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Let A be a singular K-3 surface with hypersurface singularities. If A has singularities other than rational singularities, then the minimal resolution of A is a ruled surface over a non-singular algebraic curve of genus q $(0 \leq q \leq 3)$, and further, under the additional conditions $q \neq 0$ and $\dim H^2(A; \mathbf{R}) = 1$, the global structure of M can be determined.

Introduction. Let A be a projective algebraic normal Gorenstein surface, namely, the canonical line bundle on the set of regular points of A is trivial in a neighbourhood of each singular point. Then we can define the canonical line bundle on A. We assume here that A has always singularities. Such a surface is called the singular del Pezzo surface (resp. singular K-3 surface) if the anti-canonical line bundle on A is ample (resp. trivial) on A. The study of the singular del Pezzo surface (resp. singular K-3 surface) was done by Brenton [**4**] and Hidaka-Watanabe [**7**] (resp. Umezu [**11**]). In particular, Umezu had an interesting result on the singularities of a singular K-3 surface.

On the other hand, these surfaces are also closely related to the study of a complex analytic compactification of $\mathbf{C}^3$ (see [**4**], [**5**]). Let (X, A) be a non-singular Kähler compactification of $\mathbf{C}^3$ such that A has at most isolated singularities. Since X is a non-singular 3-fold, A has at most isolated hypersurface singularities. Further, we can see that $\operatorname{Pic} A \cong \mathbf{Z}$ and A is isomorphic to either $\mathbf{P}^2$, or a singular del Pezzo surface, or a singular K-3 surface. In the case where A is isomorphic to $\mathbf{P}^2$ or a singular del Pezzo surface, the structure of (X, A) is determined in [**6**] (see also [**4**]).

Now, in this paper, we shall consider the singular K-3 surface. Let A be a projective algebraic singular K-3 surface and $\pi\colon M \to A$ be the minimal resolution of singularities of A. Then M is a non-singular K-3 surface or a ruled surface over a non-singular algebraic curve R of genus $q = \dim H^1(M; O_M)$. Let S be the set of singularities of A which are not rational singularities. Then $S \neq \varnothing$ if and only if M is a ruled surface over

R. Taking into account that $\operatorname{Pic} A \cong \mathbf{Z}$ implies $S \neq \varnothing$, we shall study here the singular K-3 surface A with $S \neq \varnothing$.

In §1, we discuss the structure of M as a ruled surface (see Proposition 3). In §2, we show that if the singularities of A are hypersurface singularities, then we have $0 \leq q \leq 3$ (see Propositions 5 and 6). Finally, in case of $q \neq 0$ and $\dim H^2(A; \mathbf{R}) = 1$, we determine the global structure of M (see Theorem).

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1. Preliminaries.

1°. Let A be a projective algebraic normal Gorenstein surface (see Introduction). Then we can define the canonical divisor K_A on A. We call A the singular K-3 surface if (i) the singular locus of A is not empty, (ii) $K_A = 0$, (iii) $H^1(A; O_A) = 0$. Let A be a singular K-3 surface and S be the set of singular points which are not rational double points. Let π: $M \to A$ be the minimal resolution of the singular points of A and put $\pi^{-1}(S) = C = \bigcup_{i=1}^{s_0} C_i$. Then we have

PROPOSITION 1 (*Umezu* [**11**]). *Assume that* $S \neq \varnothing$. *Then*

(1) *the canonical divisor* $K_M = -\sum_{i=1}^{s_0} n_i \cdot C_i$ $(n_i > 0)$ *and thus* M *is a ruled surface over a non-singular compact algebraic curve* R *of genus* $q = \dim H^1(M; O_M)$ (*namely,* M *is birationally equivalent to* $\mathbf{P}^1$*-bundle over* R).

(2) *if* $q \neq 1$, *then* S *consists of one point with* $p_g = \dim(R^1\pi_* O_M)_S = q + 1$.

(3) *if* $q = 1$, *then* S *consists of either one point with* $p_g = 2$ *or two points with* $p_g = 1$. *Moreover, in second case of* (3), *both of the two points are simple elliptic.*

REMARK 1. Let $b^+(A)$ be the dimension of positive eigenspace with respect to the cup product pairing $H^2(A; \mathbf{R}) \times H^2(A; \mathbf{R}) \to H^4(A; \mathbf{R}) \cong \mathbf{R}$. Then $b^+(A) = 1$ if $S \neq \varnothing$. In fact, if $S \neq \varnothing$, then $p_g(M) = 0$ since M is ruled. By Kodaira equality $b^+(M) = 2p_g(M) + 1$, where $p_g = \dim H^2(M; O_M)$, we have $b^+(M) = 1$. By Brenton [3], $b^+(A) = b^+(M)$, thus we have the claim.

In case of $S \neq \varnothing$, let $\overline{M}$ be the relatively minimal model of M and $\mu\colon M \to \overline{M}$ be the birational morphism. Then $\overline{M}$ is a $\mathbf{P}^1$-bundle over R. Then we have the following

PROPOSITION 2. *Assume that $S \neq \varnothing$. If $q \neq 0$, then we have either*

(1) *$M = \overline{M}$ and C is irreducible (in fact, C is a section of M),*

(2) *there exists an irreducible component C_{i_1} of C such that C_{i_1} is a section of M and the rest $\overline{C - C_{i_1}} = \bigcup_{i \neq i_1} C_i$ is contained in the singular fibres of M, or*

(3) *C consists of two disjoint irreducible components C_1 and C_2 which are the sections of M.*

LEMMA U_1 ([**11**]). *Let $M = M_0 \overset{\mu_1}{\to} M_1 \to \cdots \overset{\mu_n}{\to} M_n = \overline{M}$ be a sequence of blow-downs obtaining a relatively minimal model $\overline{M}$ of M. Then there exists $D_i \in |-K_{M_i}|$ $(0 \leq i \leq n)$ such that*

(i) supp(D_0) *is the union of the exceptional sets of π which correspond to the singular points in S,*

(ii) *μ_i is the blow-up with center at a point on* supp(D_i) *for $1 \leq i \leq n$,*

(iii) *$\mu_i(D_{i-1}) = D_i$ for $1 \leq i \leq n$.*

LEMMA U_2 ([**11**]). *Assume $q \geq 1$. Then $|-K_M|$ contains no irreducible curve.*

(*Proof of Proposition* 2). By Proposition 1, M is a ruled surface over a nonsingular compact algebraic curve R of genus $q > 0$ and $-K_M = \Sigma_i n_i C_i$ $(n_i > 0)$. Applying the adjunction formula for a general fibre f of M, we have

$$2 = (-K_M \cdot f) = \sum_i n_i (C_i \cdot f).$$

Thus we have the following

(i) There exist two irreducible components C_1, C_2 of C such that $n_1 = n_2 = 1$, $(C_1 \cdot f) = (C_2 \cdot f) = 1$, and $(C_i \cdot f) = 0$ for $i \geq 3$. Applying the adjunction formula for the curve C_i $(i = 1, 2)$, we have that the curve C_i $(i = 1, 2)$ is a non-singular elliptic curve with $(C_1 \cdot C_2) = 0$ and there exists no other irreducible component of C which intersects C_i $(i = 1, 2)$. Thus, by Proposition 1, we must have $C = C_1 \cup C_2$ and $-K_M = C_1 + C_2$.

(ii) There exists an irreducible component C_{i_1} such that $n_{i_1} = 2$, $(C_{i_1} \cdot f) = 1$ and $(C_i \cdot f) = 0$ $(i \neq i_1)$. Thus, $-K_M = 2C_{i_1} + \Sigma_{i \neq i_1} n_i C_i$.

(iii) There exists an irreducible component C_1 of C such that $n_1 = 1$, $(C_1 \cdot f) = 2$ and $(C_i \cdot f) = 0$ $(i \neq 1)$. Applying the adjunction formula for the curve C_1, we have that C_1 is a non-singular elliptic curve and there exists no other irreducible component of C which intersects C_1. Thus, by Proposition 1, we must have $C = C_1$ and $-K_M = C_1$.

By Lemma U_1, U_2, the case (iii) can not occur. Assume that $M = \overline{M}$. Then the case (i) cannot occur. In fact, since $M = \overline{M}$ is a $\mathbf{P}^1$-bundle over a non-singular elliptic curve in this case, $0 = (-K_M)^2$. Thus, $(C_1 + C_2)^2 = C_1^2 + C_2^2 = 0$. Since C is an exceptional curve, this is a contradiction. In case (ii), since $(C_i \cdot f) = 0$ $(i \neq i_1)$, C_i's $(i \neq i_1)$ are all fibres of M, which are not exceptional. Therefore we must have $C = C_{i_1}$, and this is a section of M. This proves (1). The assertions (2) and (3) follow from the above facts (i) and (ii). □

2°. We shall prepare some notations and results from the local theory of normal two dimensional singular points (see Laufer **[9]**, Yau **[13]**, **[14]**). Let A, $\pi\colon M \to A$, C be as in 1°. Let Z be the fundamental cycle of the singular points S with respect to the resolution $\pi\colon M \to A$. Let U be a strongly pseudoconvex neighbourhood of C in M. A cycle D on U is an integral combination of the C_i, $D = \sum d_i C_i$ $(1 \leq i \leq s_0)$, with d_i an integer. We let $\operatorname{supp} D = |D| = \bigcup C_i$, $d_i \neq 0$, denote the support of D. We put $O_D := O_U/O_U(-D)$ and $\chi(D) = \dim H^0(U;\, O_D) - \dim H^1(U;\, O_D)$. By the Riemann-Roch theorem **[10]**, we have

$$(1.1) \qquad \chi(D) = -\tfrac{1}{2}(D \cdot D + D \cdot K_U),$$

where K_U is the canonical divisor on U. Let g_i be the genus of the desingularization of C_i and μ_i be the "number" of nodes and cusps on C_i. Then, we have **[10]**

$$(1.2) \qquad C_i K_U = -C_i \cdot C_i + 2g_i - 2 + 2\mu_i$$

For two cycles D and E, we have, by (1.1),

$$(1.3) \qquad \chi(D + E) = \chi(D) + \chi(E) - D \cdot E.$$

3°. Next, we shall study the anti-canonical divisor $-K_M$ on M.

LEMMA 1. $K_M = K_U$.

PROPOSITION 3. *Assume that* $S \neq \emptyset$. *Then*

(I) $S = \{\textit{one point}\}$

(i) *if* $q = 0$, *then* $-K_M = Z$

(ii) *If* $q \neq 0$, *then* $-K_M = Z + C_{i_1}$, *where* C_{i_1} *is a section of* M *in Proposition* 2-(2).

(II) $S = \{$*two points*$\}$ (*thus* $q = 1$). *Then*, $-K_M = C_1 + C_2$, *where* C_1 *and* C_2 *are two disjoint sections of* M *in Proposition* 2-(3).

Proof. By a theorem of Laufer [9] and Lemma 1, we have (I)-(i). The assertion (II) follows directly from Proposition 2-(3). We shall show the assertion (I)-(ii). Since $(-K_M - C_{i_1}) \cdot C_{i_1} \leq 0$ $(1 \leq i \leq s_0)$, by definition of the fundamental cycle, $-K_M - C_{i_1} \geq Z$. Now, let us assume that $-K_M = Z + C_{i_1} + D$, where $D > 0$. For a general fiber f of M, $2 = -(K_M \cdot f) = Z \cdot f + C_{i_1} \cdot f + D \cdot f$. Since $C_{i_1} \subset |Z|$, $Z \cdot f = 1 = C_{i_1} \cdot f$ and $D \cdot f = 0$. This means that D is contained in the singular fibres of M. Since $H^2(M; O_M(-Z)) \cong H^0(M; O_M(-C_{i_1} - D)) \cong 0$ and $H^2(M; O_M) \cong 0$, by the Riemann-Roch theorem, we have

$$0 \geq -\dim H^1(M; O_M(-Z)) = \tfrac{1}{2}(Z \cdot Z + Z \cdot K_M) + 1 - q.$$

By Lemma 1, and (1.1), we have the inequality $\chi(Z) \geq 1 - q$. Since $H^0(U; O_Z) \cong \mathbf{C}$ by Laufer [9], $\chi(Z) = 1 - \dim H^1(U; O_Z) \leq 1$. Since S does not contain rational singularities, $\chi(Z) \neq 1$ by [1]. Therefore we have

$$1 - q \leq \chi(Z) \leq 0 \tag{1.4}$$

Since $1 - q = \chi(C_{i_1}) = \chi(-K_U - C_{i_1}) = \chi(Z + D) = \chi(Z) + \chi(D) - D \cdot Z$,

$$\chi(Z) = -\chi(D) + 1 - q + D \cdot Z. \tag{1.5}$$

By (1.4) and (1.5), $D \cdot Z \geq \chi(D)$. Since $D \cdot Z \leq 0$, $\chi(D) \leq 0$.

On the other hand, we have just seen that the support $|D|$ of D is contained in the singular fibres of M. We can easily find that the contraction of $|D|$ in M yields rational singularities. Thus, we have $\chi(D) \geq 1$. This is a contradiction. Therefore $D = 0$, namely, $-K_M = Z + C_{i_1}$. □

COROLLARY 1. *In the case* (I)-(ii) *of Proposition* 3, *we have*

(1) $C_{i_1} \cdot Z = 2 - 2q$

(2) $Z \cdot Z \leq C_{i_1} \cdot C_{i_1}$

(3) $Z \cdot Z \leq 2 - 2q$.

Proof. Since $-K_M = Z + C_{i_1}$, $-(C_{i_1} \cdot K_M) = C_{i_1} \cdot C_{i_1} + C_{i_1} \cdot Z$. By the adjunction formula, $C_{i_1} \cdot C_{i_1} + C_{i_1} \cdot K_M = 2q - 2$. Thus, we have

$C_{i_1} \cdot Z = 2 - 2q$. This proves (1). Since $-K_M = 2C_{i_1} + \Sigma_{i \neq i_1} \lambda_i C_i$ ($\lambda_i > 0$) (see (ii) in the proof of Proposition 2), we can represent $Z - C_{i_1} = \Sigma_{i \neq i_1} \lambda_i \cdot C_i$ ($\lambda_i > 0$). Then

$$(Z - C_{i_1})(Z + C_{i_1}) = -K_M\Big(\sum_{i \neq i_1} \lambda_i \cdot C_i\Big) = -\sum_{i \neq i_1} \lambda_i (C_i \cdot K_M) \leq 0.$$

Therefore $Z \cdot Z \leq C_{i_1} \cdot C_{i_1}$. This proves (2). By the Noether formula, $K_M \cdot K_M = Z \cdot Z + 2(Z \cdot C_{i_1}) + C_{i_1} \cdot C_{i_1}$, we have, by (1) and (2), $10 - 8q - b_2(M) \geq 2(Z \cdot Z) + 4(1 - q)$, namely,

$$(1.6) \qquad 2 \leq b_2(M) \leq 6 - 4q - 2(Z \cdot Z).$$

Therefore $-(Z \cdot Z) \geq 2q - 2$. This proves (3). □

2. Singular K-3 surfaces with hypersurface singularities.

1°. Throughout this section, we will assume that A is a singular K-3 surface with hypersurface isolated singularities. Let the notations S, M, C, C_i, Z, etc. be as in §1. Let us denote by mult $(O_{A,x})$ the multiplicity of the local ring $O_{A,x}$ at the point x of A. Then,

PROPOSITION 4. *Assume that S consists of one point $x \in A$. We put* $n = \text{mult}(O_{A,x})$. *Then,*

(1) (*Wagreich* **[12]**): $Z \cdot Z \geq -n$.

(2) (*Yau* **[14]**): $p_g \geq \frac{1}{2}(n-1)(n-2)$.

PROPOSITION 5. *Assume that $S \neq \varnothing$. Then* $0 \leq q \leq 3$.

Proof. We may assume that S consists of one point. Then $p_g = q + 1$. By Proposition 4-(2), we have

$$(2.1) \qquad 0 < n \leq \tfrac{1}{2}\left(3 + \sqrt{9 + 8q}\right).$$

By (1.6), $-2(Z \cdot Z) \geq 4q - 6 + b_2(M)$. Thus, by Proposition 4-(1), we have $2n \geq 4q - 6 + b_2(M)$. We have, together with (2.1),

$$(2.2) \qquad 2 \leq b_2(M) \leq 9 - 4q + \sqrt{9 + 8q}.$$

Thus, $9 - 4q + \sqrt{9 + 8q} \geq 2$, namely, $q \leq 3$. □

COROLLARY 2.

(1) $q = 3 \Rightarrow b_2(M) = 2$, *namely*, $M = \overline{M}$.

(2) $q = 2 \Rightarrow 2 \leq b_2(M) \leq 6$.

(3) $q = 1 \Rightarrow 3 \leq b_2(M) \leq 8$.

(4) $q = 0 \Rightarrow 11 \leq b_2(M) \leq 13$.

Proof. The assertions (1), (2) and (3) follow directly from Proposition 4-(1), (2.1) and (2.2). In case (3), $b_2(M) \neq 2$. In fact, if $b_2(M) = 2$, then $M = \overline{M}$, since $b_2(\overline{M}) = 2$. Since $q = 1$ and $M = \overline{M}$, $K_M \cdot K_M = 0$. On the other hand, by Proposition 1-(1) $K_M \cdot K_M = \Sigma_{i,j} n_i n_j (C_i C_j) < 0$, since $n_i > 0$ and the intersection matrix $(C_i \cdot C_u)$ is negative definite. This is a contradiction. Next, if $q = 0$, then $-K_M = Z$, by Proposition 3-(1). Since S is a hypersurface singularity, by Laufer [9], $0 < -(Z \cdot Z) \leq 3$. By Noether formula, $K_M \cdot K_M = 10 - b_2(M)$. Therefore $10 < b_2(M) \leq 13$. This proves (4). □

2°. Finally, we shall determine the structure of the singular K-3 surfaces with hypersurface singularities whose second Betti numbers are equal to 1. Let us denote by $\operatorname{Sing} A$ the singular locus of A. Then $\operatorname{Sing} A - S$ consists of rational double points. We put $B = \pi^{-1}(\operatorname{Sing} A) \hookleftarrow C = \cup_{i=1}^{s_0} C_i$ and $s := \dim H^2(B; \mathbf{R})$.

LEMMA 2. *If* $b_2(A) = 1$, *then* S *consists of one point and* $b_2(M) = s + 1$.

Proof. Let us consider the following exact sequence of cohomology group (see [3]):

$$\rightarrow H^1(A; \mathbf{R}) \rightarrow H^1(M; \mathbf{R}) \rightarrow H^1(B; \mathbf{R}) \rightarrow H^2(A; \mathbf{R})$$

$$\xrightarrow{\pi^*} H^2(M; \mathbf{R}) \rightarrow H^2(B; \mathbf{R}) \rightarrow 0.$$

Since $H^1(A; O_A) = 0$, we have $H^1(A; \mathbf{R}) = 0$. Since A is projective algebraic, M is also projective algebraic. Thus $1 = b_2(A) \geq b^+(A) = b^+(M) = 2p_g(M) + 1 \geq 1$, that is, $b^+(A) = 1$, and thus $\ker \pi^* = 0$. This implies $H^1(M; \mathbf{R}) \cong H^1(B; \mathbf{R})$ and $b_2(M) = s + 1$. Now, let us assume that S consists of two points with $p_g = 1$. We have then $C = C_1 \cup C_2$, and C_i's $(i = 1, 2)$ are non-singular elliptic curves (see Proposition 2 and (i) in the proof). We have also seen that C_i's are two disjoint sections there. Thus M is a ruled surface over a non-singular elliptic curve, that is, $2 = \dim H^1(M; \mathbf{R})$. On the other hand,

$$\dim H^1(M; \mathbf{R}) = \dim H^1(B; \mathbf{R}) \geq \dim H^1(C; \mathbf{R})$$

$$= \sum_{i=1}^{2} \dim H^1(C_i; \mathbf{R}) = 4.$$

This is a contradiction. Therefore S consists of one point. □

Let C_{i_1} be the section of M as in Proposition 2-(2), and put the self-intersection number $C_{i_1} \cdot C_{i_1} = e < 0$. Then, by Proposition 3, Proposition 5, Corollary 2 and Lemma 2, we have the following

PROPOSITION 6. *Assume that* $b_2(A) = 1$. *Then we have*
(1) *if* $q = 3$, *then* $Z \cdot Z = -4$ *and* $s = 1$.
(2) *if* $q = 2$, *then* $-2 \leq Z \cdot Z \leq -4$ *and*
  (i) $Z \cdot Z = -4 \Rightarrow (e, s) = (-3, 4), (-4, 5)$.
  (ii) $Z \cdot Z = -3 \Rightarrow (e, s) = (-3, 3)$
  (iii) $Z \cdot Z = -2 \Rightarrow (e, s) = (-2, 1)$
(3) $q = 1$, *then* $Z \cdot Z \geq -3$ *and*
  (i) $Z \cdot Z = -3 \Rightarrow (e, s) = (-3, 7), (-2, 6), (-1, 5)$
  (ii) $Z \cdot Z = -2 \Rightarrow (e, s) = (-2, 5), (-1, 4)$
  (iii) $Z \cdot Z = -1 \Rightarrow (e, s) = (-1, 3)$
(4) $q = 0$, *then* $Z \cdot Z \geq -3$ *and*
  (i) $Z \cdot Z = -3 \Rightarrow s = 12$
  (ii) $Z \cdot Z = -2 \Rightarrow s = 11$
  (iii) $Z \cdot Z = -1 \Rightarrow s = 10$.

Next, let us see the structure of M as a ruled surface in case of $q \neq 0$.

PROPOSITION 7. *Assume that* $b_2(A) = 1$. *If* $q \neq 0$, *then either* $M = \overline{M}$, *or there exists unique exceptional curve of the first kind in every singular fibre of M and then another irreducible components of singular fibre are all contained in B.*

Proof. Assume that $M \neq \overline{M}$. Since $q \neq 0$, by Proposition 2-(2), there exists an irreducible component C_{i_1} of C such that the rest $B - C_{i_1}$ is contained in the singular fibres of M. Let $F_1, \ldots, F_r$ be the singular fibres of M, $1 + \alpha_i$ ($\alpha_i > 0$) the "number" of the irreducible components of F_i and δ_i the "number" of the irreducible components of F_i which are not contained in B. Then we have

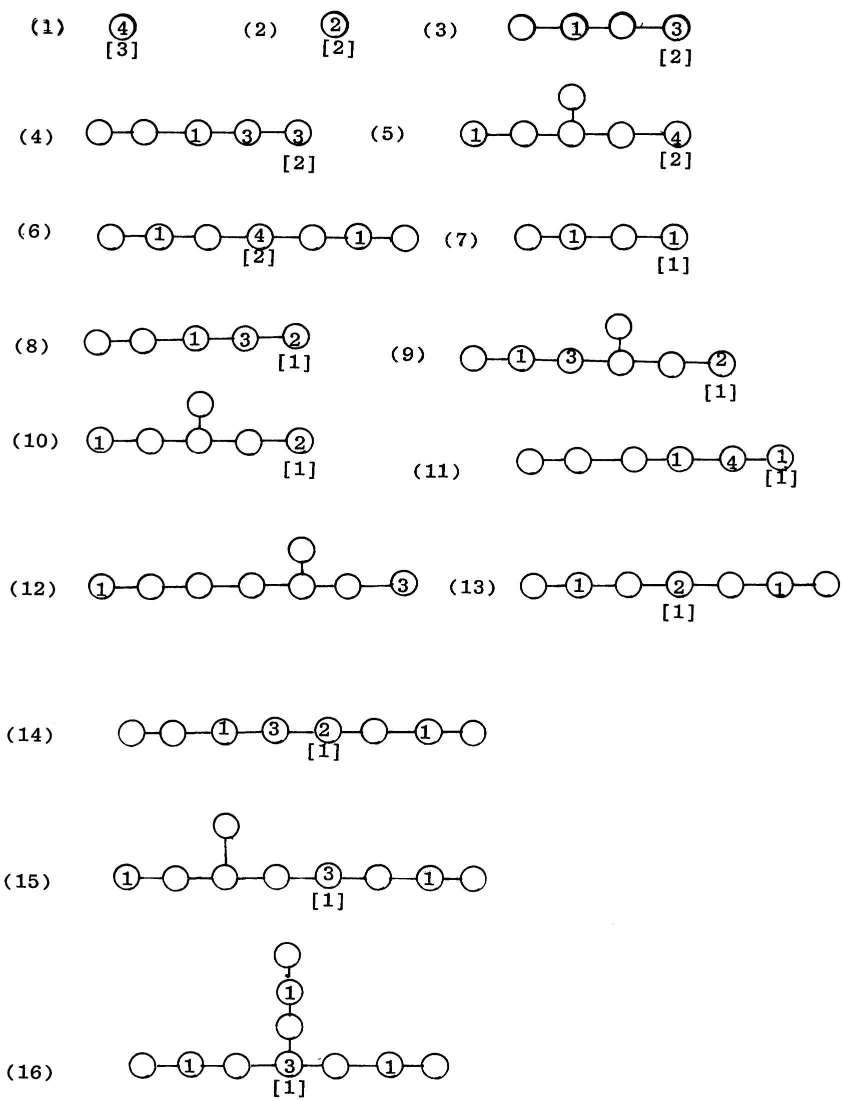
$$\begin{cases} 1 + s = b_2(M) = 2 + \sum_{i=1}^{r} \alpha_i \\ \sum_{i=1}^{r} (1 + \alpha_i - \delta_i) + 1 = s \end{cases}$$

Thus we have $\sum_{i=1}^{r}(1 - \delta_i) = 0$. Since each singular fibre F_i contains at least an exceptional curve of the first kind, we have $\delta_i \geq 1$ $(1 \leq i \leq r)$, thus $\delta_i = 1$ $(1 \leq i \leq r)$. This completes the proof. □

By Proposition 6 and Proposition 7, we have

THEOREM. *Let A be a singular K-3 surface with hypersurface singularities. Assume that $b_2(A) = 1$. Let S be the set of singular points which are not rational singular points, and π: $M \to A$ be the minimal resolution of singularities of A. Then M is a ruled surface over a non-singular compact algebraic curve R of genus q $(0 \leq q \leq 3)$, and S consists of one point. Moreover, if $q \neq 0$, then the dual graph of all the exceptional curves in M can be classified as Table* I.

TABLE I



NOTATION. In Table I, the vertex

$$\underset{[g]}{\textcircled{k}}$$

represents a non-singular compact algebraic curve of genus g with self-intersection number $-k$, $\textcircled{k}$ a non-singular rational curve with self-intersection number $-k$, and we denote $\textcircled{2}$ by $\bigcirc$.

REMARK 2. In case of $q = 0$, since $-(K_M \cdot K_M) = \Sigma n_i(C_i \cdot K_M)$ and $(K_M \cdot K_M) = -1, -2$, or -3, repeating the adjunction formula, we can determine the integers n_i's and the dual graph $\Gamma(C)$ of the exceptional curve C (see Laufer [**9**]).

REMARK 3 (see [**6**]). Let (X, A) be a non-singular Kähler compactification of $\mathbf{C}^3$ and A has at most isolated singular points. Then A is purely two dimensional compact analytic subvariety of X with hypersurface singular points and the canonical divisor $K_X = -r \cdot A$ $(1 \leq r \leq 4)$. In case of $r \geq 2$, the structure of (X, A) is determined in [**6**]. But in case of $r = 1$, it is still unknown. In that case, A is a singular K-3 surface with hypersurface singular points and $b_2(A) = 1$. Applying the theory of Iskovskih [**8**] and our theorem to the paire (X, A), we can obtain some detailed informations on (X, A). This will be discussed elsewhere.

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