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It is known that if L is a separable, finite dimensional extension of a field K and if v is a proper valuation (absolute value) on K, then each ring topology on L whose restriction to K is the topology $\mathscr{T}_v$ defined on K by v is the supremum of a finite family of valuation (absolute valued) topologies. We give a characterization of the fields K and L and the valuations (absolute values) v on K for which each ring topology on L extending $\mathscr{T}_v$ is the supremum of a family of valuation (absolute valued) topologies on K when L is an arbitrary finite dimensional extension of K.

Let R be a ring and let $\mathscr{T}$ be a ring topology on R, that is, $\mathscr{T}$ is a topology on R making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to R. A subset A of R is *bounded* for $\mathscr{T}$ if given any neighborhood U of zero, there exists a neighborhood V of zero such that $VA \cup AV \subseteq U$. $\mathscr{T}$ is a *locally bounded topology* on R if there exists a fundamental system of neighborhoods of zero for $\mathscr{T}$ consisting of bounded sets.

Recall that a *norm* N on a ring R is a function from R to the nonnegative reals satisfying $N(x) = 0$ if and only if $x = 0$, $N(x - y) \leq N(x) + N(y)$ and $N(xy) \leq N(x)N(y)$ for all x and y in R. Each norm N on R defines a locally bounded topology $\mathscr{T}_N$ on R in a natural way. In particular, if $|\cdot\cdot|$ is a proper absolute value on a field K, then there exists a locally bounded topology $\mathscr{T}_{|\cdot\cdot|}$ on K defined by $|\cdot\cdot|$. We note further that if N is a nontrivial norm on a field K, that is, $\mathscr{T}_N$ is nondiscrete, then a subset A of K is bounded in norm if and only if A is a $\mathscr{T}_N$-bounded subset of K.

Each proper valuation v on a field K defines a locally bounded topology $\mathscr{T}_v$ on K as well. If each of v and w is a valuation or an absolute value on K, then v and w are *independent* if $\mathscr{T}_v \neq \mathscr{T}_w$.

In [**11**], Rigo and Warner proved that if L is a separable, finite dimensional extension of a field K and if v is a proper valuation (absolute value) on K, then each ring topology on L inducing $\mathscr{T}_v$ on K is the supremum of a finite family of valuation (absolute valued) topologies on L (Theorem 2). In this paper we characterize the fields K and L and valuations (absolute values) v on K for which each ring topology on L

extending $\mathscr{T}_v$ is the supremum of a finite family of valuation (absolute valued) topologies on L when L is an arbitrary finite dimensional extension of K.

THEOREM 1. *Let K be a field, v a proper valuation (absolute value) on K, $\hat{K}$ the completion of K for $\mathscr{T}_v$, L a purely inseparable, finite dimensional extension of K, w the unique extension of v to L and $\hat{L}$ the completion of L for $\mathscr{T}_w$. The following are equivalent.*

1°. $[\hat{L}:\hat{K}] = [L:K]$.

2°. *$\mathscr{T}_w$ is the only ring topology on L whose restriction to K is $\mathscr{T}_v$.*

3°. *$\mathscr{T}_w$ is the only locally bounded topology on L whose restriction to K is $\mathscr{T}_v$.*

Proof. We first consider the case when $[L:K] = p$ where p is the characteristic of K.

Suppose $[\hat{L}:\hat{K}] = p$. Then there exists a in $L \setminus \hat{K}$. Hence $L = K(a)$ and the minimal polynomial of a over K is irreducible in $\hat{K}[X]$. Thus by [**11**, Corollary 2 of Theorem 1], $\mathscr{T}_w$ is the only ring topology on L whose restriction to K is $\mathscr{T}_v$.

Clearly 2° implies 3°. So it suffices to prove that if $\mathscr{T}_w$ is the only locally bounded topology on L whose restriction to K is $\mathscr{T}_v$, then $[\hat{L}:\hat{K}] = p$. If $[\hat{L}:\hat{K}] = 1$, let $a \in L \setminus K$ and let $f(X)$ be the minimal polynomial of a over K. Then $f(X) = (X-a)^p$ and $X - a \in \hat{K}[X]$. Hence by [**11**, Theorem 1], there are p ring topologies $\mathscr{T}_1, \ldots, \mathscr{T}_p$ on L inducing $\mathscr{T}_v$ on K and the completion $\hat{L}_i$ of L for $\mathscr{T}_i$ is a finite dimensional $\hat{K}$-algebra for each $i \in [1, p]$. If v is a valuation on K, let $\hat{v}$ be its extension to $\hat{K}$, let G be the order group of $\hat{v}$ and let $\{x_1, \ldots, x_n\}$ be a basis for $\hat{L}_i$ over $\hat{K}$ where $x_1 = 1$. Then $\{V_\alpha : \alpha \in G\}$ is a fundamental system of neighborhoods of zero for a Hausdorff topology on $\hat{L}_i$ compatible with the vector space structure of $\hat{L}_i$ where for each $\alpha \in G$,

$$V_\alpha = \left\{ \sum_{j=1}^{n} a_j x_j : a_j \in \hat{K}, \inf\{\hat{v}(a_j) : 1 \leq j \leq n\} \geq \alpha \right\}.$$

Hence by [**8**, Theorem 7], $\{V_\alpha : \alpha \in G\}$ is a fundamental system of neighborhoods of zero for the completion $\hat{\mathscr{T}}_i$ of L for $\mathscr{T}_i$. It follows that the restriction of $\hat{\mathscr{T}}_i$ to $\hat{K}$ is the topology defined on $\hat{K}$ by $\hat{v}$. Thus as $L \subseteq \hat{K}$, $\hat{\mathscr{T}}_i|_L$ is a locally bounded topology on L, that is, each $\mathscr{T}_i$ is a locally bounded topology on L, a contradiction. If v is an absolute value on K, then each $\mathscr{T}_i$ is normable and hence locally bounded. Indeed, by [**2**, Theorem 2, p. 27; 3, Proposition 10, p. 69 and Theorem 1, p. 70], there exist a vector space norm N on $\hat{L}_i$ and a positive number c such that

$N(xy) \leq cN(x)N(y)$ for all x and y in $\hat{L}_i$. Therefore the function $\|\cdot\cdot\|$ defined on $\hat{L}_i$ by, $\|x\| = cN(x)$, is an algebra norm on $\hat{L}_i$ defining the topology on $\hat{L}_i$. So $[\hat{L}:\hat{K}] = p$ by [**11**, Corollary 1 of Theorem 1].

Now let L be an arbitrary, purely inseparable, finite dimensional extension of K.

Suppose that $[\hat{L}:\hat{K}] = [L:K]$ and let $\mathcal{T}$ be a ring topology on L whose restriction to K is $\mathcal{T}_v$. Let K_1 be a maximal subfield of L containing K such that $\mathcal{T}|_{K_1}$ is defined by a valuation (absolute value) v_1 extending v to K_1. If $K_1 \neq L$, let $a \in L \setminus K_1$ be such that $[K_1(a):K_1] = p$. Denote $K_1(a)$ by K_2, let v_2 be the unique extension of v to K_2, let $\hat{K}_1$ be the completion of K_1 for $\mathcal{T}_{v_1}$ and let $\hat{K}_2$ be the completion of K_2 for $\mathcal{T}_{v_2}$. If $a \notin \hat{K}_1$, then by the previous argument, $\mathcal{T}|_{K_2} = \mathcal{T}_{v_2}$, contradicting the maximality of K_1. If $a \in \hat{K}_1$, then $[\hat{K}_2:\hat{K}_1] = 1$. So $[\hat{L}:\hat{K}] = [\hat{L}:\hat{K}_2][\hat{K}_1:\hat{K}] \leq [L:K_2][K_1:K] < [L:K]$, a contradiction. Hence $K_1 = L$.

Assume 3° holds. Let $[L:K] = p^n$ and let $a_1, a_2, \ldots, a_n \in L$ be such that $L = K(a_1, \ldots, a_n)$, $[K(a_1):K] = p$ and for all $i \in [1, n-1]$, $[K(a_1, \ldots, a_{i+1}): K(a_1, \ldots, a_i)] = p$. Denote K by K_0. For each $i \in [1, n]$ let $K_i = K(a_1, \ldots, a_i)$, let v_i be the unique extension of v to K_i and let $\hat{K}_i$ be the completion of K_i for $\mathcal{T}_{v_i}$. If $p^n > [\hat{L}:\hat{K}]$, then as $[\hat{L}:\hat{K}] = \prod_{i=0}^{n-1} [\hat{K}_{i+1}:\hat{K}_i]$, there exists an i such that $[\hat{K}_{i+1}:\hat{K}_i] = 1$. So by the previous argument there exists a locally bounded topology $\mathcal{T}'$ on K_{i+1} whose restriction to K_i is $\mathcal{T}_{v_i}$ but $\mathcal{T}' \neq \mathcal{T}_{v_{i+1}}$. By [**12**, Satz 1.6], $\mathcal{T}'$ extends to a locally bounded topology $\mathcal{T}$ on L. Clearly $\mathcal{T}|_K = \mathcal{T}_v$ but $\mathcal{T} \neq \mathcal{T}_w$, a contradiction. So $[\hat{L}:\hat{K}] = p^n = [L:K]$.

THEOREM 2. *Let L be a finite dimensional extension of a field K, let D be the separable closure of K in L, let v be a proper valuation (absolute value) on K and let $\{v_i: 1 \leq i \leq m\}$ be a complete family of pairwise independent valuations (absolute values) on D extending v. For each $i \in [1, m]$, let w_i be the unique extension of v_i to L, let $\hat{L}_i$ denote the completion of L for $\mathcal{T}_{w_i}$ and let $\hat{D}_i$ denote the completion of D for $\mathcal{T}_{v_i}$. The following are equivalent.*

1°. *Each ring topology on L whose restriction to K is $\mathcal{T}_v$ is the supremum of a finite family of valuation (absolute valued) topologies on L.*

2°. *Each locally bounded topology on L whose restriction to K is $\mathcal{T}_v$ is the supremum of a finite family of valuation (absolute valued) topologies on L.*

3°. *There are $2^m - 1$ locally bounded topologies on L inducing $\mathcal{T}_v$ on K, namely the topologies $\sup_{i \in M} \mathcal{T}_{w_i}$ where M runs through all nonempty subsets of $[1, m]$.*

4°. $[\hat{L}_i : \hat{D}_i] = [L : D]$ *for all* $i \in [1, m]$.

Proof. Clearly 1° implies 2° and 3° implies 2°. We first show that 2° implies 3°. Suppose that $\mathscr{T}$ is a locally bounded topology on L and $\mathscr{T} = \sup_{1 \le i \le n} \mathscr{T}_{u_i}$ where each u_i is a proper valuation (absolute value) on L and $\mathscr{T}_{u_i} \neq \mathscr{T}_{u_j}$ for $i \neq j$. Then $\mathscr{T}_v = \mathscr{T}|_K = \sup_{1 \le i \le n} \mathscr{T}_{u_i}|_K$. As the completion of K for $\mathscr{T}_v$ is a field, the Approximation Theorem [7, Theorem 3.4, p. 292] yields that each $u_i|_K$ is equivalent to v. Hence for each $i \in [1, n]$, there exists $j(i) \in [1, m]$ such that $\mathscr{T}_{u_i} = \mathscr{T}_{w_{j(i)}}$.

We next show that 3° implies 4°. Let $\mathscr{T}$ be a locally bounded topology on L whose restriction to D is $\mathscr{T}_{v_i}$ and let M be a nonempty subset of $[1, m]$ such that $\mathscr{T} = \sup_{j \in M} \mathscr{T}_{w_j}$. Note that for $\iota, j \in M$, $\iota \neq j$, $\mathscr{T}_{w_\iota}|_D \neq \mathscr{T}_{w_j}|_D$. Then $\mathscr{T}_{v_i} = \mathscr{T}|_D = \sup_{j \in M} \mathscr{T}_{w_j}|_D$ and so the Approximation Theorem implies that the cardinality of M is one. Thus $M = \{i\}$ by the definition of w_i, that is, $\mathscr{T} = \mathscr{T}_{w_i}$. As L is a purely inseparable extension of D, 4° follows from Theorem 1.

Finally suppose that 4° holds. Let $\mathscr{T}$ be a ring topology on L whose restriction to K is $\mathscr{T}_v$. By Theorems 2 and 4 of [**11**], there exist a nonempty subset M of $[1, m]$ and ring topologies $\mathscr{T}_i$ on L for each $i \in M$ such that $\mathscr{T}_i|_D = \mathscr{T}_{v_i}$ and $\mathscr{T} = \sup_{i \in M} \mathscr{T}_i$. Hence $\mathscr{T}_i = \mathscr{T}_{w_i}$ for all $i \in M$ by Theorem 1 and so 1° holds.

COROLLARY. *Let K be the field F(X) of rational functions over the field F, let L be a finite dimensional extension of K and let v be a proper valuation or absolute value on K, improper on F. Define D,* L_i *and* $\hat{D}_i$ *as in Theorem 2. Then* $[\hat{L}_i : \hat{D}_i] = [L : D]$ *for all i and each ring topology on L inducing* $\mathscr{T}_v$ *on K is a locally bounded topology.*

Proof. First note that if v is a proper valuation on K improper on F, then v is equivalent to a real-valued valuation [**1**, Example 4, p. 106]. It suffices to establish 2° of Theorem 2. Let $\mathscr{T}$ be a locally bounded topology on L whose restriction to K is $\mathscr{T}_v$. Then there exists a nonzero topological nilpotent for $\mathscr{T}_v$ and hence for $\mathscr{T}$. So by [**5**, Theorem 6.1], there exists a norm N on L such that $\mathscr{T} = \mathscr{T}_N$. As F is a bounded subset of K for $\mathscr{T}_v$ and as $\mathscr{T}_N|_K = \mathscr{T}_v$, F is bounded in norm (for N). Consequently, F is a $\mathscr{T}$-bounded subset of L as well. Thus by [**6**, Theorem 4] and the argument used to establish Theorem 3 of [**4**], $\mathscr{T}$ is the supremum of a finite family of valuation topologies on L.

In [**9**], Nagata gave an example of fields L and K, each of prime characteristic p, and a discrete valuation v on K such that $\hat{K} = L$ is a purely inseparable extension of K of degree p over K (p. 56). Thus $\hat{L} = \hat{K}$ and so conditions 1°–4° of Theorem 2 need not hold in general.

THEOREM 3. *Let K be a field and let $\mathcal{T}_0$ denote $\sup_{1 \le i \le m} \mathcal{T}_{v_i}$ where each v_i is a proper valuation or absolute value on K and for $i \neq j$, $\mathcal{T}_{v_i} \neq \mathcal{T}_{v_j}$. Let L be a finite dimensional extension of K and let D be the separable closure of K in L. For each $i \in [1, m]$, let $\{v_{ij}\colon 1 \le j \le M(i)\}$ be a complete family of pairwise independent valuations or absolute values extending v_i to D. For each $i \in [1, m]$, $j \in [1, M(i)]$, let w_{ij} denote the unique extension of v_{ij} to L, let $\hat{L}_{ij}$ denote the completion of L for $\mathcal{T}_{w_{ij}}$ and let $\hat{D}_{ij}$ denote the completion of D for $\mathcal{T}_{v_{ij}}$. The following are equivalent.*

1°. *Each ring topology on L whose restriction to K is $\mathcal{T}_0$ is the supremum of a finite family $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ of topologies on L where for each i, $\mathcal{T}_i$ is defined by a proper valuation or absolute value on L.*

2°. *Each locally bounded topology on L whose restriction to K is $\mathcal{T}_0$ is the supremum of a finite family $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ of topologies on L where for each i, $\mathcal{T}_i$ is defined by a proper valuation or absolute value on L.*

3°. *There are $\prod_{i=1}^{m}(2^{M(i)} - 1)$ locally bounded topologies on L inducing $\mathcal{T}_0$ on K, namely the topologies $\sup_{1 \le i \le m}(\sup_{j \in S(i)} \mathcal{T}_{w_{ij}})$ where $S(i)$ runs through all nonempty subsets of $[1, M(i)]$.*

4°. $[\hat{L}_{ij}\colon \hat{D}_{ij}] = [L\colon D]$ *for all $i \in [1, m]$, $j \in [1, M(i)]$.*

Proof. Clearly 1° implies 2° and 3° implies 2°. We first prove that 2° implies 3°. Let $\mathcal{T}$ be a locally bounded topology on L inducing $\mathcal{T}_0$ on K. Then $\mathcal{T} = \sup_{1 \le i \le n} \mathcal{T}_{u_i}$ where each u_i is a proper valuation or absolute value on L and $\sup_{1 \le j \le m} \mathcal{T}_{v_j} = \sup_{1 \le i \le n} \mathcal{T}_{u_i}|_K$. Suppose that there exists an i, $1 \le i \le n$, such that for all j, $1 \le j \le m$, $\mathcal{T}_{u_i}|_K \neq \mathcal{T}_{v_j}$. Without loss of generality assume that $v_1, \dots, v_r$ are valuations on K, $v_{r+1}, \dots, v_m$ are absolute values on K and $i = 1$. If u_1 is an absolute value on L, let $a \in K$ be such that $u_1(a) > 1$, $v_j(a) > 0$ for $j \in [1, r]$ and $v_j(a) < 1$ for $j \in [r+1, m]$. (The existence of a is guaranteed by [7, Theorem 3.4, p. 292].) Then $\{a^t\colon t = 1, 2, \dots\}$ is a bounded set for $\sup_{1 \le j \le m} \mathcal{T}_{v_j}$ but not for $\mathcal{T}_{u_1}|_K$, a contradiction. (Indeed, if $\{a^t\colon t = 1, 2, \dots\}$ is bounded for $\sup_{1 \le i \le n} \mathcal{T}_{u_i}|_K$, then there exists a nonzero element x in K such that $x\{a^t\colon t = 1, 2, \dots\} \subseteq \{y \in K\colon u_1(y) \le 1\}$. But $u_1(xz^t) \to \infty$ as $t \to \infty$, a contradiction.) If u_1 is a valuation on L, let G be the order group of $u_1|_K$ and for each $\alpha \in G$, let $a_\alpha \in K$ be such that $v_j(a_\alpha) > 0$ for $j = 1, 2, \dots, r$, $v_j(a_\alpha) < 1$ for $j = r+1, \dots, m$ and $u_1(a_\alpha) = \alpha$. Then $\{a_\alpha\colon \alpha \in G\}$ is a bounded set for $\sup_{1 \le j \le m} \mathcal{T}_{v_j}$ but not for $\mathcal{T}_{u_1}|_K$, a contradiction. Thus for each $i \in [1, n]$ there exists $j(i) \in [1, m]$

and $t(i) \in [1, M(j(i))]$ such that $\mathcal{T}_{u_i} = \mathcal{T}_{w_{j(i),t(i)}}$. Furthermore a similar argument establishes that for each $j \in [1, m]$, there exists an $i \in [1, n]$ such that $\mathcal{T}_{u_i}|_K = \mathcal{T}_{v_j}$.

Assume 3° holds. Suppose that there exist $i \in [1, m]$ and $j \in [1, M(i)]$ with $[\hat{L}_{ij} : \hat{D}_{ij}] < [L : D]$. By Theorem 2 there exists a locally bounded topology $\mathcal{T}$ on L whose restriction to K is $\mathcal{T}_{v_i}$ but $\mathcal{T}$ is not the supremum of a finite family of valuation or absolute valued topologies on L. Let $\mathcal{T}' = \sup(\mathcal{T}, \sup_{\iota \neq i} \mathcal{T}_{w_{\iota 1}})$. Then $\mathcal{T}'|_K = \mathcal{T}_0$ but $\mathcal{T}'$ is not the supremum of a finite family of topologies on L of the appropriate type. Indeed, if $\mathcal{T}'$ is such a supremum, then as $\mathcal{T} \subseteq \mathcal{T}'$, Theorem 4.4 of **[10]** yields that $\mathcal{T}$ is as well. Thus 4° holds.

Finally the proof that 4° implies 1° is the same as that used in Theorem 2.

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