Pacific Journal of Mathematics

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Vol. 124, No. 1 May 1986

PACIFIC JOURNAL OF MATHEMATICS
Vol. 124, No. 1, 1986

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K. Murasugi has conjectured that if every pair of the μ components of a classical link L has linking number ± 1, then the group G of L will have the property that $G_q/G_{q+1} \cong F_q/F_{q+1}$ $\forall q \geq 2$, where F is free on $\mu - 1$ generators. (The conjecture has been verified by other authors.) Here we show that this property is equivalent to the special case $G_2/G_3 \cong F_2/F_3$, and also give other equivalent conditions.

1. Introduction. A (tame) *link* in the three-sphere S^3 is the union $L = K_1 \cup \cdots \cup K_\mu$ of a finite number of oriented, pairwise disjoint, polygonal simple closed curves, its *components*. Two such links are *ambient isotopic* iff there is an orientation-preserving homeomorphism of S^3 with itself which maps one onto the other in such a way that the indexing and orientations of the components correspond.

Among the invariants of the ambient isotopy type of a link L, one of the most important is the *group* $G = \pi_1(S^3 - L)$ of L. The *lower central series* $\{G_q\}$ of G is given by $G_1 = G$ and, for $q \geq 1$, $G_{q+1} = [G, G_q]$. In this paper we will be concerned with the various *Chen groups* $G''G_q/G''G_{q+1}$, and also the quotients G_q/G_{q+1} of the lower central series; these abelian groups are invariants of L under equivalence relations much coarser than ambient isotopy [**9**; **14**], and are of interest only when $\mu > 1$.

In the simplest interesting case, $\mu = 2$, K. Murasugi [**11**] has shown that if the linking number $l = l(K_1, K_2)$ is nonzero, then the Chen groups of L are all determined (up to isomorphism) by l: $G''G_q/G''G_{q+1} \cong \mathbf{Z}_l^{q-1}$ $\forall q \geq 2$, where $\mathbf{Z}_l$ is the cyclic group of order $|l|$. In particular, these three statements are equivalent: $l = \pm 1$, $G_2/G_3 = 0$, and $G''G_q/G''G_{q+1} = 0$ $\forall q \geq 2$. (Note that if $G_2/G_3 = 0$ then $G_2 = G_3 = G_q$ $\forall q \geq 2$, and hence $G_q/G_{q+1} = 0$ $\forall q \geq 2$.)

Murasugi conjectured that analogously, a link L of $\mu > 2$ components with all linking numbers $l(K_i, K_j) = \pm 1$ would have $G_q/G_{q+1} \cong F_q/F_{q+1}$ $\forall q \geq 2$, where F is free on $\mu - 1$ generators. This conjecture has been partly verified by S. Kojima [**6**], and T. Maeda [**8**] has shown that the hypothesis that all the linking numbers $l(K_i, K_j)$ be ± 1 can be significantly weakened.

In this paper we *completely* characterize those links such that $G_q/G_{q+1} \cong F_q/F_{q+1}$ for all $q \geq 2$. To explain our characterization of such links, we need to introduce the following notation. Let the $\mu \times \binom{\mu}{2}$ matrix Λ, whose rows are indexed by $\{1, \ldots, \mu\}$ and whose columns are indexed by $\{(p, q) \mid 1 \leq p < q \leq \mu\}$, have the entries

$$\Lambda_{i(p,q)} = \begin{cases} l(K_i, K_q), & \text{if } i = p \\ -l(K_i, K_p), & \text{if } i = q \\ 0, & \text{if } p \neq i \neq q. \end{cases}$$

Let $E_{\binom{\mu-1}{2}}(\Lambda)$ be the $\binom{\mu-1}{2}$th *elementary ideal* of Λ, that is, the ideal of $\mathbf{Z}$ generated by the determinants of the various $(\mu - 1) \times (\mu - 1)$ submatrices of Λ. (Another description of this ideal is given in §3.) Equivalently, we may use elementary row and column operations to transform Λ into a matrix

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a $(\mu - 1) \times (\mu - 1)$ diagonal matrix, and then $E_{\binom{\mu-1}{2}}(\Lambda)$ is the ideal of $\mathbf{Z}$ generated by the product of the diagonal entries of D.

THEOREM 1. *Let L be a tame link of $\mu \geq 2$ components in S^3, G the group of L, and F the free group on $\mu - 1$ generators. Then any two of these statements are equivalent*:

(a) $G_2/G_3 \cong F_2/F_3$;
(b) $E_{\binom{\mu-1}{2}}(\Lambda) = \mathbf{Z}$;
(c) $G''G_q/G''G_{q+1} \cong F''F_q/F''F_{q+1}\ \forall q \geq 2$;
(d) $G_q/G_{q+1} \cong F_q/F_{q+1}\ \forall q \geq 2$; *and*
(e) *the cup-product pairing*

$$H^1(S^3 - L; \mathbf{Z}) \otimes_{\mathbf{Z}} H^1(S^3 - L; \mathbf{Z}) \to H^2(S^3 - L; \mathbf{Z})$$

is epimorphic.

That (a) implies (b) follows from the theorem of K. -T. Chen [3] that Λ is a presentation matrix of the abelian group G_2/G_3, for if $G_2/G_3 \cong F_2/F_3$ is free abelian of rank $\binom{\mu-1}{2}$, then

$$E_{\binom{\mu-1}{2}}(G_2/G_3) = E_{\binom{\mu-1}{2}}(\Lambda) = \mathbf{Z}.$$

(See [12, Chapter 3] for a discussion of the basic properties of the elementary ideals (or "Fitting invariants") of a finitely generated module over a commutative ring with unity.) That (c) is equivalent to (d) has been proven by T. Maeda [8]. (Maeda has mentioned to us that he actually announced the equivalence of (a), (c), and (d) at a conference held at the University of Sussex in 1982, but that subsequently he found a gap in the argument he'd devised to deduce (c) from (a).) Clearly (d) implies (a), so the proof of the equivalence of (a), (b), (c), and (d) is completed by verifying the implication (b) $\Rightarrow$ (c). We do this in §3, where we also present yet another condition equivalent to those of Theorem 1. We discuss (e) in §4.

Let $\nabla(L) \in \mathbf{Z}[t, t^{-1}]$ be the Hosokawa polynomial of L [5]. Using the relationship between $\nabla(L)(1)$ and the linking numbers of the components of L with each other, we also prove

PROPOSITION 1. *If* $\nabla(L)(1) = \pm 1$, *then the equivalent conditions of Theorem* 1 *hold.*

2. Some commutative algebra. In this section we present some algebraic notions that will be useful in our proof of the implication (b) $\Rightarrow$ (c). Much of the material involving I-adic filtrations, their associated graded modules, and I-adic completions appears in [17, Chapter VIII] and [1, Chapter III] (in a more general context), though our presentation differs somewhat in notation and terminology.

It will be convenient for us to use P to denote a ring that may be either the integers $\mathbf{Z}$ or a prime field (i.e., the rationals $\mathbf{Q}$ or a quotient $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, p a prime). For any such P, there is a unique homomorphism $\gamma_P\colon \mathbf{Z} \to P$ with $\gamma_P(1) = 1$. Also, if A and B are abelian groups with subgroups C and D, we use $C \otimes D$ to denote the image of $C \otimes_{\mathbf{Z}} D$ in $A \otimes_{\mathbf{Z}} B$ (that is, the subgroup of $A \otimes_{\mathbf{Z}} B$ generated by the various elements $c \otimes d$ with $c \in C$ and $d \in D$).

If L is a link of μ components in S^3, and G is the group of L, then by Alexander duality the abelianization $H = G/G_2$ is the free abelian group on certain generators $t_1, \ldots, t_\mu$, the *meridians* of L. A group ring PH may be identified, then, with the ring $P[t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}]$ of Laurent polynomials in $t_1, \ldots, t_\mu$ with coefficients from P. The *augmentation map* $\varepsilon\colon PH \to P$ is given by $\varepsilon(h) = 1$ $\forall h \in H$ (or, equivalently, $\varepsilon(f) = f(1, \ldots, 1)$ for a Laurent polynomial $f(t_1, \ldots, t_\mu)$); its kernel is the *augmentation ideal* IH of PH. Note that γ_P induces an isomorphism

$$\gamma_{P*}\colon P \otimes_{\mathbf{Z}} \mathbf{Z}H \to PH$$

under which $P \otimes IH$ is mapped onto IH.

If B is a PH-module, it is filtered by the various submodules $(IH)^s \cdot B$, $s \geq 0$. The *associated graded module* of B is the direct sum

$$\mathrm{gr}(B) = \bigoplus_{s \geq 0} (IH)^s \cdot B/(IH)^{s+1} \cdot B;$$

it is a graded module over the graded ring $\mathrm{gr}(PH)$ in a natural way. We denote by v the *order function* associated to the filtration $\{(IH)^s \cdot B\}$, that is, for $b \in \bigcap_s (IH)^s \cdot B$, $v(b) = \infty$, and for other elements b of B, $b \in (IH)^{v(b)} \cdot B - (IH)^{v(b)+1} \cdot B$. For $b \in B$ the *initial form* $\mathrm{in}(b)$ is $\mathrm{in}(b) = b + (IH)^{v(b)+1} \cdot B \in \mathrm{gr}_{v(b)}(B)$, if $v(b)$ is finite, and $\mathrm{in}(b) = 0$ if $v(b) = \infty$; this defines a function in: $B \to \mathrm{gr}(B)$.

If B and C are PH-modules, and $e\colon B \to C$ is a PH-epimorphism, it defines a $\mathrm{gr}(PH)$-epimorphism $\mathrm{gr}(e)\colon \mathrm{gr}(B) \to \mathrm{gr}(C)$ in a natural way. The kernel of $\mathrm{gr}(e)$ is the *leading submodule* of $\ker e$ in B, which we denote $\mathrm{in}(\ker e)$; it is the $\mathrm{gr}(PH)$-submodule of $\mathrm{gr}(B)$ consisting of the initial forms of the elements of $\ker e$, that is,

$$\mathrm{in}_s(\ker e) = \left((IH)^s \cdot B\right) \cap \ker e + (IH)^{s+1} \cdot B/(IH)^{s+1} \cdot B$$

for $s \geq 0$.

We say an element of PH is *homogeneous of degree s* (with respect to $t_1 - 1, \ldots, t_\mu - 1$) iff it can be expressed as a sum

$$\sum a_{r_1,\ldots,r_\mu} \cdot (t_1 - 1)^{r_1} \cdots (t_\mu - 1)^{r_\mu},$$

the sum taken over the set of all μ-tuples $(r_1, \ldots, r_\mu)$ of non-negative integers with $\Sigma r_i = s$, for suitable $a_{r_1,\ldots,r_\mu} \in P$. (Note that 0 is homogeneous of every degree.) If Z is a free PH-module with basis $\{z_i\}$, we say an element of Z is *homogeneous of degree s* (with respect to this basis) iff it can be expressed as a sum $\Sigma h_i z_i$, where each h_i is homogeneous of degree s, and we denote by $H_s(Z)$ the P-submodule of Z consisting of the homogeneous elements of degree s. Clearly then

PROPOSITION (2.1). *For every $s \geq 0$, the initial form function* in: $Z \to \mathrm{gr}(Z)$ *defines an isomorphism between $H_s(Z)$ and $\mathrm{gr}_s(Z)$.*

COROLLARY (2.2). *If Z is a free PH-module with basis $\{z_1, \ldots, z_n\}$, then for $s \geq 0$, $\mathrm{gr}_s(Z)$ is a free P-module of rank $n \cdot \binom{\mu+s-1}{s}$.*

Proof. It suffices to verify this in case $n = 1$.

Clearly $H_s(PH)$ is the free P-module with basis

$$\left\{(t_{i_1} - 1) \cdots (t_{i_s} - 1) \mid 1 \leq i_1 \leq \cdots \leq i_s \leq \mu\right\},$$

which has $\binom{\mu+s-1}{s}$ elements. □

If a submodule of a free PH-module Z is generated by homogeneous elements of Z, then its leading submodule in Z is simple to describe, as we see in the next two propositions.

LEMMA (2.3). *Let Y and Z be free PH-modules with bases $\{y_i\}$ and $\{z_j\}$, respectively, and let $f\colon Y \to Z$ be a PH-homomorphism with the property that for some fixed $r \geq 0$, $f(y_i) \in H_r(Z)\ \forall i$. Then* $\mathrm{in}_s(f(Y)) = 0$ *for $s < r$, and* $\mathrm{in}_s(f(Y)) = f(H_{s-r}(Y)) + (IH)^{s+1} \cdot Z/(IH)^{s+1} \cdot Z$ *for $s \geq r$.*

Proof. Clearly $\mathrm{in}_s(f(Y)) = 0$ for $s < r$.

Suppose $s \geq r$ and $y = \Sigma r_i y_i \in Y$ has $f(y) \in (IH)^s \cdot Z$. If $r_i \in (IH)^{s-r}\ \forall i$, then $f(y) + (IH)^{s+1} \cdot Z = f(h) + (IH)^{s+1} \cdot Z$, where h is the homogeneous element of Y whose initial form is $\mathrm{in}(y)$.

On the other hand, suppose some r_i is not in $(IH)^{s-r}$. Let $s - r - t = \min\{v(r_i)\}$. We proceed by induction on t, having just verified the case $t = 0$. For each i, let h_i be 0 (if $v(r_i) > s - r - t$), or the homogeneous element of PH whose initial form is $r_i + (IH)^{s-r-t+1}$ (if $v(r_i) = s - r - t$). Then $f(\Sigma h_i y_i)$ is a homogeneous element of Z of degree $s - t$, clearly. In addition, $r_i - h_i \in (IH)^{s-r-t+1}\ \forall i$. Since $f(y) \in (IH)^{s-t+1} \cdot Z$, it follows that $f(\Sigma h_i y_i) \in (IH)^{s-t+1} \cdot Z$, and hence $f(\Sigma h_i y_i) = 0$, by Proposition (2.1). Thus $f(y) = f(\Sigma(r_i - h_i) y_i)$, and since $r_i - h_i \in (IH)^{s-r-t+1}\ \forall i$, we may apply the inductive hypothesis to conclude that $f(y) + (IH)^{s+1} \cdot Z = f(h) + (IH)^{s+1} \cdot Z$ for some $h \in H_{s-r}(Y)$. □

Generalizing Lemma (2.3) inductively, we have

PROPOSITION (2.4). *Let Y and Z be free PH-modules with bases $\{y_i\}$ and $\{z_j\}$, respectively, and let $f\colon Y \to Z$ be a PH-homomorphism with the property that $f(y_i)$ is homogeneous $\forall i$. For $k \geq 0$, let $\{y_{ik}\}$ be the set of those y_i with $v(f(y_i)) = k$, and Y_k the submodule of Y freely generated by $\{y_{ik}\}$ (if there are no y_{ik}, $Y_k = 0$). Then*

$$\mathrm{in}_s(f(Y)) = \sum_{k=0}^{s} f(H_{s-k}(Y_k)) + (IH)^{s+1} \cdot Z/(IH)^{s+1} \cdot Z$$

for every $s \geq 0$.

Proof. Suppose $s \geq 0$, and let J be the right-hand side of the asserted equality. Obviously $\mathrm{in}_s(f(Y)) \supseteq J$, so it suffices to verify the opposite inclusion.

Every element of $\mathrm{in}_s(f(Y))$ is $f(x) + (IH)^{s+1} \cdot Z$ for some $x \in \sum_{k=0}^{s} Y_k$, say $x = \sum_{k=0}^{s} x_k$. If $v(f(x_k)) = s$ $\forall k$, then $f(x) + (IH)^{s+1} \cdot Z = \sum \mathrm{in}(f(x_k))$, and we may apply Lemma (2.3) to find, for each $k \in \{0, \dots, s\}$, an $h_k \in H_{s-k}(Y_k)$ with $\mathrm{in}(f(x_k)) = \mathrm{in}(f(h_k))$. Then $f(x) + (IH)^{s+1} \cdot Z = \sum \mathrm{in}(f(h_k)) \in J$.

On the other hand, suppose $v(f(x_k)) < s$ for some $k \in \{0, \dots, s\}$, and let $s - t = \min\{v(f(x_k))\}$. We proceed by induction on t, having just dealt with $t = 0$. For $k \in \{0, \dots, s\}$ let $h_k = 0$ if $v(f(x_k)) > s - t$, and otherwise let $h_k \in H_{s-t-k}(Y_k)$ have $\mathrm{in}(f(h_k)) = \mathrm{in}(f(x_k))$. (Such an h_k exists by Lemma (2.3).) Since $v(f(x)) = s > s - t$, it must be that $\sum \mathrm{in}(f(h_k)) = 0$, and hence $\sum f(h_k) = 0$ by Proposition (2.1). Thus $f(x) + (IH)^{s+1} \cdot Z = f(\sum_{k=0}^{s}(x_k - h_k)) + (IH)^{s+1} \cdot Z$. Since $v(f(x_k - h_k)) > s - t$ for every k, we may apply the inductive hypothesis to conclude that $f(\sum(x_k - h_k)) + (IH)^{s+1} \cdot Z \in J$. $\square$

We will call a map f that satisfies the hypothesis of Proposition (2.4) *homogeneous*; a PH-module *possesses a homogeneous presentation* (or is *homogeneously presentable*) iff it is isomorphic to the cokernel of some homogeneous map.

The augmentation ideal IH is an example of a homogeneously presentable $\mathbf{Z}H$-module. Recall [7, p. 189] that there is an exact sequence

$$Z_4 \xrightarrow{\zeta_4} Z_3 \xrightarrow{\zeta_3} Z_2 \xrightarrow{\zeta_2} Z_1 \xrightarrow{\zeta_1} IH \to 0$$

of $\mathbf{Z}H$-modules, where Z_4 is free on $\{z_{ijkl} \mid 1 \le i < j < k < l \le \mu\}$, Z_3 is free on $\{z_{ijk} \mid 1 \le i < j < k \le \mu\}$, Z_2 is free on $\{z_{ij} \mid 1 \le i < j \le \mu\}$, Z_1 is free on $\{z_i \mid 1 \le i \le \mu\}$,

$$\zeta_1(z_i) = t_i - 1, \zeta_2(z_{ij}) = (t_i - 1)z_j - (t_j - 1)z_i,$$

$$\zeta_3(z_{ijk}) = (t_i - 1)z_{jk} - (t_j - 1)z_{ik} + (t_k - 1)z_{ij},$$

and

$$\zeta_4(z_{ijkl}) = (t_i - 1)z_{jkl} - (t_j - 1)z_{ikl} + (t_k - 1)z_{ijl} - (t_l - 1)z_{ijk}.$$

PROPOSITION (2.5). *For $s > 0$, $\mathrm{in}_s(\zeta_3(Z_3))$ is a direct summand of $\mathrm{gr}_s(Z_2)$ whose rank is*

$$\binom{s+1}{2} \cdot \binom{\mu + s - 1}{s + 2}.$$

Proof. If $s = 1$,

$$\mathrm{in}_s(\zeta_3(Z_3)) = \zeta_3(H_0(Z_3)) + (IH)^2 \cdot Z_2/(IH)^2 \cdot Z_2$$

is generated by the $\binom{\mu}{3}$ elements $\zeta_3(z_{ijk}) + (IH)^2 \cdot Z_2$. To verify that these elements are linearly independent over $\mathbf{Z}$, and that $\mathrm{in}_s(\zeta_3(Z_3))$ is a direct summand of $\mathrm{gr}_s(Z_2)$, note that $\mathrm{gr}_s(Z_2)$ is a free abelian group with a basis consisting of the various elements $(t_i - 1)z_{jk} + (IH)^2 \cdot Z_2$, and we can obtain a new basis by replacing $(t_i - 1)z_{jk} + (IH)^2 \cdot Z_2$ by $\zeta_3(z_{ijk}) + (IH)^2 \cdot Z_2$ whenever $i < j$.

If $s \geq 2$,

$$\mathrm{in}_s(\zeta_3(Z_3)) = \zeta_3(H_{s-1}(Z_3)) + (IH)^{s+1} \cdot Z_2/(IH)^{s+1} \cdot Z_2$$

is generated by the various elements

$$\zeta_3\big((t_{r_1} - 1) \cdots (t_{r_{s-1}} - 1)z_{ijk}\big) + (IH)^{s+1} \cdot Z_2, \qquad 1 \leq r_1 \leq \cdots \leq r_{s-1} \leq \mu.$$

Since $\zeta_3\zeta_4 = 0$, those

$$\zeta_3\big((t_{r_1} - 1) \cdots (t_{r_{s-1}} - 1)z_{ijk}\big) + (IH)^{s+1} \cdot Z_2$$

with $r_1 \geq i$ suffice to generate $\mathrm{in}_s(\zeta_3(Z_3))$. For a given $i \in \{1, \ldots, \mu - 2\}$, the number of pairs (j, k) with $1 \leq i < j < k \leq \mu$ is $\binom{\mu - i}{2}$, and the number of $(s-1)$-tuples $(r_1, \ldots, r_{s-1})$ with $i \leq r_1 \leq \cdots \leq r_{s-1} \leq \mu$ is $\binom{\mu+s-i-1}{s-1}$. Thus the number of elements of this generating set of $\mathrm{in}_s(\zeta_3(Z_3))$ is

$$\sum_{i=1}^{\mu-2} \binom{\mu - i}{2}\binom{\mu + s - i - 1}{s - 1} = \binom{s+1}{2} \cdot \sum_{i=1}^{\mu-2} \binom{\mu + s - i - 1}{s + 1},$$

and the latter sum is easily seen to be $\binom{\mu+s-1}{s+2}$. To verify that this generating set is linearly independent over $\mathbf{Z}$, and that $\mathrm{in}_s(\zeta_3(Z_3))$ is a direct summand of $\mathrm{gr}_s(Z_2)$, note that $\mathrm{gr}_s(Z_2)$ is a free abelian group with a basis consisting of the various elements

$$(t_i - 1)(t_{r_1} - 1) \cdots (t_{r_{s-1}} - 1)z_{jk} + (IH)^{s+1} \cdot Z_2$$

with $i \leq r_1 \leq \cdots \leq r_{s-1}$, and we can obtain a new basis by replacing

$$(t_i - 1)(t_{r_1} - 1) \cdots (t_{r_{s-1}} - 1)z_{jk} + (IH)^{s+1} \cdot Z_2$$

by

$$\zeta_3\big((t_{r_1} - 1) \cdots (t_{r_{s-1}} - 1)z_{ijk}\big) + (IH)^{s+1} \cdot Z_2$$

whenever $i < j$. □

If B is a $\mathbf{Z}H$-module then for any P, $P \otimes_{\mathbf{Z}} B$ is a PH-module under the action $\gamma_{P*}(x \otimes h) \cdot (y \otimes b) = (xy) \otimes (hb)$. For every $s \geq 0$, $\gamma_{P*}(P \otimes (IH)^s) = (IH)^s$, and so certainly $(IH)^s \cdot (P \otimes_{\mathbf{Z}} B) = P \otimes (IH)^s \cdot B$. A natural homomorphism δ_P: $P \otimes_{\mathbf{Z}} \operatorname{gr}(B) \to \operatorname{gr}(P \otimes_{\mathbf{Z}} B)$ may be defined by

$$\delta_P\left(x \otimes \left(b + (IH)^{s+1} \cdot B\right)\right) = x \otimes b + (IH)^{s+1} \cdot (P \otimes_{\mathbf{Z}} B)$$

for $x \in P$ and $b \in (IH)^s \cdot B$.

PROPOSITION (2.6). *For any* $\mathbf{Z}H$*-module* B, δ_P *is an isomorphism.*

Proof. Suppose $s \geq 0$, and let c: $(IH)^s \cdot B \to \operatorname{gr}_s(B)$ be the canonical map onto the quotient. Then

$$\operatorname{id}_P \otimes c\colon P \otimes_{\mathbf{Z}} (IH)^s \cdot B \to P \otimes_{\mathbf{Z}} \operatorname{gr}_s(B)$$

is an epimorphism whose kernel is $P \otimes (IH)^{s+1} \cdot B$, by the right exactness of tensor products. Let i: $(IH)^s \cdot B \to B$ be the inclusion map, and c_P: $(IH)^s \cdot (P \otimes_{\mathbf{Z}} B) \to \operatorname{gr}_s(P \otimes_{\mathbf{Z}} B)$ the canonical map onto the quotient. Since

$$(\operatorname{id}_P \otimes i)\left(P \otimes_{\mathbf{Z}} (IH)^s \cdot B\right) = (IH)^s \cdot (P \otimes_{\mathbf{Z}} B),$$

$$c_P(\operatorname{id}_P \otimes i)\colon P \otimes_{\mathbf{Z}} (IH)^s \cdot B \to \operatorname{gr}_s(P \otimes_{\mathbf{Z}} B)$$

is an epimorphism; since

$$(\operatorname{id}_P \otimes i)\left(P \otimes (IH)^{s+1} \cdot B\right) = (IH)^{s+1} \cdot (P \otimes_{\mathbf{Z}} B) = \ker c_P,$$

this epimorphism induces an isomorphism

$$\delta_P\colon P \otimes_{\mathbf{Z}} \operatorname{gr}_s(B) \to \operatorname{gr}_s(P \otimes_{\mathbf{Z}} B). \qquad \square$$

If B is a PH-module, the order function v defines a metric on $B/\bigcap_s (IH)^s \cdot B$ by

$$d(\bar{a}, \bar{b}) = \exp(-v(a - b)),$$

where $\bar{a}$ and $\bar{b}$ are the elements of $B/\bigcap_s (IH)^s \cdot B$ determined by a and b. The completion $\hat{B}$ of this metric space is the *IH-adic completion* of B; it is a $\widehat{PH}$-module in a natural way. We denote the natural mapping $B \to \hat{B}$ by h; in case h is injective (i.e., if $\bigcap_s (IH)^s \cdot B = 0$) we may suppress it, and regard B as a subset of $\hat{B}$. If B is finitely generated, there is an isomorphism $\widehat{PH} \otimes_{PH} B \to \hat{B}$ under which the image of $x \otimes b$ is $x \cdot h(b)$ [**1**, III, §3.4]; consequently, if B is the free PH-module on a finite set of generators, $\hat{B}$ is the free $\widehat{PH}$-module on the same set.

Just as PH may be identified with the ring $P[t_1,\dots,t_\mu,t_1^{-1},\dots,t_\mu^{-1}]$ of Laurent polynomials, its IH-adic completion may also be identified with a familiar ring: the imbedding of PH in the formal power series ring $P[[u_1,\dots,u_\mu]]$ given by $t_i \mapsto 1-u_i$ extends to an isomorphism between $\widehat{PH}$ and $P[[u_1,\dots,u_\mu]]$ (see [**9**, §3] for the case $P=\mathbf{Z}$; the general argument is similar).

Let $\widehat{IH}$ be the ideal of $\widehat{PH}$ generated by IH. Then every $\widehat{PH}$-module possesses an $\widehat{IH}$-adic filtration defined in the obvious way, and this filtration produces an associated graded module as before. If B is a finitely generated PH-module the natural mapping $h\colon B \to \hat{B}$ defines an isomorphism $\operatorname{gr}(h)\colon \operatorname{gr}(B) \to \operatorname{gr}(\hat{B})$ of graded modules over the graded ring $\operatorname{gr}(PH) \cong \operatorname{gr}(\widehat{PH})$.

We close this section with two properties of elementary ideals. Let R be a commutative ring with unity, X and Y free R-modules of finite ranks m and n, respectively, with $m \le n$. Let $f\colon X \to Y$ be an R-homomorphism, M an $m \times n$ matrix that represents f with respect to some bases of X and Y, and $E_{n-m}(M)$ the ideal of R generated by the determinants of the various $m \times m$ submatrices of M.

LEMMA (2.7). *If $E_{n-m}(M)$ is not annihilated by any nonzero element of R, f is a monomorphism. Furthermore, f is a split monomorphism iff $E_{n-m}(M)=R$.*

Proof. Let $M=(m_{ij})$ be the matrix of f with respect to the bases $\{x_1,\dots,x_m\}$ of X and $\{y_1,\dots,y_n\}$ of Y.

Suppose, first, that $\operatorname{ann} E_{n-m}(M)=0$, and $x=\Sigma a_i x_i \in \ker f$. Then for every $j \in \{1,\dots,n\}$, $\Sigma a_i m_{ij}=0$; that is, the a_i are the coefficients in a relation between the rows of M. It follows that $a_i \det N=0$ for every i and every $m \times m$ submatrix N of M, so $a_1,\dots,a_m \in \operatorname{ann} E_{n-m}(M)$; thus $x=0$.

Suppose $E_{n-m}(M)=R$. Then f is a monomorphism, and furthermore, $\operatorname{coker} f$ is a projective R-module [**12**, p. 122], so f is split. Conversely, suppose f is a split monomorphism, and let $Y=f(X)\oplus C$. Then X and $f(X)$ are isomorphic R-modules, so they have the same elementary ideals, or "Fitting invariants". (See [**12**, Chapter 3] for an account of the basic properties of these invariants.) It follows that $R=E_n(Y)=E_{n-m}(C)$, and since M is a presentation matrix for C, $E_{n-m}(C)=E_{n-m}(M)$. □

3. The completion of the Alexander invariant. The *Alexander invariant* B_L of a link L is the abelianization $G'/G''=B_L$ of the commutator subgroup of its group, considered as an H-module with the action given by $(gG')\cdot(cG'') = gcg^{-1}G''$ for $g \in G$ and $c \in G'$. This module has the property that

$$G''G_q/G''G_{q+1} = (IH)^{q-2}\cdot B_L/(IH)^{q-1}\cdot B_L = \mathrm{gr}_{q-2}(B_L),$$

and hence $G''G_q/G''G_{q+1} \cong \mathrm{gr}_{q-2}(\hat{B}_L)$, for $q \geq 2$ [**9**].

Let $Y = Y_0$ be the free $\mathbf{Z}H$-module on the set $\{x_1, \ldots, x_\mu\}$, and let $\lambda_0\colon Y_0 \to Z_2$ be the $\mathbf{Z}H$-homomorphism whose matrix (with respect to the bases $\{x_i\}$ and $\{z_{ij}\}$) is the matrix Λ defined in the introduction. For $1 \leq r \leq \mu$, let Y_r be the submodule of Y_0 generated by those x_i other than x_r, and λ_r the restriction of λ_0 to Y_r. For each r, let $\hat{\lambda}_r\colon \hat{Y}_r \to \hat{Z}_2$ be the unique continuous extension of λ_r.

It was proven in [**16**] that there is a $\widehat{\mathbf{Z}H}$-homomorphism $\nu_0\colon \hat{Y}_0 \to \hat{Z}_2$ such that for every $r \in \{0, \ldots, \mu\}$ there is an exact sequence

$$\hat{Y}_r \oplus \hat{Z}_3 \xrightarrow{\langle \nu_r, \hat{\zeta}_3\rangle} \hat{Z}_2 \xrightarrow{\tau} \hat{B}_L \to 0 \tag{3.1}$$

where ν_r is the restriction of ν_0 to $\hat{Y}_r$. It will not be necessary for us to recall the definition of this map ν_0, but two properties will be useful: $\nu_0(x_i) \equiv \hat{\lambda}_0(x_i)$ (modulo $\widehat{IH}\cdot \hat{Z}_2$) $\forall i$; and if L happens to be the unlink of μ components, then $\nu_0 = 0$.

The second property leads directly to a well-known formula.

PROPOSITION (3.2). *Let Φ be the free group on μ generators. Then for $q \geq 2$, $\Phi''\Phi_q/\Phi''\Phi_{q+1}$ is a free abelian group of rank $(q-1)\cdot\binom{\mu+q-2}{q}$.*

Proof. Let L_0 be the unlink of μ components; its group is Φ. In the exact sequence (3.1), $\nu_0 = 0$, so τ is a $\widehat{\mathbf{Z}H}$-epimorphism whose kernel is $\hat{\zeta}_3(\hat{Z}_3)$. For every $s \geq 0$, then, $\mathrm{gr}_s(\tau)$ is an epimorphism of $\mathrm{gr}_s(\hat{Z}_2) = \mathrm{gr}_s(Z_2)$ onto $\mathrm{gr}_s(\hat{B}_{L_0}) \cong \Phi''\Phi_{s+2}/\Phi''\Phi_{s+3}$ whose kernel is $\mathrm{in}_s(\hat{\zeta}_3(\hat{Z}_3)) = \mathrm{in}_s(\zeta_3(Z_3))$. By Corollary (2.2) and Proposition (2.5), it follows that $\Phi''\Phi_{s+2}/\Phi''\Phi_{s+3}$ is free abelian of rank

$$\begin{aligned}
&\binom{\mu}{2}\cdot\binom{\mu+s-1}{s} - \binom{s+1}{2}\cdot\binom{\mu+s-1}{s+2} \\
&= \binom{\mu+s-1}{s+2}\cdot\left(\frac{(s+1)(s+2)}{(\mu-2)(\mu-1)}\binom{\mu}{2} - \binom{s+1}{2}\right) \\
&= \binom{\mu+s-1}{s+2}\cdot\left(\frac{s+1}{\mu-2}\right)\cdot\left(\frac{(s+2)\mu}{2} - \frac{s(\mu-2)}{2}\right) \\
&= \left(\frac{s+1}{\mu-2}\right)\cdot\binom{\mu+s-1}{s+2}\cdot(\mu+s) \\
&= (s+1)\cdot\binom{\mu+s}{s+2}.
\end{aligned}$$

□

We conclude that if F is the free group on $\mu - 1$ generators, then for $q \geq 2$, $F''F_q/F''F_{q+1}$ is a free abelian group of rank $(q-1) \cdot \binom{\mu+q-3}{q}$.

For $r \in \{0, \ldots, \mu\}$, let $\langle \lambda_r, \zeta_3 \rangle$: $Y_r \oplus Z_3 \to Z_2$ be the obvious map, and let $B_\Lambda = \operatorname{coker}\langle \lambda_r, \zeta_3 \rangle$. (Note that since $\Sigma \lambda_0(x_i) = 0$, the image of $\langle \lambda_r, \zeta_3 \rangle$ is independent of r; we will generally take $r = \mu$ in this situation, if only for definiteness' sake.) Our strategy in verifying the implication (b) $\Rightarrow$ (c) is to describe $\operatorname{gr}(B_\Lambda)$ when (b) holds, and then to show that (b) implies that $\operatorname{gr}(B_\Lambda)$ and $\operatorname{gr}(B_L)$ are isomorphic.

It is convenient to give a more explicit description of the ideal $E_{\binom{\mu-1}{2}}(\Lambda)$.

Proposition (3.3). *The ideal* $E_{\binom{\mu-1}{2}}(\Lambda)$ *of* $\mathbf{Z}$ *is generated by the collection of all products*

$$\prod_{k=1}^{\mu-1} l(K_{i_k}, K_{j_k})$$

such that $j_1, \ldots, j_{\mu-1}$ *are pairwise distinct and* $i_k \notin \{j_k, \ldots, j_{\mu-1}\}$ $\forall k$.

Proof. Let E be the ideal generated by these products.

Suppose M is a $(\mu-1) \times (\mu-1)$ submatrix of Λ with $\det M \neq 0$. Note that each column of Λ either has no nonzero entries, or else has two nonzero entries which are negatives of each other. M must have some column with precisely one nonzero entry; that is, there must be $i_1 \neq j_1 \in \{1, \ldots, \mu\}$ such that M involves the (i_1, j_1) or (j_1, i_1) column of Λ, and the j_1th row, but not the i_1th row. The submatrix of M obtained by deleting this row and column must also have nonzero determinant, so it too must have a column with precisely one nonzero entry; that is, there must be $i_2 \neq j_2 \in \{1, \ldots, \mu\}$ such that this submatrix involves the (i_2, j_2) or (j_2, i_2) column of Λ, and the j_2th row, but not the i_2th row. Note that j_2 cannot be i_1 or j_1, since this submatrix involves neither the i_1th nor the j_1th row of Λ. Continuing in this vein, we conclude that there are $i_1, \ldots, i_{\mu-1}, j_1, \ldots, j_{\mu-1} \in \{1, \ldots, \mu\}$ such that $j_1, \ldots, j_{\mu-1}$ are pairwise distinct, $i_k \notin \{j_k, \ldots, j_{\mu-1}\}$ $\forall k$, M involves the j_kth row of Λ $\forall k$, and M involves the (i_k, j_k) or (j_k, i_k) column of Λ $\forall k$. It follows that

$$\det M = \pm \prod_{k=1}^{\mu-1} l(K_{i_k}, K_{j_k}) \in E.$$

Thus $E_{\binom{\mu-1}{2}}(\Lambda) \subseteq E$.

To verify that $E_{\binom{\mu-1}{2}}(\Lambda) \supseteq E$, note that if $i_1, \ldots, i_{\mu-1}, j_1, \ldots, j_{\mu-1}$ are as described then the $(\mu-1) \times (\mu-1)$ submatrix of Λ which involves the j_kth row and (i_k, j_k) or (j_k, i_k) column $\forall k$ has determinant equal to

$$\pm \prod_{k=1}^{\mu-1} l(K_{i_k}, K_{j_k}). \qquad \square$$

Let Γ be the graph with vertices $\gamma_1, \ldots, \gamma_\mu$, which has an edge $\{\gamma_i, \gamma_j\}$ whenever $i \neq j$ and $l(K_i, K_j) \neq 0$. Then we may rephrase Proposition (3.3) in the following way: $E_{\binom{\mu-1}{2}}(\Lambda) \neq 0$ iff Γ is connected, and if so then $E_{\binom{\mu-1}{2}}(\Lambda)$ is the ideal of $\mathbf{Z}$ generated by the set of all products

$$\prod_{k=1}^{\mu-1} l(K_{i_k}, K_{j_k})$$

such that $\{\gamma_{i_1}, \gamma_{j_1}\}, \ldots, \{\gamma_{i_{\mu-1}}, \gamma_{j_{\mu-1}}\}$ are the edges of a spanning tree in Γ.

COROLLARY (3.4). *If* $E_{\binom{\mu-1}{2}}(\Lambda) = \mathbf{Z}$, *then* $\mathrm{id} \otimes (\zeta_2\lambda_\mu)\colon P \otimes_{\mathbf{Z}} Y_\mu \to P \otimes_{\mathbf{Z}} Z_1$ *is injective for every* P.

Proof. Note that $\Sigma\lambda_0(x_i) = 0$, and so if $1 \leq r \leq \mu$ then

$$\left(\mathrm{id} \otimes (\zeta_2\lambda_\mu)\right)\left(\sum_{i<\mu} w_i \otimes a_i x_i\right) = 0$$

iff

$$\left(\mathrm{id} \otimes (\zeta_2\lambda_r)\right)\left(\sum_{r\neq i<\mu} w_i \otimes a_i x_i - \sum_{i\neq r} w_r \otimes a_r x_i\right) = 0.$$

It suffices, then, to show that $\mathrm{id} \otimes (\zeta_2\lambda_r)$ is injective for some r. Also, note that for each i,

$$\zeta_2\lambda_0(x_i) = \sum_{j\neq i} l(K_i, K_j) \cdot \left((t_i - 1)z_j - (t_j - 1)z_i\right).$$

By the proposition, there must be $i_1, \ldots, i_{\mu-1}, j_1, \ldots, j_{\mu-1}$ with $j_1, \ldots, j_{\mu-1}$ pairwise distinct, $i_k \notin \{j_k, \ldots, j_{\mu-1}\}$ $\forall k$, and $l(K_{i_k}, K_{j_k}) \not\equiv 0$ (modulo $\operatorname{char} P$) $\forall k$. We claim that $\mathrm{id} \otimes (\zeta_2\lambda_{i_1})$ is injective; by Lemma (2.7), it suffices to show that if M is the matrix representing $\zeta_2\lambda_0$ with respect to the bases $\{x_i\}$ and $\{z_i\}$, then the submatrix N obtained from M by deleting its i_1th row and column has $\gamma_{P*}(1 \otimes \det N) \neq 0$; or equivalently, that if $p = \operatorname{char} P$ then $\det N \not\equiv 0$ (modulo p).

For $2 \leq m \leq \mu$, let N_m be the submatrix of M obtained by deleting its i_1th row and column, and its j_kth row and column for $k \geq m$. (In particular, $N_\mu = N$.) Note that $\det N_2 \equiv l(K_{i_1}, K_{j_1}) \cdot (1 - t_{i_1}) \not\equiv 0$ (modulo $(p, t_{j_1} - 1, \ldots, t_{j_{\mu-1}} - 1)$). Proceeding inductively, suppose $m \geq 2$ and $\det N_m \not\equiv 0$ (modulo $(p, t_{j_{m-1}} - 1, \ldots, t_{j_{\mu-1}} - 1)$). Note that

$$\det N_{m+1} \equiv \left(\sum_{j \neq j_m} l(K_j, K_{j_m})(1 - t_j) \right) \cdot \det N_m \left(\text{modulo}\left(t_{j_m} - 1\right)\right),$$

and since $i_m \notin \{ j_m, \ldots, j_{\mu-1} \}$, the former factor is not congruent to zero modulo $(p, t_{j_m} - 1, \ldots, t_{j_{\mu-1}} - 1)$, so

$$\det N_{m+1} \not\equiv 0 \left(\text{modulo}\left(p, t_{j_m} - 1, \ldots, t_{j_{\mu-1}} - 1\right)\right).$$

We conclude that $N = N_\mu$ has $\det N \not\equiv 0$ (modulo $(p, t_{j_{\mu-1}} - 1)$), so certainly $\det N \not\equiv 0$ (modulo p).

Alternatively, an inductive argument using Torres' second relation [**15**] can be used to show that the Alexander polynomial Δ of L has $v(\gamma_{P*}(1 \otimes \Delta)) = \mu - 2$. Since

$$\Delta \cdot IH + (IH)^\mu = E_1(M) + (IH)^\mu,$$

and this equation remains true if the i_1th row of M is deleted [**16**, Corollary (4.3), with $q = 2$], the matrix representing $\zeta_2 \lambda_{i_1}$ must have some $(\mu - 1) \times (\mu - 1)$ submatrix N' with $v(\gamma_{P*}(1 \otimes \det N')) = \mu - 1 < \infty$, and so by Lemma (2.7), $\mathrm{id} \otimes (\zeta_2 \lambda_{i_1})$ must be injective. □

It is worth noting that for a particular P, the proof of Corollary (3.4) only requires the hypothesis $E_{\binom{\mu-1}{2}}(\Lambda) \not\equiv 0$ (modulo $\operatorname{char} P$). The same is true of

COROLLARY (3.5). *If* $E_{\binom{\mu-1}{2}}(\Lambda) = \mathbf{Z}$, *then*

$$\begin{aligned} &\mathrm{in}_s((\mathrm{id} \otimes \zeta_3)(P \otimes_{\mathbf{Z}} Z_3)) \cap \mathrm{in}_s\left((\mathrm{id} \otimes \lambda_\mu)(P \otimes_{\mathbf{Z}} Y_\mu)\right) \\ &\qquad = 0 \subseteq \mathrm{gr}_s(P \otimes_{\mathbf{Z}} Z_2) \end{aligned}$$

for every $s \geq 0$.

Proof. If $s = 0$, $\mathrm{in}_s((\mathrm{id} \otimes \zeta_3)(P \otimes_{\mathbf{Z}} Z_3)) = 0$.
If $s \geq 1$, recall that by Lemma (2.3)

$$\begin{aligned} &\mathrm{in}_s\left((\mathrm{id} \otimes \zeta_3)(P \otimes_{\mathbf{Z}} Z_3)\right) \\ &\quad = (\mathrm{id} \otimes \zeta_3)\left(H_{s-1}(P \otimes_{\mathbf{Z}} Z_3)\right) \\ &\qquad + (IH)^{s+1} \cdot (P \otimes_{\mathbf{Z}} Z_2)/(IH)^{s+1} \cdot (P \otimes_{\mathbf{Z}} Z_2), \end{aligned}$$

and

$$\begin{aligned}\operatorname{in}_s\big((\mathrm{id}\otimes\lambda_\mu)(P\otimes_{\mathbf{Z}} Y_\mu)\big)&\\ =\big(\mathrm{id}\otimes\lambda_\mu\big)\big(H_s(P\otimes_{\mathbf{Z}} Y_\mu)\big)&\\ +(IH)^{s+1}\cdot(P\otimes_{\mathbf{Z}} Z_2)/(IH)^{s+1}\cdot(P\otimes_{\mathbf{Z}} Z_2).&\end{aligned}$$

Suppose $z\in H_{s-1}(P\otimes_{\mathbf{Z}} Z_3)$ and $y\in H_s(P\otimes_{\mathbf{Z}} Y_\mu)$ have

$$\operatorname{in}\big((\mathrm{id}\otimes\zeta_3)(z)\big)=\operatorname{in}\big((\mathrm{id}\otimes\lambda_\mu)(y)\big).$$

Since $(\mathrm{id}\otimes\zeta_3)(z)$ and $(\mathrm{id}\otimes\lambda_\mu)(y)$ are homogeneous elements of $P\otimes_{\mathbf{Z}} Z_2$, it follows from Proposition (2.1) that $(\mathrm{id}\otimes\zeta_3)(z)=(\mathrm{id}\otimes\lambda_\mu)(y)$, and hence that $(\mathrm{id}\otimes(\zeta_2\lambda_\mu))(y)=0$. By Corollary (3.4), then, $y=0$. $\square$

Recall that B_Λ is the cokernel of the map $\langle\lambda_\mu,\zeta_3\rangle\colon Y_\mu\oplus Z_3\to Z_2$.

PROPOSITION (3.6). *If $E_{\binom{\mu}{2}-1}(\Lambda)=\mathbf{Z}$, then for every P and every $s\geq 0$, $P\otimes_{\mathbf{Z}}\operatorname{gr}_s(B_\Lambda)$ is a free P-module of rank $(s+1)\cdot\binom{\mu+s-1}{s+2}$.*

Proof. Suppose $s\geq 0$. Note that $\operatorname{gr}_s(B_\Lambda)$ is a finitely generated abelian group (it is isomorphic to $\operatorname{gr}_s(Z_2)/\operatorname{in}_s(\langle\lambda_\mu,\zeta_3\rangle(Y_\mu\oplus Z_3))$), and so it suffices to verify the assertion for $P\neq\mathbf{Z}$.

Suppose, then, that $P\neq\mathbf{Z}$. By Proposition (2.6), $P\otimes_{\mathbf{Z}}\operatorname{gr}_s(B_\Lambda)\cong\operatorname{gr}_s(P\otimes_{\mathbf{Z}} B_\Lambda)$. By the right exactness of tensor products, $P\otimes_{\mathbf{Z}}B_\Lambda$ is isomorphic to the cokernel of $\mathrm{id}\otimes\langle\lambda_\mu,\zeta_3\rangle\colon P\otimes_{\mathbf{Z}}(Y_\mu\oplus Z_3)\to P\otimes_{\mathbf{Z}} Z_2$, so $P\otimes_{\mathbf{Z}}\operatorname{gr}_s(B_\Lambda)$ is isomorphic to

$$\operatorname{gr}_s(P\otimes_{\mathbf{Z}} Z_2)/\operatorname{in}_s\big((\mathrm{id}\otimes\langle\lambda_\mu,\zeta_3\rangle)\big(P\otimes_{\mathbf{Z}}(Y_\mu\oplus Z_3)\big)\big).$$

By Proposition (2.4),

$$\begin{aligned}&\operatorname{in}_s\big((\mathrm{id}\otimes\langle\lambda_\mu,\zeta_3\rangle)\big)\big(P\otimes_{\mathbf{Z}}(Y_\mu\oplus Z_3)\big)\\ &\quad=\big(\mathrm{id}\otimes\lambda_\mu\big)\big(H_s(P\otimes_{\mathbf{Z}}Y_\mu)\big)+(\mathrm{id}\otimes\zeta_3)\big(H_{s-1}(P\otimes_{\mathbf{Z}} Z_3)\big)\\ &\qquad+(IH)^{s+1}\cdot(P\otimes_{\mathbf{Z}} Z_2)/(IH)^{s+1}\cdot(P\otimes_{\mathbf{Z}} Z_2)\\ &\quad=\operatorname{in}_s\big((\mathrm{id}\otimes\lambda_\mu)(P\otimes_{\mathbf{Z}} Y_\mu)\big)+\operatorname{in}_s\big((\mathrm{id}\otimes\zeta_3)(P\otimes_{\mathbf{Z}} Z_3)\big),\end{aligned}$$

and by Corollary (3.5), the latter sum is direct. Thus $P\otimes_{\mathbf{Z}}\operatorname{gr}_s(B_\Lambda)$ is a vector space of dimension $d-d_1-d_2$ over P, where d, d_1, and d_2 are the dimensions of $\operatorname{gr}_s(P\otimes_{\mathbf{Z}} Z_2)$, $\operatorname{in}_s((\mathrm{id}\otimes\lambda_\mu)(P\otimes_{\mathbf{Z}} Y_\mu))$, and $\operatorname{in}_s((\mathrm{id}\otimes\zeta_3)(P\otimes_{\mathbf{Z}} Z_3))$.

By Corollary (2.2), $d=\binom{\mu}{2}\cdot\binom{\mu+s-1}{s}$.

Since the sum of the rows of Λ is 0, its elementary ideals are unchanged by the removal of any one of its rows. The matrix representing λ_μ (that is, the matrix obtained from Λ by removing its last row) must then be a matrix of integers with $E_{\binom{\mu-1}{2}} = \mathbf{Z}$, so considering it as a matrix with entries from $\mathbf{Z}H$, it has $E_{\binom{\mu-1}{2}} = \mathbf{Z}H$. By Lemma (2.7), then, λ_μ is a split monomorphism of $\mathbf{Z}H$-modules, and hence $\mathrm{id} \otimes \lambda_\mu$ is a split monomorphism of PH-modules. It follows that $\mathrm{gr}(\mathrm{id} \otimes \lambda_\mu)\colon \mathrm{gr}(P \otimes_{\mathbf{Z}} Y_\mu) \to \mathrm{gr}(P \otimes_{\mathbf{Z}} Z_2)$ is a split monomorphism of graded $\mathrm{gr}(PH)$-modules, so $\mathrm{gr}_s(\mathrm{id} \otimes \lambda_\mu)(\mathrm{gr}_s(P \otimes_{\mathbf{Z}} Y_\mu)) \cong \mathrm{gr}_s(P \otimes_{\mathbf{Z}} Y_\mu)$ is a vector space of dimension $(\mu - 1)\binom{\mu+s-1}{s}$ over P. By Lemma (2.3),

$$\begin{aligned}
\mathrm{in}_s\big((\mathrm{id} \otimes \lambda_\mu)(P \otimes_{\mathbf{Z}} Y_\mu)\big)
&= (\mathrm{id} \otimes \lambda_\mu)\big(H_s(P \otimes_{\mathbf{Z}} Y_\mu)\big) \\
&\quad + (IH)^{s+1} \cdot (P \otimes_{\mathbf{Z}} Z_2)/(IH)^{s+1} \cdot (P \otimes_{\mathbf{Z}} Z_2) \\
&= \mathrm{gr}_s(\mathrm{id} \otimes \lambda_\mu)\big(\mathrm{gr}_s(P \otimes_{\mathbf{Z}} Y_\mu)\big),
\end{aligned}$$

so $d_1 = (\mu - 1) \cdot \binom{\mu+s-1}{s}$.

By Proposition (2.5), $\mathrm{gr}_s(\mathrm{coker}\,\zeta_3) \cong \mathrm{gr}_s(Z_2)/\mathrm{in}_s(\zeta_3(Z_3))$ is a free abelian group of rank $d - \binom{s+1}{2} \cdot \binom{\mu+s-1}{s+2}$. By Proposition (2.6),

$$\mathrm{gr}_s(\mathrm{coker}(\mathrm{id} \otimes \zeta_3)) \cong \mathrm{gr}_s(P \otimes_{\mathbf{Z}} Z_2)/\mathrm{in}_s((\mathrm{id} \otimes \zeta_3)(P \otimes_{\mathbf{Z}} Z_3))$$

must then be a vector space over P of dimension $d - \binom{s+1}{2} \cdot \binom{\mu+s-1}{s+2}$; necessarily then $d_2 = \binom{s+1}{2} \cdot \binom{\mu+s-1}{s+2}$.

Thus $P \otimes_{\mathbf{Z}} \mathrm{gr}_s(B_\Lambda)$ is a vector space over P of dimension

$$\begin{aligned}
d - d_1 - d_2
&= \binom{\mu}{2} \cdot \binom{\mu+s-1}{s} - (\mu - 1) \cdot \binom{\mu+s-1}{s} \\
&\quad - \binom{s+1}{2} \cdot \binom{\mu+s-1}{s+2} \\
&= \binom{\mu+s-1}{s} \cdot \left[\binom{\mu-1}{2} - \frac{(\mu-2)(\mu-1)}{(s+1)(s+2)} \cdot \binom{s+1}{2}\right] \\
&= \binom{\mu+s-1}{s} \cdot \binom{\mu-1}{2} \cdot \left(1 - \frac{s}{s+2}\right) \\
&= \binom{\mu+s-1}{s} \cdot \frac{(\mu-1)(\mu-2)}{s+2} = \binom{\mu+s-1}{s+2} \cdot (s+1). \qquad \square
\end{aligned}$$

Recalling Proposition (3.2), we conclude that if F is a free group on $\mu - 1$ generators then condition (b) of Theorem 1 implies that $\mathrm{gr}_s(B_\Lambda) \cong F''F_{s+2}/F''F_{s+3}$ $\forall s \geq 0$. Since $G''G_q/G''G_{q+1} = \mathrm{gr}_{q-2}(B_L) \cong \mathrm{gr}_{q-2}(\hat{B}_L)$

for $q \geq 2$ (as we noted at the beginning of this section), to complete the proof of (b) $\Rightarrow$ (c) it suffices to show that when (b) holds, $\operatorname{gr}(B_\Lambda) \cong \operatorname{gr}(\hat{B}_L)$. This isomorphism actually exists under a slightly weaker hypothesis, as we shall see in Proposition (3.8).

LEMMA (3.7). *If* $E_{\binom{\mu-1}{2}}(\Lambda) \neq 0$, *then*

$$\operatorname{in}\big(\nu_\mu(\hat{Y}_\mu)\big) = \operatorname{in}\big(\lambda_\mu(Y_\mu)\big) \subseteq \operatorname{gr}(Z_2) = \operatorname{gr}(\hat{Z}_2).$$

Proof. Note that we are identifying $\operatorname{gr}(Z_2)$ and $\operatorname{gr}(\hat{Z}_2)$ via the isomorphism induced by the inclusion $Z_2 \subseteq \hat{Z}_2$.

By Lemma (2.3),

$$\operatorname{in}_s\big(\lambda_\mu(Y_\mu)\big) = \lambda_\mu\big(H_s(Y_\mu)\big) + (IH)^{s+1} \cdot Z_2/(IH)^{s+1} \cdot Z_2$$

for every $s \geq 0$. Since $\nu_\mu(y) \equiv \hat{\lambda}_\mu(y)$ (modulo $\widehat{IH} \cdot \hat{Z}_2$) $\forall y \in \hat{Y}_\mu$, certainly $\nu_\mu(h) \equiv \lambda_\mu(h)$ (modulo $(\widehat{IH})^{s+1} \cdot \hat{Z}_2$) for all $h \in H_s(Y_\mu)$ for every $s \geq 0$, and so $\operatorname{in}_s(\lambda_\mu(Y_\mu)) \subseteq \operatorname{in}_s(\nu_\mu(\hat{Y}_\mu))$ for every $s \geq 0$.

On the other hand, suppose $y = \Sigma a_i x_i \in \hat{Y}_\mu$ and $\nu_\mu(y) \in (\widehat{IH})^s \cdot \hat{Z}_2$; let $s - t = \min\{v(a_i)\}$. If $t = 0$, then $\operatorname{in}(\nu_\mu(y)) = \operatorname{in}(\hat{\lambda}_\mu(y)) \in \operatorname{in}_s(\lambda_\mu(Y_\mu))$. Suppose instead that $t > 0$. For each $i < \mu$, let $h_i \in H_{s-t}(\mathbf{Z}H)$ be the homogeneous element with $\operatorname{in}(h_i) = \operatorname{in}(a_i)$ (if $v(a_i) = s - t$) or $h_i = 0$ (if $v(a_i) > s - t$). Then $a_i - h_i \in (\widehat{IH})^{s-t+1}$ $\forall i$, so $\nu_\mu(\Sigma(h_i - a_i)x_i) \in (\widehat{IH})^{s-t+1} \cdot \hat{Z}_2$; since $\nu_\mu(y) \in (\widehat{IH})^{s-t+1} \cdot \hat{Z}_2$, then, $\nu_\mu(\Sigma h_i x_i) \in (\widehat{IH})^{s-t+1} \cdot \hat{Z}_2$. Since $\nu_\mu(x_i) \equiv \lambda_\mu(x_i)$ (modulo $\widehat{IH} \cdot \hat{Z}_2$) $\forall i$, it follows that $\lambda_\mu(\Sigma h_i x_i) \in (\widehat{IH})^{s-t+1} \cdot \hat{Z}_2$. But $\lambda_\mu(\Sigma h_i x_i)$ is homogeneous of degree $s - t$, so $\lambda_\mu(\Sigma h_i x_i) = 0$, by Proposition (2.1). By Corollary (3.4), applied with $P = \mathbf{Z}$ (see the comment preceding Corollary (3.5)), λ_μ is injective; necessarily then $h_i = 0$ $\forall i$. This is impossible, though, for by the definition of t, $v(a_i) = v(h_i) = s - t < \infty$ for some i. Thus it must be that $t = 0$. □

PROPOSITION (3.8). *If* $E_{\binom{\mu-1}{2}}(\Lambda) \neq 0$, *then*

$$\begin{aligned}\operatorname{in}\big(\langle \nu_\mu, \hat{\zeta}_3\rangle(\hat{Y}_\mu \oplus \hat{Z}_3)\big) &= \operatorname{in}\big(\langle \lambda_\mu, \zeta_3\rangle(Y_\mu \oplus Z_3)\big)\\ &\subseteq \operatorname{gr}(Z_2) = \operatorname{gr}(\hat{Z}_2),\end{aligned}$$

and consequently $\operatorname{gr}(\hat{B}_L) \cong \operatorname{gr}(B_\Lambda)$.

Proof. We assert that $\operatorname{in}(\langle \nu_\mu, \hat{\zeta}_3\rangle(\hat{Y}_\mu \oplus \hat{Z}_3)) = \operatorname{in}(\nu_\mu(\hat{Y}_\mu)) + \operatorname{in}(\hat{\zeta}_3(\hat{Z}_3))$. Obviously the former contains the latter, so to verify this equality it suffices to show that whenever $y \in \hat{Y}_\mu$ and $z \in \hat{Z}_3$ have $v(\nu_\mu(y) + \hat{\zeta}_3(z)) = s$, there are $\nu_\mu(y'), \hat{\zeta}_3(z') \in (\widehat{IH})^s \cdot \hat{Z}_2$ with $\mathrm{in}(\nu_\mu(y) + \hat{\zeta}_3(z)) = \mathrm{in}(\nu_\mu(y') + \hat{\zeta}_3(z'))$. If either of $\nu_\mu(y)$, $\hat{\zeta}_3(z)$ is in $(\widehat{IH})^s \cdot \hat{Z}_2$, then both are, and we may take $y = y'$ and $z = z'$. If neither is, then we must have $v = v(\nu_\mu(y)) = v(\hat{\zeta}_3(z)) < s$ and $\mathrm{in}(\nu_\mu(y)) = -\mathrm{in}(\hat{\zeta}_3(z))$. By Lemma (3.7), then, $\mathrm{in}(\hat{\zeta}_3(z)) \in \mathrm{in}_v(\lambda_\mu(Y_\mu))$. By Corollary (3.5) (with $P = \mathbf{Z}$), this implies that $\mathrm{in}(\hat{\zeta}_3(z)) = 0$, an impossibility since $v(\hat{\zeta}_3(z)) < \infty$.

Having verified our assertion, we may cite Lemma (2.3), Proposition (2.4), and Lemma (3.7) to verify that for every $s \geq 0$,

$$\begin{aligned}
\mathrm{in}_s\big(\langle \nu_\mu, \hat{\zeta}_3\rangle(\hat{Y}_\mu \oplus \hat{Z}_3)\big) &= \mathrm{in}_s\big(\nu_\mu(\hat{Y}_\mu)\big) + \mathrm{in}_s(\hat{\zeta}_3(\hat{Z}_3)) \\
&= \mathrm{in}_s\big(\lambda_\mu(Y_\mu)\big) + \mathrm{in}_s(\zeta_3(Z_3)) \\
&= \lambda_\mu\big(H_s(Y_\mu)\big) + \zeta_3(H_{s-1}(Z_3)) + (IH)^{s+1} \cdot Z_2/(IH)^{s+1} \cdot Z_2 \\
&= \mathrm{in}_s\big(\langle \lambda_\mu, \zeta_3\rangle(Y_\mu \oplus Z_3)\big).
\end{aligned}$$

The isomorphism of the statement follows immediately, since

$$\mathrm{gr}(\hat{B}_L) \cong \mathrm{gr}(\hat{Z}_2)/\mathrm{in}\big(\langle \nu_\mu, \hat{\zeta}_3\rangle(\hat{Y}_\mu \oplus \hat{Z}_3)\big) \quad \text{and}$$

$$\mathrm{gr}(B_\Lambda) \cong \mathrm{gr}(Z_2)/\mathrm{in}\big(\langle \lambda_\mu, \zeta_3\rangle(Y_\mu \oplus Z_3)\big). \qquad \square$$

Proposition (3.8) completes our proof of the fact that condition (b) of Theorem 1 implies condition (c). As we noted in the introduction, the implication (a) $\Rightarrow$ (b) and the equivalence (c) $\Leftrightarrow$ (d) follow from the work of K.-T. Chen [3] and T. Maeda [**8**]; since (a) is simply a special case of (d), we may now conclude that each of (a), (b), (c), and (d) is equivalent to any other.

Before proceeding to consider (e), we present another condition equivalent to those of Theorem 1. To motivate this condition, note that by [**16**, Corollary (4.5)] the completion $\hat{B}_L$ of the Alexander invariant has the property that its elementary ideal $E_{\binom{\mu-1}{2}-1}(\hat{B}_L)$ is contained in $\widehat{IH}$. Since a $\widehat{\mathbf{Z}H}$-module generated by some k of its elements must have $E_k = \widehat{\mathbf{Z}H}$ [**12**, Theorem 3.2], this implies that no set of fewer than $\binom{\mu-1}{2}$ elements could possibly generate $\hat{B}_L$.

THEOREM (3.9). *The conditions of Theorem* 1 *hold if, and only if,* $\hat{B}_L$ *can be generated* (*as a* $\widehat{\mathbf{Z}H}$*-module*) *by some* $\binom{\mu-1}{2}$ *of its elements.*

Proof. (We actually prove that this condition is equivalent to (b).)

First, suppose that $\hat{B}_L$ can be generated by some $\binom{\mu-1}{2}$ of its elements. Then $E_{\binom{\mu-1}{2}}(\hat{B}_L) = \widehat{\mathbf{Z}H}$, so certainly $\varepsilon E_{\binom{\mu-1}{2}}(\hat{B}_L) = \mathbf{Z}$. Let $\mathcal{N}$ be the matrix representing the map ν_0: $\hat{Y}_0 \to \hat{Z}_2$ that appears in the exact sequence (3.1). Then by [**16**, Theorem (4.4)] (with $q = 1$),

$$\varepsilon E_{\binom{\mu-1}{2}}(\hat{B}_L) = \varepsilon E_{\binom{\mu-1}{2}}(\mathcal{N}) = E_{\binom{\mu-1}{2}}(\varepsilon(\mathcal{N}));$$

as $\varepsilon(\mathcal{N})$ is precisely the matrix Λ, this shows that (b) holds.

Conversely, suppose (b) holds. Let Λ_μ and $\mathcal{N}_\mu$ be the matrices obtained from Λ and $\mathcal{N}$ by deleting the last row of each. Since the sum of the rows of Λ is zero, (b) implies that $E_{\binom{\mu-1}{2}}(\Lambda_\mu) = \mathbf{Z}$ too. Since $\varepsilon(\mathcal{N}) = \Lambda$, $\varepsilon(\mathcal{N}_\mu) = \Lambda_\mu$, and so $\varepsilon E_{\binom{\mu-1}{2}}(\mathcal{N}_\mu) = \mathbf{Z}$; thus $E_{\binom{\mu-1}{2}}(\mathcal{N}_\mu)$ is an ideal of $\widehat{\mathbf{Z}H}$ which contains some element whose image under ε is 1. If we identify $\widehat{\mathbf{Z}H}$ with the power series ring $\mathbf{Z}[[u_1, \ldots, u_\mu]]$ (as discussed in §2), this element will be a power series whose constant term is 1; a well-known property of the power series ring is that such an element must be a unit. Thus $E_{\binom{\mu-1}{2}}(\mathcal{N}_\mu)$ must be all of $\widehat{\mathbf{Z}H}$, so by Lemma (2.7), the map ν_μ: $\hat{Y}_\mu \to \hat{Z}_2$ must be a split monomorphism, since it is represented by the matrix $\mathcal{N}_\mu$. It follows immediately from this and the exactness of (3.1) that $\hat{B}_L$ can be generated by $\binom{\mu}{2} - (\mu - 1) = \binom{\mu-1}{2}$ of its elements. □

When μ is two or three and the equivalent conditions of Theorem 1 hold, this additional condition allows us to determine the structure of the $\widehat{\mathbf{Z}H}$-module $\hat{B}_L$. If $\mu = 2$, $\hat{B}_L$ can be generated by $\binom{1}{2} = 0$ of its elements; that is, $\hat{B}_L = 0$. If $\mu = 3$, $\hat{B}_L$ can be generated by $\binom{2}{2} = 1$ element, so it is isomorphic to $\widehat{\mathbf{Z}H}/E_0(\hat{B}_L)$ [**12**, Theorem 3.5]; by [**16**, Theorem (4.1)], $E_0(\hat{B}_L)$ is the principal ideal of $\widehat{\mathbf{Z}H}$ generated by the Alexander polynomial of L.

4. Proof that (e) ⇔ (a). We will re-phrase the assertion that statements (e) and (a) of Theorem 1 are equivalent as follows:

PROPOSITION (4.1). *Let L be a tame link of μ components in S^3 with group G and complementary space $X = S^3 - L$. Then G_2/G_3 is a free abelian group of rank $\binom{\mu-1}{2}$ if and only if the cup product pairing (with integer coefficients)*

$$H^1(X) \otimes_{\mathbf{Z}} H^1(X) \to H^2(X) \tag{4.1}$$

is epimorphic.

Proof. First of all, we assert that if G_2/G_3 is free abelian of rank $\binom{\mu-1}{2}$, then the link L is unsplittable. For, it is readily verified that if the link L is splittable, then $E_{\binom{\mu-1}{2}}(\Lambda) = 0$. Since conditions (a) and (b) are equivalent, the assertion follows.

Similarly, we assert that if the cup product pairing (4.1) is epimorphic, then the link L is unsplittable. To prove this assertion, one proves that if the link L is splittable, then the image of the cup product pairing (4.1) is a subgroup of $H^2(X)$ which has rank at most $\mu - 2$ (the proof uses the Mayer-Vietoris cohomology exact sequence, the naturality of cup products, and the fact that (by Alexander duality) $H^2(X)$ is free abelian of rank $\mu - 1$).

In view of these two assertions, we see that we may assume that the link L is unsplittable. As a consequence, the complementary space X is an Eilenberg-MacLane complex, $K(G, 1)$ [**13**, p. 19], and $H_i(X) = H_i(G)$ for all i.

To prove the proposition, we will use the following five-term exact sequence:

$$H_2(G) \xrightarrow{p_*} H_2(G/G_2) \to G_2/G_3 \to H_1(G) \xrightarrow{1} H_1(G/G_2) \to 0$$

(cf. p. 205 of [**4**]). In this exact sequence, arrow number 1 is obviously an isomorphism, hence the sequence simplifies to the following:

$$H_2(G) \xrightarrow{p_*} H_2(G/G_2) \to G_2/G_3 \to 0.$$

Now G/G_2 is a free abelian group of rank μ, hence

$$K(G/G_2, 1) = (S^1)^\mu,$$

the cartesian product of μ copies of the circle S^1. Therefore $H_2(G/G_2)$ is a free abelian group of rank $\binom{\mu}{2}$. Since $H_2(G) = H_2(X)$ is free abelian of rank $\mu - 1$, we see that G_2/G_3 is free abelian of rank $\binom{\mu-1}{2}$ if and only if p_* is a split monomorphism. Next, recall that the integral homology groups of G and G/G_2 are all free abelian of finite rank. From this and the universal coefficient theorem [**4**, p. 222] it follows that p_* is a split monomorphism if and only if the dual homomorphism in integral cohomology,

$$p^*\colon H^2(G/G_2) \to H^2(G),$$

is a split epimorphism. Now observe that

$$p^*\colon H^1(G/G_2) \to H^1(G)$$

is an isomorphism, and that the following diagram is commutative:

$$\begin{array}{ccc} H^1(G/G_2) \otimes_{\mathbf{Z}} H^1(G/G_2) & \xrightarrow{\cup} & H^2(G/G_2) \\ \downarrow p^* \otimes p^* & & \downarrow p^* \\ H^1(G) \otimes_{\mathbf{Z}} H^1(G) & \xrightarrow{\cup} & H^2(G) \end{array}$$

(the horizontal arrows denote cup product homomorphisms). Since the cup product in the top line of this diagram is epimorphic (this follows readily from a cohomological version of the Künneth theorem [**10**, §VIII.11]), we conclude that

$$p^*\colon H^2(G/G_2) \to H^2(G)$$

is epimorphic if and only if the cup product in the bottom line is epimorphic. This completes the proof.

5. The Hosokawa polynomial. Let L be a tame link of μ components in S^3, and l the $\mu \times \mu$ matrix with entries given by $l_{ii} = -\Sigma_{k \neq i} l(K_i, K_k)$, and for $i \neq j$, $l_{ij} = l(K_i, K_j)$. Then $l = \Lambda M$ for an appropriate $\binom{\mu}{2} \times \mu$ integral matrix M, and so (by the functoriality of the exterior algebra), the determinant of a square submatrix of l can be expressed as an integral linear combination of the determinants of the various square submatrices of Λ of the same size; that is, $E_{\mu-m}(l) \subseteq E_{\binom{\mu}{2}-m}(\Lambda)\ \forall m$.

In particular, $E_1(l) \subseteq E_{\binom{\mu-1}{2}}(\Lambda)$. The Hosokawa polynomial $\nabla(L)$ has the property that $\nabla(L)(1)$ generates the ideal $E_1(l)$; in fact, every $(\mu - 1) \times (\mu - 1)$ submatrix of l has determinant $\pm\nabla(L)(1)$ [**5**]. Thus we conclude

PROPOSITION (5.1). $\nabla(L)(1) \in E_{\binom{\mu-1}{2}}(\Lambda)$.

Consequently, if $\nabla(L)(1) = \pm 1$ then $E_{\binom{\mu-1}{2}}(\Lambda) = \mathbf{Z}$.

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Received March 10, 1985.

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Pacific Journal of Mathematics

Vol. 124, No. 1 May, 1986