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TANGENTS TO A MULTIPLE PLANE CURVE

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The limiting behavior of the tangents and the flexes are computed as a reduced plane curve degenerates into a multiple plane curve.

0. Introduction. In this paper, we consider the degeneration of a reduced irreducible plane curve to a multiple plane curve. We study the associated degeneration of tangent lines by viewing a line as a linear imbedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^2$ and studying deformations of this imbedding. We compute the limiting behavior of the dual curve and the flexes. A similar computation yields the limiting behavior of the bitangents; this will appear later in a separate paper. The main result is stated as Proposition (2.1).

The author takes this opportunity to thank Professor William Fulton for introducing him to this problem.

1. The dual of a multiple curve. Let $C \subset \mathbf{P}^2_{\mathbf{C}}$ be a smooth curve of degree d. $C^* \subset \mathbf{P}^{2*}$ will denote the dual curve of tangents to C.

Let n be a positive integer, $n \geq 2$. Let

$$G^n + tF = 0 \tag{1.1}$$

be a generic pencil of plane curves, with $\deg G = d$, $\deg F = nd$. We will freely abuse notation by using the same letter to denote a polynomial or its zero locus. Here, generic means that G, F are smooth, and meet transversely at their nd^2 points of intersection, the base points of the pencil. G^* is assumed to have only nodes and cusps as singularities. The pencil (1.1) will be denoted by C_t. Let $C_0^* = \lim_{t \to 0} C_t^*$. The goal of this section is to prove the following.

(1.2) PROPOSITION. *C_0^* is the union of G^* with multiplicity n, together with the nd^2 pencils of lines through the base points, each pencil having multiplicity $(n-1)$.*

REMARKS. (1) Proposition (1.2) is quite elementary. It is not much more difficult than the case $n = 2$, $d = 1$ implicitly worked out in [**4**]. The value in this method of proof lies purely in its expository value as a prelude to §2.

(2) By a standard formula for plane curves ([2], for example) $\deg C_t^* = nd(nd-1)$ for $t \neq 0$, while $\deg C_0^* = nd(d-1) + (n-1)nd^2 = nd(nd-1)$.

The techniques used are a variant of the techniques of [3], which were inspired by the work of Clemens. Given a line $L \subset \mathbf{P}^2$, we look for a family of lines L_s with $L_0 = L$ and L_s tangent to C_t with $t = s^r$ for a positive integer r. Then L would correspond to a general point of a multiplicity r component of C_0^* with cyclic local monodromy.

We choose an isomorphism $\alpha\colon \mathbf{P}^1 \to L$ given by three homogeneous linear forms $\alpha = (\alpha_0(u,v), \alpha_1(u,v), \alpha_2(u,v))$, where (u,v) are homogeneous coordinates on $\mathbf{P}^1$. We single out $(1,0) \in \mathbf{P}^1$ as the candidate for a point of tangency of L with C_0. We look for an extension of α to $\alpha(s)$, holomorphic in s for $|s| < \varepsilon$, with $\alpha(0) = \alpha$, and satisfying

$$(1.3) \qquad (G^n + s^r F) \circ \alpha(s) \equiv 0 \; (v^2) \quad \text{for } |s| < \varepsilon.$$

We attempt to solve (1.3) by power series in s. We show that this is possible when either L is tangent to G or when L passes through a base point. In the former case, for general L, we must take $r = n$, while in the latter case, we take $r = n - 1$. By consideration of degrees, i.e. Remark (2), no other components are present, proving Proposition (1.2).

We now fix some more notation. Let P_k denote the vector space of homogeneous forms of degree k on $\mathbf{P}^1$. There is a linear map

$$(1.4) \qquad \Phi_G\colon P_1^3 \to P_d, \; \Phi_G(\sigma_0, \sigma_1, \sigma_2) = \sum_{j=0}^{2} \sigma_j \left(\frac{\partial G}{\partial X_j} \circ \alpha \right)$$

and for each integer $k \geq 0$, the related map

$$(1.5) \qquad \Phi_G^{(k)}\colon P_1^3 \overset{\Phi_G}{\to} P_d \to P_d/(v^{k+1}).$$

(1.6) Lemma. *For any L, $\Phi_G^{(1)}$ is surjective (hence also $\Phi_G^{(0)}$).*

Proof. Since G is smooth, we may change coordinates so $\psi = \partial G/\partial X_0 \circ \alpha \not\equiv 0\,(v)$, so that ψ is a unit in the graded ring $R_1 = \bigoplus_j P_j/(v^2)$. Thus any $Q \in P_d/(v^2)$ can be divided by ψ (mod v^2) to yield $\sigma \in P_1$; then $\Phi_G^{(1)}(\sigma, 0, 0) = Q$. □

We introduce some more notation to facilitate higher order computations. Let

$$\alpha^{(r)} = \left(\left. \frac{d^r\alpha_i}{ds^r} \right|_{s=0} \right)_{i=0,1,2}, \quad G_{ij}\alpha^{(r)}\alpha^{(s)} = \sum_{i,j} \left(\frac{\partial^2 G}{\partial X_i \partial X_j} \right) \alpha_i^{(r)} \alpha_j^{(s)}.$$

We also note that homogeneous polynomials of degree j in (u, v) can be viewed as polynomials of degree $\leq j$ in v; we will hence usually view $P_j/(v^{k+1}) \subset \mathbf{C}[v]/(v^{k+1})$, and speak of constant terms, linear terms, etc. We also freely divide truncated polynomials.

We start by specializing to the case $n = 2$ to fix ideas.

(1.7) PROPOSITION. (1.2) *is true for* $n = 2$.

Proof. We set $n = 2$, $r = 1$ (so that $s = t$) in (1.3), and let $t = 0$ to obtain

$$G^2 \equiv 0\ (v^2) \tag{1.8}$$

where we have abused notation by viewing G as a form on $\mathbf{P}^1$ via α. This gives

$$G \equiv 0\ (v). \tag{1.9}$$

We continue by differentiating (1.3) with respect to t and setting $t = 0$.

$$2G\Phi_G\alpha' + F \equiv 0\ (v^2) \tag{1.10}$$

Using (1.9), (1.10) forces $F = 0\ (v)$, i.e.

(1.11) L passes through a base point.

To show that the pencil containing L indeed has multiplicity 1 in C_0^*, we may take L general, and so assume G is not tangent to $L \simeq \mathbf{P}^1$ at $(1, 0)$. We then obtain from (1.10)

$$\Phi_G^{(0)}\alpha' = -F/2G. \tag{1.12}$$

and Lemma 1.6 implies that we can solve (1.12) for α'. Thus the pencils through the base points deform to first order; these pencils are the only candidates for a multiplicity 1 component of C_0^*.

For the second order obstruction, we take the second derivative of (1.3) with respect to t and set $t = 0$ to obtain

$$2G\Phi_G\alpha'' + 2GG_{ij}\alpha'\alpha' + 2\left(\Phi_G\alpha'\right)^2 + 2\Phi_F\alpha' \equiv 0\ (v^2). \tag{1.13}$$

In order for (1.13) to have a solution for α'', we must require that

$$(1.14)\qquad 2\left(\Phi_G\alpha'\right)^2 + 2\Phi_F(\alpha') \equiv 0\ (v).$$

This can be accomplished by the following lemma.

(1.15) LEMMA. *$\Phi_F^{(0)}|_{\ker\Phi_G^{(1)}}$: $\ker\Phi_G^{(1)} \to P_{nd}/(v)$ is surjective.*

Proof. Since $\dim P_{nd}/(v) = 1$, the lemma can fail to hold only if $\ker\Phi_G^{(1)} \subset \ker\Phi_F^{(0)}$. But since F and G intersect transversally, we can change coordinates in $\mathbf{P}^2$ so that $X_0 = 0$ is tangent to F, and $X_1 = 0$ is tangent to G at $\alpha(1,0)$. So we may assume that, in the affine coordinate v near $(1,0) \in \mathbf{P}^1$, $(\partial G/\partial X_0)(\alpha(v)) \equiv av\ (v^2)$, $(\partial G/\partial X_1)(\alpha(v)) \equiv b\ (v)$, where $b \neq 0$. Then $(-bu, av, 0) \in \ker\Phi_G^{(1)} - \ker\Phi_F^{(0)}$.

Now we can replace α' with $\alpha' - \tilde{\alpha}$, where $\tilde{\alpha} \in \ker\Phi_G^{(1)}$ and $\Phi_F^{(0)}\tilde{\alpha} \equiv (\Phi_G\alpha')^2\ (v)$, by the lemma. Then (1.12) still holds, but now the left-hand side of (1.13) is divisible by G, since (1.14) now holds. After dividing (1.13) by G, we can now solve for α'' by using lemma (1.6) again.

For simplicity, we introduce the symbol Q_j to stand for any expression involving α only through $\alpha', \alpha'', \dots, \alpha^{(j)}$. The higher order obstructions are now handled by the following easily established lemma.

(1.16) LEMMA. *For $n \geq 2$, the nth obstruction to* (1.3) *is*

$$\frac{d^n}{dt^n}(G^2 + tF)\bigg|_{t=0} \equiv 2G\Phi_G\alpha^{(n)} + n\Phi_F\alpha^{(n-1)}$$
$$+ 2n\Phi_G\alpha'\Phi_G\alpha^{(n-1)} + GQ_{n-1} + Q_{n-2} \equiv 0\ (v^2). \qquad \square$$

We inductively complete the power series solution of (1.3). We suppose that we have solved for $\alpha', \dots, \alpha^{(n-1)}$. Then using Lemma 1.15, we modify $\alpha^{(n-1)}$ so that (1.16) becomes divisible by G. After dividing by G, we use Lemma (1.6) once more to solve for $\alpha^{(n)}$.

This procedure gives a formal power series solution of (1.3). By Artin's theorem [**1**] there is a holomorphic solution of (1.3) for $|t| < \varepsilon$. Thus, the pencils through the base points are each multiplicity 1 components of C_0^*.

REMARK. The solution for $\alpha^{(n)}$ is far from unique; in fact, the computation above shows that the ambiguity lies in $\ker\Phi_G^{(0)} \cap \ker\Phi_F^{(0)}$, a 4-dimensional vector space. Let $B \subset \mathrm{GL}(2)$ denote the isotropy group of

$(1,0)$, so that $\dim B = 3$. This is the ambiguity arising by representing L as $(\mathbf{P}^1, (1,0))$. The difference between 4 and 3 reflects that a curve (the pencil) is deforming.

The other component $2G^*$ is found by letting $n = 2$, $t = s^2$ in (1.3). The order zero obstruction again leads to (1.9), which holds for a tangent to G (in fact, $G \equiv 0\ (v^2)$). The first order obstruction is

$$(1.17) \qquad 2G\Phi_G\alpha' \equiv 0\ (v^2)$$

which is again automatic, and puts no restrictions on α'.

The second order obstruction is

$$(1.18) \qquad 2G\Phi_G\alpha'' + 2(\Phi_G\alpha')^2 + 2GG_{ij}\alpha'\alpha' + 2F \equiv 0\ (v^2).$$

This equation can be solved for α'' provided that

$$(1.19) \qquad (\Phi_G\alpha')^2 \equiv -F\ (v^2).$$

We can assume that L does not pass through a base point (i.e. $F \not\equiv 0\ (v)$). After taking a square root, Lemma (1.6) ensures that we can find such an α', and (1.18) imposes no conditions on α''. For the higher order obstructions we need an easy lemma.

(1.20) LEMMA. *For $n \geq 2$, the nth obstruction is*

$$\left.\frac{d^n}{ds^n}(G^2 + s^2F)\right|_{s=0} \equiv 2G\Phi_G\alpha^{(n)} + 2n\Phi_G\alpha'\Phi_G\alpha^{(n-1)} + GQ_{n-1} + Q_{n-2} \equiv 0\ (v^2). \qquad \square$$

Since $\Phi_G\alpha'$ is a unit in $P_d/(v^2)$, we can choose $\alpha^{(n-1)}$ to ensure that there is no nth obstruction, using Lemma (1.6). Thus, there is a formal power series solution of (1.3) with $t = s^2$, and Artin's Theorem finishes the proof of Proposition (1.7). $\square$

REMARK. In the case of tangents, the ambiguity lies in $\ker \Phi_G^{(1)}$, which is as before a 4-dimensional vector space.

Proof of Proposition (1.2). We start by letting $t = s^{n-1}$ in (1.3), and attempt to deform a pencil through a base point. There are clearly no obstructions through order $n-2$. The $(n-1)$st obstruction is (since

$G^j \equiv 0\,(v^2)$ for $j \geq 2$)

$$n!G\left(\Phi_G\alpha'\right)^{n-1} + (n-1)!F \equiv 0\,(v^2). \tag{1.21}$$

We may assume L is not tangent to G or F; then we can solve (1.21) for α'.

The nth order obstruction is seen to be

$$\begin{aligned} n!\binom{n}{2}G\left(\Phi_G\alpha'\right)^{n-2}\Phi_G\alpha'' + n!\left(\Phi_G\alpha'\right)^{n} + n!\Phi_F\alpha' \\ + GQ_n \equiv 0\,(v^2). \end{aligned} \tag{1.22}$$

We now can use Lemma (1.15) to modify α' so that (1.22) is consistent. After dividing (1.22) by G, and noting that $\Phi_G^{(1)}\alpha'$ is a unit, we can then solve for α''.

For the higher order obstructions, we note that for $r \geq n+1$

$$\begin{aligned} &\frac{d^r}{ds^r}(G^n + s^{n-1}F)\bigg|_{s=0} \\ &\equiv n(n-1)\frac{r!}{(r-n+2)!}G\left(\Phi_G\alpha'\right)^{n-2}\Phi_G\alpha^{(r-n+2)} \\ &+n\frac{r!}{(r-n+2)!}\left(\Phi_G\alpha'\right)^{n-1}\Phi_G\alpha^{(r-n+1)} \\ &+\frac{r!}{(r-n+1)!}\Phi_F\alpha^{(r-n+1)} + GQ_{r-n+1} + Q_{r-n} \equiv 0\,(v^2). \end{aligned} \tag{1.23}$$

As before, we can use Lemma (1.15) inductively to modify $\alpha^{(r-n+1)}$ to ensure the consistency of (1.23), then solve for $\alpha^{(r-n+2)}$ using Lemma (1.6). Finally, Artin's Theorem shows that a pencil through a base point is a multiplicity $(n-1)$ component of C_0^*.

Turning next to the tangents to G (so that $G \equiv 0\,(v^2)$), we let $t = s^n$ in (1.3). There are clearly no obstructions through order $(n-1)$.

The nth order obstruction yields

$$n!\left(\Phi_G\alpha'\right)^{n} + n!F \equiv 0\,(v^2). \tag{1.24}$$

Assuming that L does not pass through a base point, we can solve (1.24) for α'.

For the higher order obstructions, we note that for $r \geq n+1$

$$\begin{aligned} \frac{d^r}{ds^r}(G^n + s^nF)\bigg|_{s=0} \equiv n\frac{r!}{(r-n+1)!}\left(\Phi_G\alpha'\right)^{n-1}\Phi_G\alpha^{(r-n+1)} \\ + Q_{r-n} \equiv 0\,(v^2). \end{aligned} \tag{1.25}$$

As $\Phi_G \alpha'$ is a unit, we can solve for $\alpha^{(r-n+1)}$. Artin's Theorem completes the proof. □

2. Flexes on a multiple curve. In the situation of §1, we look at the limiting behavior of the flexes of C_t.

(2.1) PROPOSITION. *The flexes of C_t degenerate to the flexes of G, the tangents to F at a base point, and the tangents to G at a base point, with multiplicities n, $n-2$, $2n-1$ respectively.*

Proof. By a standard formula for plane curves [2], C_t has $3nd(nd-2)$ flexes; g has $3d(d-2)$ flexes and nd^2 base points. Also $3nd(nd-2) = n(3d(d-2)) + (n-2)nd^2 + (2n-1)nd^2$. So as in §1, it suffices to construct deformations of the claimed limits with the indicated multiplicities.

We now need to solve

$$(G^n + s^r F) \circ \alpha(s) \equiv 0 \ (v^3) \quad \text{for } |s| < \varepsilon \tag{2.2}$$

for $r = n$ in the case of a flex of G, for $r = n-2$ in the case of a tangent to F at a base point, and for $r = 2n-1$ in the case of a tangent to G at a base point.

We first check the flexes of G, starting with a lemma.

(2.3) LEMMA. *If L is an ordinary inflectional tangent to G, then $\Phi_G^{(2)}$ is surjective.*

Proof. We can change coordinates so that L has equation $X_1 = 0$, and G has an equation of the form $X_1 f + X_0^3 g$, where $f(0,0,1)$, $g(0,0,1) \neq 0$. We may as well let α: $\mathbf{P}^1 \to L$ be $\alpha(u,v) = (v,0,u)$. Then, using subscript notation for partial derivatives, we find that

$$G_0 \circ \alpha = 3v^2 g + v^3 g_0 \qquad G_1 \circ \alpha = f + v^3 g_1$$

and so $\Phi_G^{(2)}$ is surjective by inspection. □

The proof of the case of flexes is now completed by mimicking the computation of the component nG^* of §1, using Lemma (2.3) in place of Lemma (1.6).

We turn next to the case of a tangent to F at a base point, i.e. $G \equiv 0$ (v), $F \equiv 0$ (v^2), $t = s^{n-2}$.

There are clearly no obstructions through order $n-3$. For the order $n-2$ obstruction, we note that

$$(2.4)\qquad \frac{d^{n-2}}{ds^{n-2}}(G^n + s^{n-2}F)\bigg|_{s=0} \equiv \frac{n!}{2}G^2\left(\Phi_G\alpha'\right)^{n-2} + (n-2)!F \equiv 0\ (v^3)$$

and since F, G have order exactly 2, 1 respectively as polynomials in v, F/G^2 is a unit, so we can extract an $(n-2)$ root and solve for $\Phi_G^{(0)}\alpha'$ in (2.4).

The higher order obstructions are given by

$$(2.5)\qquad \frac{d^k}{ds^k k}(G^n + s^{n-2}F)\bigg|_{s=0}$$
$$= \frac{n(n-1)(n-2)}{2}\frac{k!}{(k+3-n)!}G^2\left(\Phi_G\alpha'\right)^{n-3}\Phi_G\alpha^{(k+3-n)}$$
$$+ G^2Q_{k+2-n} + n(n-1)\frac{k!}{(k+2-n)!}G\left(\Phi_G\alpha'\right)^{n-2}\Phi_G\alpha^{(k+2-n)}$$
$$+ \frac{k!}{(k+2-n)!}\Phi_F\alpha^{(k+2-n)} + Q_{k+1-n}\ (v^3).$$

This case is finished by a couple of lemmas.

(2.6) LEMMA. $\Phi_F^{(1)}|_{\ker\Phi_G^{(0)}}\colon \ker\Phi_G^{(0)} \to P_{nd}/(v^2)$ *is surjective.*

Proof. Lemma 1.15 says that $\dim\ker\Phi_G^{(1)} \cap \ker\Phi_F^{(0)} = 3$. Reversing the roles of F and G yields the lemma. □

(2.7) LEMMA. *After solving for the kth obstruction, we have* ∞^3 *solutions for* $\alpha^1, \ldots, \alpha^{(k+2-n)}$, *and* $\Phi_G^{(0)}\alpha^{(k+3-n)}$ *is determined.*

Proof. Inductively, we equate the linear plus constant term of (2.5) to $0\ (v^2)$, using Lemma (2.6) to modify $\alpha^{(k+2-n)}$. $\Phi_G^{(0)}\alpha^{(k+3-n)}$ is now found by Lemma (1.6). □

An application of Artin's Theorem completes the proof of the case of a tangent to F at a base point.

Finally, we turn to a tangent to G at a base point, i.e. $G \equiv 0\ (v^2)$, $F \equiv 0\ (v)$, $t = s^{2n-1}$.

There are clearly no obstructions through order $n-2$. The order $n-1$ obstruction is

$$(2.8)\qquad \frac{d^{n-1}}{ds^{n-1}}(G^n + s^{2n-1}F)\bigg|_{s=0} \equiv n!G\left(\Phi_G\alpha'\right)^{n-1} \equiv 0\ (v^3)$$

which forces

$$\Phi_G^{(0)}\alpha' = 0. \tag{2.9}$$

We change notation slightly, putting $G^{(j)} = d^j(G \circ \alpha(s))/ds^j|_{s=0}$, noting that $G^{(j)} = \Phi_G \alpha^{(j)} + Q_{j-1}$. With the additional information (2.9), we now see that there are no obstructions through order $2n - 3$. The order $2n - 2$ obstruction is given by

$$\begin{aligned} (2.10)\quad & \frac{d^{2n-2}}{ds^{2n-2}}(G^n + s^{2n-1}F)\Big|_{s=0} \\ &= \frac{n(2n-2)!}{2^{n-1}} G(G'')^{n-1} + \frac{n(n-1)(2n-2)!}{2^{n-1}}(G')^2(G'')^{n-2} \\ &\equiv 0\ (v^3). \end{aligned}$$

This leads to

$$G'' \equiv -(n-1)(G')^2/G\ (v). \tag{2.11}$$

The order $2n - 1$ obstruction is

$$\begin{aligned} (2.12)\quad & \frac{d^{2n-1}}{ds^{2n-1}}(G^n + s^{2n-1}F)\Big|_{s=0} \\ &= \frac{n(n-1)(2n-1)!}{6 \cdot 2^{n-2}} G(G'')^{n-2}G''' \\ &\quad + \frac{n(n-1)(n-2)(2n-1)!}{6 \cdot 2^{n-2}}(G')^2(G'')^{n-3}G''' \\ &\quad + \frac{n(2n-1)!}{2^{n-1}} G'(G'')^{n-1} + (2n-1)!F \\ &\equiv 0\ (v^3) \end{aligned}$$

looking at the linear term, and using (2.11), we find

$$(G')^{2n-1}/G^{n-1} \equiv (-1)^n \frac{2^{n-1}}{n(n-1)^{n-1}} F\ (v^2). \tag{2.13}$$

(2.12) implies that we can solve for $\Phi_G^{(1)}\alpha'$, and that G'' is a unit, using (2.11) again.

Turning to the quadratic term of (2.12), we see that we must solve for $G'''(v)$, or equivalently, for $\Phi_G^{(0)}\alpha'''$. This is possible exactly when the expression multiplying G''' in (2.12) is divisible by v^2, but not by v^3. But this expression is a multiple of

$$(G'')^{n-3}\left[GG'' + (n-2)(G')^2\right] \tag{2.14}$$

which satisfies the indicated requirement, by (2.11) and the fact that G'' is a unit.

Notice that $\Phi_G^{(0)}\alpha''$ depends only on α', while $\Phi_G^{(0)}\alpha'''$ depends on $\Phi_G^{(1)}\alpha''$ and α'; however, it is a non-trivial linear expression in the linear term of $\Phi_G^{(1)}\alpha''$, as revealed by a examination of our solution of (2.12).

The higher order obstructions are given by

$$\begin{aligned}
(2.15)\quad & \frac{d^k}{ds^k}(G^n + s^{2n-1}F)\Big|_{s=0} \\
&= \frac{n(n-1)k!}{2^{n-2}(k+4-2n)!}G(G'')^{n-2}G^{(k+4-2n)} \\
&+ \frac{n(n-1)(n-2)k!}{2^{n-2}(k+4-2n)!}(G')^2(G'')^{n-3}G^{(k+4-2n)} + GQ_{k+3-2n} \\
&+ (G')^2\tilde{Q}_{k+3-2n} + \frac{n(n-1)k!}{2^{n-2}(k+3-2n)!}G'(G'')^{n-2}G^{(k+3-2n)} \\
&+ G'Q_{k+2-2n} + \frac{nk!}{2^{n-1}(k+2-2n)!}(G'')^{n-1}G^{(k+2-2n)} \\
&+ Q_{k+1-2n} + \frac{k!}{(k+1-2n)!}\Phi_F(\alpha^{(k+1-2n)}) \\
&\equiv 0\ (v^3).
\end{aligned}$$

Equation (2.15) can be solved inductively.

(2.16) LEMMA. *After solving for the kth obstruction, we have ∞^3 solutions for $\alpha', \ldots, \alpha^{(k+1-2n)}$, we have found $\Phi_G^{(1)}\alpha^{(k+2-2n)}$, and we have found $\Phi_G^{(0)}\alpha^{(k+4-2n)}$. This last depends non-trivially and linearly on the linear term of $\Phi_G\alpha^{(k+3-2n)}$, and on terms of lower order.*

Proof. By induction. We start by examining the constant term of (2.15). We observe that the constant term of $G^{(k+2-2n)}$ depends on $\Phi_G^{(1)}\alpha^{(k+1-2n)}$ and lower derivatives of α. Also we note that the expression Q_{k+1-2n} in (2.15) depends on $\Phi_G^{(0)}\alpha^{(k+1-2n)}$ and lower derivatives of α. So Lemma (1.15) applies to allow for the modification of $\alpha^{(k+1-2n)}$ as before.

Next, we consider the linear term of (2.15). We observe that the constant term of Q_{k+2-2n} depends on $\Phi_G^{(0)}\alpha^{(k+2-2n)}$ and lower derivatives of α, while inductively the constant term of $G^{(k+3-2n)}$ depends non-trivially and linearly on the linear term of $\Phi_G\alpha^{(k+2-2n)}$ and on lower order terms, so that after equating the linear term of (2.15) to 0, we can first solve for the linear term of $\Phi_G\alpha^{(k+2-2n)}$ (hence for $\Phi_G^{(1)}\alpha^{(k+2-2n)}$, as we inductively know the constant term). Lemma (1.16) allows us to solve for $\alpha^{(k+2-2n)}$.

Finally, we turn to the quadratic term. Exactly as in the order $2n - 1$ obstruction, we see that $\Phi_G^{(0)}\alpha^{(k+4-2n)}$ is multiplied by a constant multiple of (2.14), which we have seen is divisible by v^2, but not by v^3. So we can solve for $\Phi_G^{(0)}\alpha^{(k+4-2n)}$, and apply Lemma (1.6). Note that the quadratic term of (2.15) involves $\alpha^{(k+3-2n)}$ only non-trivially and linearly through the linear term of $\Phi_G\alpha^{(k+3-2n)}$, completing the induction. □

An application of Artin's Theorem now finishes the case of tangents to G through a base point, as well as the proof of Proposition (2.1). □

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