(\varphi)$ is an open subset of $U \setminus \overline{D \setminus D_s}$ such that, if $\zeta \in U_s(\varphi)$, the level set $L_\zeta(\varphi)$ of φ through ζ is all contained in $U_s(\varphi)$. Moreover we set:

$$U(\varphi) = \left\{\zeta \in U;\, |\varphi(\zeta)| > \max_K |\varphi|\right\}.$$

Since $\{\overline{D \setminus D_s}\}_{s=1}^\infty$ is a decreasing sequence of compact neighbourhoods of K in $\overline{D}$ such that $K = \bigcap_{s=1}^\infty \overline{D \setminus D_s}$, it follows that $U_1(\varphi) \subset U_2(\varphi) \cdots$, and

$$U(\varphi) = \bigcup_{s=1}^\infty U_s(\varphi). \tag{2.1}$$

Moreover, since $\hat{K}_{\overline{D}} = \bigcap_{U \supset \overline{D}} \hat{K}_U$ (where U ranges over the open neighbourhoods of $\overline{D}$), the assumption of Theorem 1 implies:

$$\overline{D} \setminus K \subset \bigcup_{U \supset \overline{D}} \bigcup_{\varphi \in \mathcal{O}(U)} U(\varphi). \tag{2.2}$$

Next, for every U, φ, s as above and $h \in \mathcal{O}_\varphi^n(U \times U)$ (cf. (1.1)), consider the complex-valued function F_h^s on $U_s(\varphi) \setminus \partial D$ given by

$$F_h^s(\zeta) = \int_{\Gamma_s} f\omega(\zeta) - \int_{\partial\Gamma_s} f\Phi_h(\zeta), \tag{2.3}$$

where $\Phi_h(\zeta)$ is the $\bar{\partial}$-primitive (1.2) of the Martinelli form $\omega(\zeta)$, Γ_s is oriented as a part of ∂D and $\partial\Gamma_s$ as the boundary of Γ_s.[7] Since, for $\zeta \in U_s(\varphi)$ and $z \in \partial\Gamma_s$, $|\varphi(\zeta)| > |\varphi(z)|$ (because $\partial\Gamma_s \subset \overline{D \setminus D_s}$), the singular set $L_\zeta(\varphi)$ of $\Phi_h(\zeta)$ does not meet $\partial\Gamma_s$, so that F_h^s is indeed defined, and real analytic, on $U_s(\varphi) \setminus \Gamma_s = U_s(\varphi) \setminus \partial D$.

PROPOSITION 2.4. *Suppose there exists at least a function F as in the statement of Theorem* 1. *Then, for every U, φ, h, s as above,*

$$F = F_h^s \quad \text{on } D \cap U_s(\varphi).$$

As a consequence, on account of (2.1) *and* (2.2), *if such a F actually exists, it is necessarily unique.*

Proof. Clearly $D \cap U_s(\varphi) \subset D_s$, and, by assumption, $F \in C^0(\overline{D}_s) \cap \mathcal{O}(D_s)$ and $F = f$ on Γ_s. Therefore, since, by the Martinelli formula, for $\zeta \in D_s$, we have:

$$F(\zeta) = \int_{\Gamma_s} f\omega(\zeta) + \int_{K_s} F\omega(\zeta),$$

we are required to show that, for $\zeta \in D \cap U_s(\varphi)$, we also have:

$$(*) \qquad \int_{K_s} F\omega(\zeta) = -\int_{\partial\Gamma_s} f\Phi_h(\zeta).$$

Since F is continuous on $\overline{D} \setminus K$ and holomorphic on D, the forms $F\omega(\zeta)$, $F\Phi_h(\zeta)$ are both continuous on $(\overline{D} \setminus K) \setminus L_\zeta(\varphi)$, real analytic on $D \setminus L_\zeta(\varphi)$, and on $D \setminus L_\zeta(\varphi)$ satisfy $F\omega(\zeta) = d(F\Phi_h(\zeta))$. Moreover, since $\zeta \in U_s(\varphi)$, it follows that $K_s \subset (\overline{D} \setminus K) \setminus L_\zeta(\varphi)$. Then consider the restrictions $(F\omega(\zeta))\,|\,_{K_s}$, $(F\Phi_h(\zeta))\,|\,_{K_s}$; these are continuous on K_s, regular on $K_s \setminus \partial K_s$ and on $K_s \setminus \partial K_s$ satisfy $(F\omega(\zeta))\,|\,_{K_s} = d[(F\Phi_h(\zeta))\,|\,_{K_s}]$. Hence Stokes' theorem for regular forms on a manifold with boundary (cf. Whitney **[11]**, p. 109) implies:

$$\int_{K_s} F\omega(\zeta) = \int_{\partial K_s} F\Phi_n(\zeta).$$

Finally, since $\partial K_s = -\partial\Gamma_s$ ($= \partial\Gamma_s$ with the opposite orientation), $(*)$ follows.

The above proposition disposes of the uniqueness' assertion in Theorem 1 and, further, implies that the proof of the existence of a holomor-

[7]In this paper we take as the canonical orientation of $\mathbf{C}^n$ and of D the one given by the volume-form $(i/2)^n\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$.

phic continuation of f on D shall be a matter of showing that the F_h^s's do in fact define a holomorphic function F on D such that, for each $z^0 \in \partial D \setminus K$, $F(\zeta) \to f(z^0)$ as $\zeta \to z^0$ in D. In the first place we have:

PROPOSITION 2.5. *The functions $F_h^{s'}s$ are each other coherent and holomorphic. Hence there is a unique holomorphic function F on*

$$\left(\bigcup_{U \supset \overline{D}} \bigcup_{\varphi \in \mathcal{O}(U)} U(\varphi) \right) \setminus \partial D$$

such that, for every U, φ, h, s,

$$F = F_h^s \quad \text{on } U_s(\varphi) \setminus \partial D.$$

Proof. We first prove the coherence. This means that, for every U, φ, h, s and U', φ', h', s', we have:

$$(*) \qquad F_h^s = F_{h'}^{s'} \quad \text{on } U_s(\varphi) \cap U'_{s'}(\varphi') \setminus \partial D.$$

We may assume that $s \geq s'$. Then $(*)$ will be a consequence of the following two equalities:

(i) $F_{h'}^s = F_{h'}^{s'}$ on $U'_{s'}(\varphi') \setminus \partial D$;

(ii) $F_h^s = F_{h'}^s$ on $U_s(\varphi) \cap U'_s(\varphi') \setminus \partial D$

(recall that $U_{s'}(\varphi) \subset U_s(\varphi)$ and $U'_{s'}(\varphi') \subset U'_s(\varphi')$). To prove (i) (in case $s > s'$), consider the $(2n-1)$-chain of $\partial D \setminus K$, of class C^1, $c_{2n-1} = \Gamma_s - \Gamma_{s'}$. If ζ is any point in $U'_{s'}(\varphi') \setminus \partial D$, it is plain that

$$F_{h'}^s(\zeta) - F_{h'}^{s'}(\zeta) = \int_{c_{2n-1}} f\omega(\zeta) - \int_{\partial c_{2n-1}} f\Phi_{h'}(\zeta);$$

moreover, since $\mathrm{Supp}(c_{2n-1}) \subset \overline{D_s \setminus D_{s'}} \subset \overline{D \setminus D_{s'}}$ and $L_\zeta(\varphi') \subset U'_{s'}(\varphi') \subset U' \setminus \overline{D \setminus D_{s'}}$, it follows that $\mathrm{Supp}(c_{2n-1})$ is contained in $U' \setminus L_\zeta(\varphi')$, where $\omega(\zeta)$, $\Phi_{h'}(\zeta)$ are both defined and satisfy $\omega(\zeta) = \bar{\partial}\Phi_{h'}(\zeta)$. Then, if we take a $(n, n-2)$-form μ of class C^∞ on all of $\mathbf{C}^n$ and equal to $\Phi_{h'}(\zeta)$ on an open neighbourhood of $\mathrm{Supp}(c_{2n-1})$, we may replace $\omega(\zeta)$, $\Phi_{h'}(\zeta)$, in the right side of the above equality, respectively by $\bar{\partial}\mu$, μ. Hence Proposition 1.9 gives at once that $F_{h'}^s(\zeta) = F_{h'}^{s'}(\zeta)$.

Next we prove (ii). On account of (1.3), (1.4), we have, for each $\zeta \in U_s(\varphi) \cap U'_s(\varphi') \setminus \partial D$:

$$F_h^s(\zeta) - F_{h'}^s(\zeta)$$

$$= \begin{cases} -\displaystyle\int_{\partial \Gamma_s} f \bar{\partial} \mathrm{X}_{h,h'}(\zeta) & \text{if } n \geq 3, \\ \dfrac{1}{(2\pi i)^2} \displaystyle\int_{\partial \Gamma_s} f(z) \frac{(h_1 h'_2 - h_2 h'_1)\, dz_1 \wedge dz_2}{(\varphi(z) - \varphi(\zeta))(\varphi'(z) - \varphi'(\zeta))} & \text{if } n = 2. \end{cases}$$

In case $n \geq 3$, we may replace $X_{h,h'}(\zeta)$, in the integral on the right side, by any $(n, n-3)$-form $\tilde{X}$ of class C^∞ on all of $\mathbf{C}^n$ and equal to $X_{h,h'}(\zeta)$ on an open neighbourhood of $\partial\Gamma_s$. Hence Proposition 1.9 (for $q = n-1$, $c_{n+q} = \Gamma_s$ and $\mu = \bar{\partial}\tilde{X}$) implies that $F_h^s(\zeta) = F_{h'}^s(\zeta)$.

In case $n = 2$, we have to argue differently. Since $\zeta \in U_s(\varphi) \cap U_s'(\varphi')$ and $\partial\Gamma_s \subset \overline{D \setminus D_s}$, it follows that, for each $z \in \partial\Gamma_s$, $|\varphi(\zeta)| > \max_{\overline{D\setminus D_s}} |\varphi| \geq |\varphi(z)|$, and hence $|\varphi(z)/\varphi(\zeta)| < 1$. Similarly, $|\varphi'(z)/\varphi'(\zeta)| < 1$. Therefore we may write, for $z \in \partial\Gamma_s$:

$$\frac{1}{(\varphi(z) - \varphi(\zeta))(\varphi'(z) - \varphi'(\zeta))}$$
$$= \frac{1}{\varphi(\zeta)\varphi'(\zeta)} \cdot \frac{1}{(1 - \varphi(z)/\varphi(\zeta))(1 - \varphi'(z)/\varphi'(\zeta))}$$
$$= \frac{1}{\varphi(\zeta)\varphi'(\zeta)} \sum_{\alpha,\beta}^{0,\infty} \left(\frac{\varphi(z)}{\varphi(\zeta)}\right)^\alpha \left(\frac{\varphi'(z)}{\varphi'(\zeta)}\right)^\beta,$$

with the double series absolutely uniformly convergent on $\partial\Gamma_s$. It follows that

$$\int_{\partial\Gamma_s} f(z) \frac{(h_1 h_2' - h_2 h_1')\, dz_1 \wedge dz_2}{(\varphi(z) - \varphi(\zeta))(\varphi'(z) - \varphi'(\zeta))}$$
$$= \sum_{\alpha,\beta}^{0,\infty} \frac{1}{(\varphi(\zeta))^{\alpha+1}(\varphi'(\zeta))^{\beta+1}} \int_{\partial\Gamma_s} f\mu_{\alpha,\beta},$$

where

$$\mu_{\alpha,\beta} = (h_1 h_2' - h_2 h_1')(\varphi(z))^\alpha (\varphi'(z))^\beta\, dz_1 \wedge dz_2 \qquad (\alpha, \beta = 0, 1, 2, \ldots).$$

Now, since every $\mu_{\alpha,\beta}$ is a holomorphic 2-form on $U \cap U'$, so that $\bar{\partial}\mu_{\alpha,\beta} = 0$, Proposition 1.9 implies:

$$\int_{\partial\Gamma_s} f\mu_{\alpha,\beta} = 0 \qquad (\alpha, \beta = 0, 1, 2, \ldots).$$

Therefore also for $n = 2$ we have: $F_h^s(\zeta) = F_{h'}^s(\zeta)$.

It remains to show that every F_h^s is holomorphic, i.e. that, for each $\zeta \in U_s(\varphi) \setminus \partial D$,

$$\frac{\partial F_h^s}{\partial \bar{\zeta}_\alpha}(\zeta) = 0 \qquad (\alpha = 1, \ldots, n).$$

Clearly, we have:

$$\frac{\partial F_h^s}{\partial \bar{\zeta}_\alpha}(\zeta) = \int_{\Gamma_s} f \frac{\partial \omega}{\partial \bar{\zeta}_\alpha}(\zeta) - \int_{\partial\Gamma_s} f \frac{\partial \Phi_h}{\partial \bar{\zeta}_\alpha}(\zeta);$$

further, on account of (1.5), (1.6), (1.7), we may rewrite the right side of this equality as:

$$(*) \qquad \int_{\Gamma_s} f\bar{\partial}\left[\frac{\partial\Omega_\alpha}{\partial\bar{\zeta}_\alpha}(\zeta)\right] - \int_{\partial\Gamma_s} f\frac{\partial\Omega_\alpha}{\partial\bar{\zeta}_\alpha}(\zeta) + I,$$

where

$$I = \begin{cases} \int_{\partial\Gamma_s} f\bar{\partial}\Psi_h^\alpha(\zeta) & \text{if } n \geq 3, \\ 0 & \text{if } n = 2. \end{cases}$$

Since $[\partial\Omega_\alpha/\partial\bar{\zeta}_\alpha](\zeta)$ is defined on all of $\mathbf{C}^n \setminus \zeta$, Proposition 1.9 implies that the difference of integrals in $(*)$ is zero. Moreover, by Proposition 1.9 again, I is zero also in case $n \geq 3$, since $\Psi_h^\alpha(\zeta)$ may be replaced by any $(n, n-3)$-form $\tilde{\Psi}^\alpha$ of class C^∞ on all of $\mathbf{C}^n$ and equal to $\Psi_h^\alpha(\zeta)$ on an open neighbourhood of $\partial\Gamma_s$. Hence $[\partial F_h^s/\partial\bar{\zeta}_\alpha](\zeta) = 0$.

The proof of Proposition 2.5 is then completed.

Next, we have:

PROPOSITION 2.6. *Let V be an open neighbourhood of $\partial D \setminus K$, contained in $\bigcup_{U \supset \bar{D}} \bigcup_{\varphi \in \mathcal{O}(U)} U(\varphi)$, such that $V \setminus (\partial D \setminus K) = V_+ \cup V_-$, where V_+, V_- are connected separated open sets and $V_- \subset \mathbf{C}^n \setminus \bar{D}$.*[8] *Then $F = 0$ on V_-.*

Proof. We first point out that, given an open neighbourhood U of $\bar{D}$ and a function $\varphi \in \mathcal{O}(U)$, if ζ is a point in U such that $|\varphi(\zeta)| > \max_{\bar{D}}|\varphi|$ (which obviously implies that $\zeta \in U_1(\varphi) \setminus \bar{D}$), then $F(\zeta) = 0$. As a matter of fact, if $h \in \mathcal{O}_\varphi^n(U \times U)$, we have:

$$F(\zeta) = F_h^1(\zeta) = \int_{\Gamma_1} f\omega(\zeta) - \int_{\partial\Gamma_1} f\Phi_h(\zeta),$$

and, since $\bar{D} \subset U \setminus L_\zeta(\varphi)$, on an open neighbourhood of $\bar{D}$ $\omega(\zeta)$, $\Phi_h(\zeta)$ are both defined and satisfy $\omega(\zeta) = \bar{\partial}\Phi_h(\zeta)$. Hence Proposition 1.9 implies that $F(\zeta) = 0$.

Now, take U and φ such that $U(\varphi) \cap D \neq \varnothing$; then $\max_{\bar{D}}|\varphi| > \max_K|\varphi|$, so that φ is not constant on the connected component of U containing $\bar{D}$ and, further, any point $\zeta^0 \in \partial D$ where $|\varphi|$ attains the value

[8] Such a V does exist, because $\partial D \setminus K$ is connected. For example, we may take as V a small tubular neighbourhood of $\partial D \setminus K$ in $\mathbf{C}^n \setminus K$.

$\max_{\bar{D}}|\varphi|$ must belong to $\partial D \setminus K$. One can actually find such a point ζ^0 by the well known "maximum principle". Then ζ^0 is a limit point of the open set $W = \{\zeta \in U; |\varphi(\zeta)| > \max_{\bar{D}}|\varphi|\}$ (by the maximum principle again), and, since $\zeta^0 \in \partial D \setminus K$, this obviously implies that $W \cap V_- \neq \varnothing$. But we already know that F is zero on $W \cap V_-$; it follows that F is zero on all of V_-, because V_- is connected.

Finally, we are in a position to prove that F is a continuous extension of f to $\bar{D} \setminus K$, i.e., the following holds:

PROPOSITION 2.7. *For every point $z^0 \in \partial D \setminus K$ we have:*

$$\lim_{\zeta \to z^0} F(\zeta) = f(z^0),$$

the limit being evaluated for $\zeta \in D$.

Proof. For every $w \in \partial D \setminus K$, denote by $\vec{\nu}(w)$ the unit vector perpendicular to $\partial D \setminus K$ at w, inward pointing with respect to D. We first prove that

$$(*) \qquad \lim_{t \to 0^+} F(w + t\vec{\nu}(w)) = f(w),$$

with the limit uniform on compact subsets of $\partial D \setminus K$. Given $w \in \partial D \setminus K$, we can find an open neighbourhood U of $\bar{D}$, a function $\varphi \in \mathcal{O}(U)$ and a positive integer s such that $w \in U_s(\varphi) \cap (\Gamma_s \setminus \partial\Gamma_s)$. Then, for $t > 0$ small enough, we have:

$$w + t\vec{\nu}(w) \in U_s(\varphi) \cap D, \qquad w - t\vec{\nu}(w) \in U_s(\varphi) \cap V_-,$$

with V_- as in Proposition 2.6, and hence, if $h \in \mathcal{O}_\varphi^n(U \times U)$, it follows that

$$\begin{aligned} F(w + t\vec{\nu}(w)) &= F_h^s(w + t\vec{\nu}(w)), \\ F(w - t\vec{\nu}(w)) &= F_h^s(w - t\vec{\nu}(w)) = 0. \end{aligned}$$

Therefore we may write:

$$\begin{aligned} F(w + t\vec{\nu}(w)) &= F_h^s(w + t\vec{\nu}(w)) - F_h^s(w - t\vec{\nu}(w)) \\ &= I_1(w,t) - I_2(w,t), \end{aligned}$$

where

$$I_1(w,t) = \int_{\Gamma_s} f[\omega(w + t\vec{\nu}(w)) - \omega(w - t\vec{\nu}(w))],$$

$$I_2(w,t) = \int_{\partial\Gamma_s} f[\Phi_h(w + t\vec{\nu}(w)) - \Phi_h(w - t\vec{\nu}(w))].$$

Now, it can be shown that, for any $f \in C^0(\Gamma_s)$ (not necessarily a CR-function) and $w \in \Gamma_s \setminus \partial\Gamma_s$,

$$\lim_{t\to 0^+} I_1(w,t) = f(w),$$

with the limit uniform on compact subsets of $\Gamma_s \setminus \partial\Gamma_s$. A similar result can be found in Harvey-Lawson [**4**], pp. 251–252, and the proof given there (based on a suitable estimate for $\|\omega(w + t\vec{v}(w)) - \omega(w - t\vec{v}(w))\|$) works essentially for the present case as well.[9] Next, since the function $\zeta \mapsto \int_{\partial\Gamma_s} f\Phi_h(\zeta)$ is defined and real analytic on all of $U_s(\varphi)$, it is plain that, for $w \in U_s(\varphi) \cap (\Gamma_s \setminus \partial\Gamma_s)$,

$$\lim_{t\to 0^+} I_2(w,t) = 0,$$

with the limit uniform on compact subsets of $U_s(\varphi) \cap (\Gamma_s \setminus \partial\Gamma_s)$. Hence $(*)$ follows.

After that, it is easy to prove Proposition 2.7. Given $\varepsilon > 0$, let N_{z^0} be an open neighbourhood of z^0 in $\partial D \setminus K$ such that $|f(w) - f(z^0)| < \varepsilon/2$, for every $w \in N_{z^0}$, and $N_{z^0} \Subset \partial D \setminus K$. Further, let $t_0 > 0$ be such that $|F(w + t\vec{v}(w)) - f(w)| < \varepsilon/2$, for every $t \leq t_0$ and $w \in \overline{N}_{z^0}$. Clearly, if ζ is a point of D close enough to z^0, there exist exactly a point $w \in N_{z^0}$ and a positive number $t \leq t_0$ such that $\zeta = w + t\vec{v}(w)$. It follows that

$$|F(\zeta) - f(z^0)| \leq |F(w + t\vec{v}(w)) - f(w)| + |f(w) - f(z^0)| < \varepsilon,$$

which proves Proposition 2.7.

Now the proof of Theorem 1 is completed.

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[10]Added in proof.

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