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ORDER IDEALS IN CATEGORIES

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In the program to develop enriched category theory in a topos \mathcal{E} it seems worthwhile to study the two particular bases Ω and \mathbf{R}^+ ; that is, the ordered objects of truth values and of non-negative extended reals with their appropriate monoidal structures. Categories in \mathcal{E} enriched in Ω are ordered objects in \mathcal{E} , and it is this example we wish to study here.

Categories in \mathcal{E} enriched in \mathbf{R}^+ are metric spaces in \mathcal{E} [8] and the relevant \mathbf{R}^+ has been studied in [10]. Since ordered objects occur at the very foundations of elementary topos theory, they have already been extensively studied (especially by Mikkelsen [9] and Brook [3]). However, our purpose is to emphasize the enriched-category viewpoint to give a guide to further development of the program.

Ordered objects can be defined without Ω , of course, and much of the theory can be developed in a category \mathcal{E} much more general than a topos. Our first two sections take this general approach. The first section deals with *ideals* in a regular category; from the enriched-category viewpoint these are the *modules* (= bimodules = profunctors = distributors). There is a bicategory $\text{Idl}(\mathcal{E})$ of ordered objects and ideals. The first key result is that an ideal has a right adjoint if and only if it is locally principal. This means that locally principal ideals play the role that cauchy sequences play in metric space theory [8]. The question of whether every ordered object is "cauchy complete" thus becomes the question of whether locally principal implies principal. We show that this is true precisely when \mathcal{E} satisfies the axiom of choice. The remainder of the first section deals with completeness of ordered objects.

The purpose of the second section is to construct, for ordered objects A, B , an object $[A, B]^*$ of order-preserving arrows from A to B with right adjoints and an object $[A, B]**$ of order-preserving arrows from A to B with right adjoints which have right adjoints. This requires \mathcal{E} to be cartesian closed.

For the final section, \mathcal{E} is required to be an elementary topos. For an ordered object A , we construct the object $\mathcal{P}A$ of order ideals in A which, in enriched-category terms, is the object appropriate for receiving the yoneda embedding. After developing sufficiently the properties of $\mathcal{P}A$, we

show that the cauchy completion of an ordered object A is $\mathcal{Q}A = [\Omega, \mathcal{P}A]**$ (where the subobject classifier Ω is the value of \mathcal{P} at the terminal object of \mathcal{E}).

For any bicategory \mathcal{B} , we write \mathcal{B}^* for the sub-bicategory with the same objects and with the arrows which have right adjoints. We write r^* for the right adjoint of a relation r when it exists. Although we do consider right adjoints for order-preserving arrows and for ideals, we do not use the superscript $*$ for the right adjoints in these cases.

1. Order ideals. A relation $r: A \rightarrow B$ in a category \mathcal{E} is a diagram (r_0, R, r_1) .

$$A \begin{array}{c} \leftarrow R \rightarrow \\ r_0 \quad r_1 \end{array} B$$

such that, for all arrows $x, y: U \rightarrow R$, if $r_0x = r_0y$ and $r_1x = r_1y$ then $x = y$. An arrow $a: U \rightarrow A$ is *r-related* to an arrow $b: U \rightarrow B$ when there exists $x: U \rightarrow R$ with $r_0x = a$, $r_1x = b$; we write $a(r)b$.

An arrow $e: V \rightarrow U$ in \mathcal{E} is called *strong epic* when, for all relations $r: A \rightarrow B$ and arrows $a: U \rightarrow A$, $b: U \rightarrow B$, if $ae(r)be$ then $a(r)b$. A strong epic which is monic is invertible. Strong epic implies epic if \mathcal{E} has pullbacks.

An *ordered object* A of \mathcal{E} consists of an object A_0 together with a relation $d = d_A = (d_0, A_1, d_1): A_0 \rightarrow A_0$ such that, for all $a, b, c: U \rightarrow A_0$, the following conditions hold:

$$a(d_A)a, \\ a(d_A)b, b(d_A)c \text{ imply } a(d_A)c.$$

An *order-preserving arrow* (or *functor*) $f: A \rightarrow B$ is an arrow $f: A_0 \rightarrow B_0$ in \mathcal{E} such that $a(d_A)a'$ implies $fa(d_B)fa'$. For order-preserving $f, f': A \rightarrow B$, put $f \leq f'$ when $f(d_B)f'$. With the obvious composition, we obtain the bicategory $\text{Ord}(\mathcal{E})$ of ordered objects in \mathcal{E} .

Objects of \mathcal{E} are identified with ordered objects A for which d_A is the equality relation. When \mathcal{E} has pullbacks, each arrow $h: V \rightarrow U$ in \mathcal{E} gives an ordered object $E(h) = (V, d)$ where $x(d)y$ when $hx = hy$. Then $h: E(h) \rightarrow U$ is order preserving.

For ordered objects A, B in \mathcal{E} , an *ideal* $r: A \rightarrow B$ is a relation $r: A_0 \rightarrow B_0$ such that $a'(d_A)a, a(r)b, b(d_B)b'$ imply $a'(r)b'$.

In order to be able to compose relations and ideals usefully, we need conditions on the category. A category \mathcal{E} is called *regular* when:

- R1. pullbacks exist;

- R2. for all arrows $a: U \rightarrow A$, $b: U \rightarrow B$, there exists a relation $r = (r_0, R, r_1): A \rightarrow B$ and a strong epic $e: U \rightarrow R$ with $r_0e = a$, $r_1e = b$;
- R3. each pullback of each strong epic is strong epic.

For a regular category \mathcal{E} , there is a bicategory $\text{Rel}(\mathcal{E})$ with the same objects as \mathcal{E} , with relations $r: A \rightarrow B$ as arrows, with a 2-cell $r \leq r'$ if $r_0(r')r_1$, and composition of relations $r: A \rightarrow B$, $s: B \rightarrow C$ given by: $a(sr)c$ iff there exist b and strong epic e with $ae(r)b$ and $b(s)ce$.

Each $f: A \rightarrow B$ in \mathcal{E} can be identified with $(1, A, f): A \rightarrow B$ in $\text{Rel}(\mathcal{E})$. It is proved in [6] that an arrow r in $\text{Rel}(\mathcal{E})$ has a right adjoint r^* iff r is isomorphic to an arrow in \mathcal{E} . The following result of André Joyal shows that our regular categories are regular in the sense of Barr [1].

PROPOSITION 1. *Each strong epic in a regular category is a coequalizer.*

Proof. Let p, q be the kernel pair of a strong epic e (that is, the pullback of e, e). Then $ee^* = 1$ and $e^*e = qp^*$ in $\text{Rel}(\mathcal{E})$. To show e is the coequalizer of p, q , take h with $hp = hq$. Put $r = he^*$ in $\text{Rel}(\mathcal{E})$. Then $r(eh^*) = he^*eh = hqp^*h^* = (hp)(hq)^* \leq 1$ and $1 = ee^* \leq eh^*he^* = (eh^*)r$. So $eh^* = r^*$ and $r \cong k$ where k is in \mathcal{E} . Also $ke \leq re = he^*e = hqp^* = hpp^* \leq h$ implies $ke = h$ since ke, h are in \mathcal{E} . Since e is epic, k is unique with $ke = h$. \square

COROLLARY 2. *An arrow r in $\text{Rel}(\mathcal{E})$ has a right adjoint iff there exists a strong epic e in \mathcal{E} such that re is isomorphic to an arrow in \mathcal{E} .*

Proof. If r has a right adjoint then e can be taken to be the identity. Conversely, if $re \cong h$ with h in \mathcal{E} then $hp = hq$ for p, q forming the kernel pair of e . By Proposition 1, $h = ge$ for some g in \mathcal{E} . So $r \cong ree^* \cong he^* \cong gee^*g$. \square

For a regular category \mathcal{E} , there is also a bicategory $\text{Idl}(\mathcal{E})$ whose objects are the ordered objects in \mathcal{E} , whose arrows are ideals, and whose 2-cells and compositions are as for relations. The identity ideal of A is $d_A: A \rightarrow A$.

Each order-preserving arrow $f: A \rightarrow B$ yields an ideal $d_Bf: A \rightarrow B$ which has a right adjoint $f^*d_B: B \rightarrow A$ in $\text{Idl}(\mathcal{E})$. An ideal $r: A \rightarrow B$ is called *principal* when there exists an order-preserving arrow $f: A \rightarrow B$ such that $r \cong d_Bf$. In general, not every ideal with a right adjoint is principal; however, Corollary 2 generalizes.

PROPOSITION 3. *An ideal $r: A \rightarrow B$ has a right adjoint iff there exists a strong epic $e: U \rightarrow A_0$ with U in \mathcal{E} and $re: U \rightarrow B$ principal.*

Proof. Suppose $r \dashv s$ in $\text{Idl}(\mathcal{E})$. The unit condition $d_A \leq sr$ amounts to: $a(d_A)a'$ implies there exist b and strong epic e with $ae(r)b$ and $b(s)a'e$. The counit condition $rs \leq d_B$ amounts to: $b(s)a, a(r)b'$ imply $b(d_B)b'$. The unit condition with $a = a' = 1$ gives e strong epic and f with $e(r)f, f(s)e$. The counit condition using $fx(s)ex$, together with the ideal condition for r using $ex(r)fx$, yield that $ex(r)b'$ precisely when $fx(d_B)b'$. Hence $x(re)b'$ precisely when $x(d_Bf)b'$. So $re \cong d_Bf$. Since the source of f is in \mathcal{E} , order-preservingness is automatic. So re is principal.

Conversely, suppose $re \cong d_Bf$ with e strong epic. Put $s = ef^*$. Then $rs = ref^* \cong d_Bff^* \leq d_B$. So, to prove $r \dashv s$ it remains to prove $d_A \leq sr$. Suppose $a(d_A)a'$ and let x, e' form a pullback for e, a . Now $x(d_Bf)fx$ implies $x(re)fx$ which implies $ex(r)fx$. Since $fx(s)ex, ex = ae', ae'(d_A)a'e'$ and s is an ideal, we have $fx(s)a'e'$. So we have $ex(sr)a'e'$. So $a(sr)a'$ because $ex = ae'$ and e' is strong epic by R3. \square

Using the terminology of enriched category theory [3], we call an ideal *cauchy* when it has a right adjoint. An ordered object X is *cauchy complete* when every cauchy ideal into it is principal; it follows from Proposition 3 that we only need to check for cauchy ideals with sources in \mathcal{E} . Thinking of strong epics as *covers*, we can interpret Proposition 3 as saying: an ideal is cauchy precisely when it is locally principal. We say that \mathcal{E} satisfies the *axiom of choice* when every strong epic is a retraction.

COROLLARY 4. *The following three conditions on \mathcal{E} are equivalent:*

- (i) *the axiom of choice;*
- (ii) *every ordered object is cauchy complete;*
- (iii) *every equivalence is $\text{Idl}(\mathcal{E})$ is principal.*

Proof. (i) \Rightarrow (ii) If e is a retraction then re principal implies r principal, so Proposition 3 gives the result.

(ii) \Rightarrow (iii) Trivial.

(iii) \Rightarrow (i) If $e: V \rightarrow U$ is a strong epic then $e: E(e) \rightarrow U$ is an equivalence in $\text{Idl}(\mathcal{E})$. So $e^*: U \rightarrow E(e)$ is principal by (iii). Then $e^* \cong e^*ef$ with f in \mathcal{E} ; so $ef \cong ee^*ef \cong ee^* \cong 1$; so e is a retraction. \square

The homomorphism

$$\text{Ord}(\mathcal{E})^{\text{co}} \rightarrow \text{Idl}(\mathcal{E})^*,$$

which is the identity on objects and takes f to d_Bf , is generally not a biequivalence (it is iff \mathcal{E} satisfies the axiom of choice). Since $e: E(e) \rightarrow U$

is an equivalence in $\text{Idl}(\mathcal{E})$ when e is a strong epic, we obtain

$$\text{Ord}(\mathcal{E})(E(e), A)^{\text{op}} \rightarrow \text{Idl}(\mathcal{E})^*(E(e), A) \simeq \text{Idl}(\mathcal{E})^*(U, A)$$

taking $h: E(e) \rightarrow A$ to $d_A h e^*: U \rightarrow A$. Let CovU denote the ordered set whose elements are strong epics $e: V \rightarrow U$ (covers) with $e \leq e'$ when there exists an arrow $f: V \rightarrow V'$ such that $e = e'f$. Notice that $(\text{CovU})^{\text{op}}$ is a directed set by R3, and $e \mapsto E(e)$ gives a functor $E: \text{CovU} \rightarrow \text{Ord}(\mathcal{E})$. Thus we have a cone of ordered sets:

$$\text{Ord}(\mathcal{E})(E - , A)^{\text{op}} \rightarrow \text{Idl}(\mathcal{E})^*(U, A).$$

COROLLARY 5. *For U in \mathcal{E} and A in $\text{Ord}(\mathcal{E})$ the above cone induces an equivalence of ordered sets*

$$\text{colim}_{e \in \text{CovU}} \text{Ord}(\mathcal{E})(E(e), A)^{\text{op}} \simeq \text{Idl}(\mathcal{E})^*(U, A).$$

Proof. To obtain the inverse assignment, take a cauchy ideal $r: U \rightarrow A$. Proposition 3 gives $re \cong d_A h$ for some $h: V \rightarrow A$ and strong epic $e: V \rightarrow U$. Then $he^*e \leq d_A he^*e \cong ree^*e \cong re \cong d_A h$; so $h: E(e) \rightarrow A$ is order preserving. \square

For ideals $r: A \rightarrow C$, $s: B \rightarrow C$, we write $C(r, s): B \rightarrow A$ for the ideal characterized by the property:

$$t \leq C(r, s) \quad \text{iff} \quad rt \leq s$$

for all ideals $t: B \rightarrow A$. For a general \mathcal{E} , the ideal $C(r, s)$ may not exist for all r, s . If r is cauchy then $C(r, s)$ is the composite of s with the right adjoint for r ; in particular, if $u: A \rightarrow C$ is order preserving then $C(d_C u, s) \cong u^*s$.

PROPOSITION 6. *If \mathcal{E} is finitely complete and each \mathcal{E}/U is cartesian closed then $C(r, s)$ exists for all ideals $r: A \rightarrow C$, $s: B \rightarrow C$.*

Proof. For ordered objects A, B , the inclusion of $\text{Idl}(\mathcal{E})(A, B)$ in $\text{Rel}(\mathcal{E})(A_0, B_0)$ whose value at a relation $r: A_0 \rightarrow B_0$ is the relation (which happens to be an ideal) $A_0 \rightarrow B_0$ obtained from the internal hom in $\mathcal{E}/A_0 \times B_0$ of the objects $d_1 \times d_0: A_0 \times B_0 \rightarrow A_0 \times B_0$ and

$$\begin{pmatrix} r_0 \\ r_1 \end{pmatrix}: R \rightarrow A_0 \times B_0.$$

It is well known [7] that, under our conditions on \mathcal{E} , for each span $r: U \rightarrow W$, the functor $\text{Spn}(\mathcal{E})(V, U) \rightarrow \text{Spn}(\mathcal{E})(V, W)$ obtained by composing with r has a right adjoint. When r is a relation this right adjoint induces a right adjoint to the functor $\text{Rel}(\mathcal{E})(V, U) \rightarrow \text{Rel}(\mathcal{E})(V, W)$ given by composing with r in $\text{Rel}(\mathcal{E})$.

For ideals $r: A \rightarrow C$, $s: B \rightarrow C$, the desired ideal $C(r, s)$ is the value of the right adjoint to

$$\text{Idl}(\mathcal{E})(B, A) \rightarrow \text{Rel}(\mathcal{E})(B_0, A_0) \xrightarrow{\text{r-}} \text{Rel}(\mathcal{E})(B_0, C_0)$$

at s . □

Suppose $r: A \rightarrow B$ is an ideal and $f: B \rightarrow X$ is order preserving. An *r-weighted limit* for f is an order-preserving arrow $\text{lim}(r, f): A \rightarrow X$ such that $\text{lim}(r, f) * d_X \cong B(r, f * d_X)$.

PROPOSITION 7. *An ordered object X is Cauchy complete iff X admits all limits weighted by cauchy ideals.*

Proof. For a cauchy ideal $r: A \rightarrow B$ with right adjoint s , we have $B(r, f * d_X) \cong sf * d_X$ which is a right adjoint for $d_X fr: A \rightarrow X$.

If X is cauchy complete then $d_X fr \cong d_X g$ for some order preserving g ; so $g * d_X \cong B(r, f * d_X)$ and $g \cong \text{lim}(r, f)$.

If X admits the indicated limits consider such r with $B = X$. Let $g = \text{lim}(r, 1_X)$ so that $g * d_X \cong s1_X^* d_X \cong s$. So $r \cong d_X g$ is principal. □

PROPOSITION 8. *For any $j: A \rightarrow B$ in $\text{Ord}(\mathcal{E})$, the functor*

$$\text{Ord}(\mathcal{E})(j, 1): \text{Ord}(\mathcal{E})(B, X) \rightarrow \text{Ord}(\mathcal{E})(A, X)$$

*has right adjoint at $f: A \rightarrow X$ given by $\text{lim}(j * d_B, f)$ if this limit exists. If j is fully faithful (i.e. $d_A \cong j * d_B j$) then $\text{lim}(j * d_B, f) j \cong f$.*

Proof. $g \leq \text{lim}(j * d, f)$ iff $g * d \leq \text{lim}(j * d, f) * d \cong B(j * d, f * d)$ iff $j * dg * d \leq f * d$ iff $(gj) * d \leq f * d$ iff $gj \leq f$. If j is fully faithful then $j * d_B j f * d_X \leq f * d_X$, so $j f * d_X \leq A(j * d_B, f * d_X) \cong \text{lim}(j * d_B, f) * d_X$, so $f * d_X \leq j * \text{lim}(j * d_B, f) * d_X$, so $f \leq \text{lim}(j * d_B, f) j$. □

These results relate our work to that of Bunge-Paré [5], Bunge [4] and Street [14].

2. Objects of adjunctions. Suppose the category \mathcal{E} is finitely complete and cartesian closed:

$$\mathcal{E}(U \times X, Y) \cong \mathcal{E}(U, [X, Y]).$$

For ordered objects A, B in \mathcal{E} , form the pullback

$$\begin{array}{ccc} [A, B]_0 & \rightarrow & [A_0, B_0] \\ \downarrow & & \downarrow \begin{pmatrix} [d_0, 1] \\ [d_1, 1] \end{pmatrix} \\ [A_1, B_1] & \xrightarrow{\begin{pmatrix} [1, d_0] \\ [1, d_1] \end{pmatrix}} & [A_1, B_0] \times [A_1, B_0] \end{array}$$

in which the horizontal arrows are monic. The order d_B on B_0 induces an order on $[A_0, B_0]$ and hence on $[A, B]_0$ yielding an ordered object $[A, B]$ satisfying:

$$\text{Ord}(\mathcal{E})(C \times A, B) \cong \text{Ord}(\mathcal{E})(C, [A, B]).$$

Indeed, $\text{Ord}(\mathcal{E})$ is finitely complete and cartesian closed as a 2-category.

PROPOSITION 9. *For ordered objects A, B in \mathcal{E} , there exists an ordered object $[A, B]^*$ with a natural equivalence of ordered sets:*

$$\mathcal{E}(U, [A, B]^*) \simeq \text{Ord}(\mathcal{E}/U)^*(U \times A, U \times B)$$

(where, of course, $U \times A, U \times B$ are regarded as objects of $\text{Ord}(\mathcal{E}/U)$ by means of first projection onto U).

Proof. (This kind of result is folklore from the '60's; we indicate the proof for lack of a suitable reference.) The identity $[A, B] \rightarrow [A, B]$ corresponds to "evaluation" $\text{ev}_A: [A, B] \times A \rightarrow B$, and the composite

$$[B, C] \times [A, B] \times A \xrightarrow{1 \times \text{ev}_A} [B, C] \times B \xrightarrow{\text{ev}_B} C$$

corresponds to "composition" $\text{comp}_B: [B, C] \times [A, B] \rightarrow [A, C]$. The projection $1 \times A \rightarrow A$ gives $\text{id}_A: 1 \rightarrow [A, A]$. Let $h: H \rightarrow [B, A] \times [A, B]$ denote the inserter (or subequalizer) of the pair of arrows

$$\begin{array}{ccc} & 1 & \\ & \nearrow & \searrow \text{id}_A \\ [B, A] \times [A, B] & \xrightarrow{\text{comp}_B} & [A, A] \end{array}$$

in $\text{Ord}(\mathcal{E})$; this means that an arrow $U \rightarrow H$ amounts to order-preserving arrows $f: U \times A \rightarrow B, g: U \times B \rightarrow A$ such that $pr_3 \leq g(U \times f)$. Let $k: K \rightarrow [A, B] \times [B, A]$ denote the inserter of the pair of arrows.

$$\begin{array}{ccc} [A, B] \times [B, A] & \xrightarrow{\text{comp}_A} & [B, B] \\ & \searrow & \nearrow \text{id}_B \\ & 1 & \end{array}$$

in $\text{Ord}(\mathcal{E})$. Form the pullback

$$\begin{array}{ccc} [A, B]_0^* & \rightarrow & K_0 \\ \downarrow & & \downarrow k \\ H_0 & \xrightarrow{h} & [B, A] \times [A, B] \cong [A, B] \times [B, A]. \end{array}$$

It is easy to see that the composite of the above square with each projection onto $[A, B]$ and onto $[B, A]$ is monic. Let $[A, B]^*$ be the object $[A, B]_0^*$ enriched by the order induced from $[A, B]$ via the monic. The natural equivalence is easily verified. \square

There are order-preserving monics

$$[A, B]^* \rightarrow [A, B] \quad \text{and} \quad [A, B]^* \rightarrow [B, A]^{\text{op}}$$

induced by the inclusion

$$\text{Ord}(\mathcal{E}/U)^*(U \times A, U \times B) \rightarrow \text{Ord}(\mathcal{E}/U)(U \times A, U \times B)$$

and the right-adjoint-assigning monic

$$\text{Ord}(\mathcal{E}/U)^*(U \times A, U \times B) \rightarrow \text{Ord}(\mathcal{E}/U)(U \times B, U \times A).$$

The object $[A, B]**$ defined by the pullback

$$\begin{array}{ccc} [A, B]** & \rightarrow & [B, A]**^{\text{op}} \\ \downarrow & & \downarrow \\ [A, B]^* & \rightarrow & [B, A]^{\text{op}} \end{array}$$

and the natural equivalence of ordered sets

$$\mathcal{E}(U, [A, B]**) \simeq \text{Ord}(\mathcal{E}/U)**(U \times A, U \times B)$$

will be used to construct the Cauchy completion of an ordered object.

Notice that these universal properties of $[A, B]^*$, $[A, B]**$ do determine them up to isomorphism. This follows because $\mathcal{E} \rightarrow \text{Ord}(\mathcal{E})$ is dense; in fact, $\mathcal{E} \rightarrow \text{Cat}(\mathcal{E})$ is dense as can be seen using the extended Yoneda lemma [12; p. 287].

3. Cauchy completion. For this Section we assume that \mathcal{E} is an elementary topos. Then \mathcal{E} satisfies the assumptions of the earlier sections, including those of Proposition 6. The subobject classifier Ω is regarded as an ordered object via that order which gives a natural equivalence of ordered sets:

$$\text{Rel}(\mathcal{E})(X, Y) \simeq \mathcal{E}(X \times Y, \Omega).$$

For ordered objects A, B , this equivalence (with $X = A_0, Y = B_0$) enriches to a natural equivalence

$$\text{Idl}(\mathcal{E})(A, B) \simeq \text{Ord}(\mathcal{E})(A^{\text{op}} \times B, \Omega)$$

where A^{op} denotes A_0 with the reverse order (d_1, A_0, d_0) . Putting $A = [\mathcal{P}A^{\text{op}}, \Omega]$, we obtain a natural equivalence

$$\text{Idl}(\mathcal{E})(A, B) \simeq \text{Ord}(\mathcal{E})(B, \mathcal{P}A);$$

compare [11; pp. 172–5].

The identity of $\mathcal{P}A$ in $\text{Ord}(\mathcal{E})$ corresponds to an ideal $\in_A: A \rightarrow \mathcal{P}A$ called *membership*. The last natural equivalence is then given by: the ideal $r: A \rightarrow B$ corresponds to the order-preserving arrow $f: B \rightarrow \mathcal{P}A$ when $r \cong f^* \in_A$; that is,

$$a(r)b \quad \text{iff} \quad a(\in_A)fb.$$

The *yoneda embedding* $y_A: A \rightarrow \mathcal{P}A$ is the order-preserving arrow defined by $d_A \cong y_A^* \in_A$; that is,

$$a \leq a' \quad \text{iff} \quad a(\in_A)y_A a'.$$

PROPOSITION 10. (i) $a(\in_A)f$ iff $y_A a \leq f$.

(ii) If $y_A a \leq f$ implies $y_A a \leq f'$ for all a then $f \leq f'$.

(iii) $\in_A y_A^* \cong d_{\mathcal{P}A}$.

(iv) The left extension of y_A along y_A is $1_{\mathcal{P}A}$ in $\text{Ord}(\mathcal{E})$.

Proof. (i) $y_A a \leq f$ iff $a^* y_A^* \in_A \leq f^* \in_A$ iff $a^* d_A \leq f^* \in_A$ iff $fa^* d_A \leq \in_A$ iff $fa^* \leq \in_A$ (since \in_A is an ideal) iff $a(\in_A)f$.

(ii) $f \leq f'$ iff $f^* \in_A \leq f'^* \in_A$ iff $(a^* \leq f^* \in_A \Rightarrow a^* \leq f'^* \in_A)$ iff $(fa^* \leq \in_A \Rightarrow f'a^* \leq \in_A)$ iff $(a(\in_A)f \Rightarrow a(\in_A)f')$ iff $(y_A a \leq f \Rightarrow y_A a \leq f')$.

(iii) $p(\in_A y_A^*)q$ iff $(pe = y_A a, a(\in_A)qe$ for some a and epic e) iff $(pe = y_A a, y_A a \leq qe$ for some a and epic e) iff $p \leq q$ iff $p(d_{\mathcal{P}A})q$.

(iv) $y_A \leq ky_B$ iff $1(\in_A)ky_A$ iff $1(\in_A y_A^*)k$ iff $1(d_{\mathcal{P}A})k$ iff $1 \leq k$. \square

An ordered object X is called *complete* when it admits all limits weighted by all ideals.

Put $\mathcal{P}^\dagger A = (\mathcal{P}A^{\text{op}})^{\text{op}} = [A, \Omega]^{\text{op}}$ and $y_A^\dagger = (y_{A^{\text{op}}})^{\text{op}}: A \rightarrow \mathcal{P}^\dagger A$. Then we have an ideal $\ni_A: \mathcal{P}^\dagger A \rightarrow A$ which induces an equivalence

$$\text{Ord}(\mathcal{E})(B, \mathcal{P}^\dagger A)^{\text{op}} \simeq \text{Idl}(\mathcal{E})(B, A).$$

PROPOSITION 11. For all $C \in \text{Ord}(\mathcal{E})$ the ordered objects $\mathcal{P}C$ and $\mathcal{P}^\dagger C$ are both complete.

Proof. Limits in $\mathcal{P}^\dagger C$ are obtained from *composition of ideals*. To see this take an ideal $r: A \rightarrow B$ and a functor $f: B \rightarrow \mathcal{P}^\dagger C$. Let $s: B \rightarrow C$ be the ideal corresponding to f : this means $b(s)c$ iff $fb \leq y_C^\dagger c$. The composite ideal $sr: A \rightarrow C$ gives a functor $g: A \rightarrow \mathcal{P}^\dagger C$ with $ga \leq y_C^\dagger c$ iff $a(sr)c$. We claim that $g = \text{lim}(r, f)$. Twice using Proposition 10(ii), we have $rt \leq f*d$ iff $(p(t)a, a(r)b \Rightarrow p \leq fb)$ iff $(p(t)a, a(r)b, fb \leq y_C^\dagger c \Rightarrow p \leq y_C^\dagger c)$ iff $(p(t)a, a(r)b, b(s)c \Rightarrow p \leq y_C^\dagger c)$ iff $(p(t)a, a(sr)c \Rightarrow p \leq y_C^\dagger c)$ iff $(p(t)a, ga \leq y_C^\dagger c \Rightarrow p \leq y_C^\dagger c)$ iff $(p(t)a \Rightarrow p \leq ga)$ iff $t \leq g*d$.

Limits in $\mathcal{P}C$ are obtained from *right liftings of ideals* (which exist by Proposition 6). To see this, take an ideal $r: A \rightarrow B$ and a functor $f: B \rightarrow \mathcal{P}C$. Let $s: C \rightarrow B$ be the ideal corresponding to f . Let $g: A \rightarrow \mathcal{P}C$ be the functor corresponding to the ideal $B(r, s): C \rightarrow A$. One easily verifies that $g = \text{lim}(r, f)$. □

PROPOSITION 12. *The following conditions on an ordered object X are equivalent:*

- (a) X is complete;
- (b) $y_X^\dagger: X \rightarrow \mathcal{P}^\dagger X$ has a right adjoint;
- (c) X^{op} is complete;
- (d) $y_X: X \rightarrow \mathcal{P}X$ has a left adjoint.

Proof. (a) \Rightarrow (b) $\text{lim}(\ni_X, 1_X): \mathcal{P}^\dagger X \rightarrow X$ can be verified to be a right adjoint for y_X^\dagger .

(b) \Rightarrow (c) Condition (b) means that $y_{X^{\text{op}}}: X^{\text{op}} \rightarrow \mathcal{P}(X^{\text{op}})$ has a left adjoint. Since $y_{X^{\text{op}}}$ is fully faithful and $\mathcal{P}(X^{\text{op}})$ admits all limits (Proposition 11), a familiar argument gives that X^{op} admits all limits and they are preserved by $y_{X^{\text{op}}}$.

- (c) \Rightarrow (d) Apply (a) \Rightarrow (b) to X^{op} .
- (d) \Rightarrow (a) Apply (b) \Rightarrow (c) to X^{op} . □

PROPOSITION 13. *If X is complete then composition with $y_A: A \rightarrow \mathcal{P}A$ gives an equivalence*

$$\text{Ord}(\mathcal{E})^*(\mathcal{P}A, X) \simeq \text{Ord}(\mathcal{E})(A, X).$$

Furthermore, $[\mathcal{P}A, X]^* \simeq [A, X]$.

Proof. Composition with $y_A: A \rightarrow \mathcal{P}A$ gives a functor

$$\text{Ord}(\mathcal{E})(\mathcal{P}A, X) \rightarrow \text{Ord}(\mathcal{E})(A, X)$$

which has a left adjoint by Proposition 8 and 12; the left adjoint in fact lands in $\text{Ord}(\mathcal{E})^*(\mathcal{P}A, X)$ since its value $\hat{f}: A \rightarrow X$ at $f: A \rightarrow X$ has a

right adjoint $X \rightarrow \mathcal{P}A$ which corresponds to the ideal $d_{Xf}: A \rightarrow X$. Since y_A is fully faithful, this left adjoint $\text{Ord}(\mathcal{E})(A, X) \rightarrow \text{Ord}(\mathcal{E})^*(\mathcal{P}A, X)$ is fully faithful; since y_A is dense (Proposition 10), it is surjective up to isomorphism. Thus we have the first equivalence. To obtain the second, apply the first in the topos \mathcal{E}/U in place of \mathcal{E} , and use the denseness of $\mathcal{E} \rightarrow \text{Ord}(\mathcal{E})$ with Proposition 9. \square

Using Proposition 10(iii), we see that we have a homomorphism of bicategories

$$\text{Idl}(\mathcal{E})^{\text{op}} \rightarrow \text{Ord}(\mathcal{E})^*$$

which is given on objects by \mathcal{P} and on homs is the equivalence

$$\text{Idl}(\mathcal{E})(A, B) \simeq \text{Ord}(\mathcal{E})(B, \mathcal{P}A) \simeq \text{Ord}(\mathcal{E})^*(\mathcal{P}B, \mathcal{P}A).$$

Since homomorphisms preserve adjunctions, we deduce that there is an equivalence

$$\text{Idl}(\mathcal{E})^*(A, B) \simeq \text{Ord}(\mathcal{E})^{**}(\mathcal{P}A, \mathcal{P}B).$$

Apply this now to the ordered objects U and $U \times A$ in the topos \mathcal{E}/U to obtain:

$$\begin{aligned} \text{Idl}(\mathcal{E})^*(U, A) &\simeq \text{Idl}(\mathcal{E}/U)^*(U, U \times A) \\ &\simeq \text{Ord}(\mathcal{E}/U)^{**}(U \times \Omega, U \times \mathcal{P}A)^{\text{op}} \\ &\simeq \mathcal{E}(U, [\Omega, \mathcal{P}A]^{**})^{\text{op}}. \end{aligned}$$

THEOREM 14. *Each ordered object A in an elementary topos \mathcal{E} has a cauchy completion $\mathcal{Q}A$. In fact, $\mathcal{Q}A = [\Omega, \mathcal{P}A]^{**}$ is cauchy complete and there exists a fully faithful functor $n_A: A \rightarrow \mathcal{Q}A$ which, for all cauchy complete X , induces an equivalence of ordered sets*

$$\text{Ord}(\mathcal{E})(\mathcal{Q}A, X) \simeq \text{Ord}(\mathcal{E})(A, X).$$

Proof. The following equivalence is proved above:

$$(a) \quad \mathcal{E}(U, \mathcal{Q}A) \simeq \text{Idl}(\mathcal{E})^*(U, A)^{\text{op}}.$$

The natural functors

$$\mathcal{E}(U, A) \rightarrow \text{Idl}(\mathcal{E})^*(U, A)^{\text{op}} \rightarrow \text{Idl}(\mathcal{E})(A, U)$$

(the first takes f to $d_A f$ and the second takes a cauchy ideal to its right adjoint) induce fully faithful functors

$$A \xrightarrow{n_A} \mathcal{Q}A \xrightarrow{m_A} \mathcal{P}A$$

between the representing objects such that $m_A n_A \cong y_A$. The equivalence (a) therefore takes $f: U \rightarrow \mathcal{Q}A$ to the left adjoint of the ideal $f^* m_A \in \mathcal{A}$. There will be no ambiguity in omitting the subscripts from m_A, n_A , and so on.

We shall show that the fully faithful functor

(b)
$$\text{Idl}(\mathcal{E})^*(U, A) \rightarrow \text{Idl}(\mathcal{E})^*(U, \mathcal{Q}A),$$

which takes r to dnr , is an equivalence. To see this, take a cauchy ideal $s: U \rightarrow \mathcal{Q}A$. By Proposition 3, there exist a (strong) epic $e: V \rightarrow U$ and arrow $f: V \rightarrow \mathcal{Q}A$ with $se \cong df$. Under (a) the arrow gives a cauchy ideal $t: V \rightarrow A$ whose right adjoint ideal is $f^* m^* \in \cdot$. So we obtain a cauchy ideal $r \cong te^*: U \rightarrow A$. Now $f^* m^* \in n^* d \cong f^* m^* \in n^* m^* m d \cong f^* m^* \in y^* m d \cong f^* m^* d m d \cong f^* d$, so, taking left adjoint ideals, we obtain $dnt \cong df \cong se$; so $s \cong dnte^* \cong dnr$. So s is, up to isomorphism, in the image of (b).

Combining (a), (b), we obtain the equivalence

(c)
$$\mathcal{E}(U, \mathcal{Q}A)^{\text{op}} \simeq \text{Idl}(\mathcal{E})^*(U, \mathcal{Q}A).$$

Thus $\mathcal{Q}A$ is cauchy complete.

Next we show that the ideal $n^* d: \mathcal{Q}A \rightarrow A$ is cauchy. Composition with dn is a fully faithful functor

$$\text{Idl}(\mathcal{E})(U, A) \rightarrow \text{Idl}(\mathcal{E})(U, \mathcal{Q}A)$$

whose right adjoint is composition with $n^* d$. Furthermore, this adjunction restricts to the equivalence (b). It follows that composition with $n^* d$ gives the inverse equivalence for (b). Thus $n^* d f$ is cauchy for all functors $f: U \rightarrow \mathcal{Q}A$. By Proposition 3, $n^* d$ is cauchy.

If X is cauchy complete then it admits limits weighted by $n^* d$ (Proposition 7). The functor

(d)
$$\text{Ord}(\mathcal{E})(\mathcal{Q}A, X) \rightarrow \text{Ord}(\mathcal{E})(A, X)$$

given by composition with n thus has a right adjoint (Proposition 8) which is fully faithful since n is. It remains to show that (d) reflects isomorphisms. Since every X has a fully faithful functor $y_X: X \rightarrow \mathcal{P}X$ into a complete object, it suffices to prove (d) reflects isomorphisms for X complete.

Let $m^\dagger: A \rightarrow \mathcal{P}^\dagger A$ denote the fully faithful functor induced by the inclusion

$$\text{Idl}(\mathcal{E})^*(U, A)^{\text{op}} \rightarrow \text{Idl}(\mathcal{E})(U, A)^{\text{op}};$$

then $y^\dagger: A \rightarrow \mathcal{P}^\dagger A$ is isomorphic to $m^\dagger n$. Suppose $g, h: \mathcal{Q}A \rightarrow X$ are functors with $g \leq h$ and $gn \cong kn$. Assuming X complete, we have right extensions $f', h': \mathcal{P}^\dagger A \rightarrow X$ of g, h along m^\dagger in $\text{Ord}(\mathcal{E})$ with $g'm^\dagger \cong g$, $h'm^\dagger \cong h$. So $g'y^\dagger \cong g'm^\dagger n \cong gn \cong kn \cong h'm^\dagger n \cong h'y^\dagger$. By the dual of Proposition 13, we have $g' \cong h'$. Hence $g \cong g'm^\dagger \cong h'm^\dagger \cong k$. \square

COROLLARY 15. (a) $\mathcal{P}\mathcal{Q}A \cong \mathcal{P}A$. (b) $\mathcal{P}A \cong \mathcal{P}B$ iff $\mathcal{Q}A \cong \mathcal{Q}B$.

Proof. (a) Since $\mathcal{P}^\dagger B$ is cauchy complete, we have

$$\begin{aligned} \text{Ord}(\mathcal{E})(B, \mathcal{P}\mathcal{Q}A) &\cong \text{Idl}(\mathcal{E})(\mathcal{Q}A, B) \\ &\cong \text{Ord}(\mathcal{E})(\mathcal{Q}A, \mathcal{Q}^\dagger B)^{\text{op}} \cong \text{Ord}(\mathcal{E})(A, \mathcal{P}^\dagger B)^{\text{op}} \\ &\cong \text{Idl}(\mathcal{E})(A, B) \cong \text{Ord}(\mathcal{E})(B, \mathcal{P}A). \end{aligned}$$

So (a) follows.

(b) From the formula $\mathcal{Q} - = [\Omega, \mathcal{P} -]^{**}$ we see that $\mathcal{P}A \cong \mathcal{P}B$ implies $\mathcal{Q}A \cong \mathcal{Q}B$. The converse follows from (a). \square

REFERENCES

- [1] M. Barr, *Exact Categories*, Lecture Notes in Math., **236** (Springer, Berlin-New York, 1971), 1–120.
- [2] R. Betti, A. Carboni, R. Street and R. Walters, *Variation through enrichment*, J. Pure and Appl. Algebra, **29** (1983), 109–127.
- [3] T. Brook, *Order and Recursion in Topoi*, Notes on Pure Mathematics, **9** (Australian National University, Canberra, 1977), 226 pp.
- [4] M. Bunge, *Stack completions and Morita equivalence for categories in a topos*, Cahiers de topologie et géométrie différentielle, **20** (1979), 401–435.
- [5] M. Bunge and R. Paré, *Stacks and equivalence of indexed categories*, Cahiers de topologie et géométrie différentielle, **20** (1979), 373–399.
- [6] A. Carboni, S. Kasangian and R. Street, *Bicategories of spans and relations*, J. Pure and Appl. Algebra, **33** (1984), 259–267.
- [7] B. J. Day, *Limit Spaces and Closed Span Categories*, Lecture Notes in Math., **420** (Springer, Berlin-New York, 1974), 65–74.
- [8] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano, **43** (1973), 135–166.
- [9] C. J. Mikkelsen, *Lattice-theoretic and logical aspects of elementary topoi*, Aarhus Universitet Various Publication Series, **25** (1976).
- [10] J. Z. Reichman, *Semicontinuous real numbers in a topos*, J. Pure and Appl. Algebra, **28** (1983), 81–91.

- [11] R. Street, *Elementary Cosmoi*, Lecture Notes in Math., **420** (Springer, Berlin-New York, 1974), 134–180.
- [12] ———, *Cosmoi of internal categories*, Trans. Amer. Math. Soc., **258** (1980), 271–318.
- [13] ———, *Cauchy characterization of enriched categories*, Rend. Sem. Mat. Fis. Milano, **51** (1981), 217–233.
- [14] ———, *Characterization of Bicategories of Stacks*, Lecture Notes in Math., **962** (Springer, Berlin-New York, 1982), 282–291.
- [15] ———, *Enriched categories and cohomology*, Quaestiones Math., **6** (1983), 265–283.

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