4-DIMENSIONAL G-MANIFOLDS WITH 3-DIMENSIONAL ORBITS

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We describe the smooth G-actions of compact 4-manifolds with 3-dimensional principal orbits. We obtain equivariant and topological classifications of such actions. We include two lists: the manifolds organized by the group action, and the manifolds ordered by topological type, sorted on the first homology.

0. Notions and notation. In §0, we review the classification of codimension 1 actions. Section 1 lists the potential Lie groups and subgroups, and §2 gives the equivariant classification, and identities the simpler actions. Section 3 calculates the fundamental groups of the manifolds, and graphs the groups of several infinite fundamental groups. Section 4 constructs equivariant fibrations, showing that the SU(2) and SO(3) manifolds with finite fundamental group are fibrations, with fiber and base S^2 or $\mathbf{RP}^2$. Section 5 classifies such fibrations topologically, and §6 applies the classification to our examples.

All manifolds in the paper are assumed to be smooth, closed, and connected. Without loss of generality, we may assume that all our group actions are effective. All groups are compact, connected Lie groups, and all actions are smooth. The action of the group G on the manifold M is a map

$$G \times M \to M$$
$$g \times m \to gm \quad \text{for } g \in G, \, m \in M.$$

The action is *effective* if and only if for each g in G, there is an x in M such that $gx \neq x$. The action is *almost effective* if for all g outside a finite subgroup of G, there is an x such that $gx \neq x$.

We represent the isotropy subgroup of G that fixes x by G_x, and the orbit of x under the action of G by $G(x)$.

A diffeomorphism

$$f: M \to N$$

between G manifolds is *equivariant* if $f(gm) = g(fm)$ for all $m \in M$ and $g \in G$. The map f is a *weak equivariant equivalence* if

$$f(gm) = h(g)(f(m)),$$

where h is an automorphism of G.

We let M^* be the space of orbits of the G action on M, and let p be the projection map

$$p\colon M \to M^*.$$

Codimension 1 Actions

The original equivariant classification for codimension 1 actions is due to Mostert [Mo], and we will call such actions Mostert actions.

We summarize the relevant results below: they may also be found in Neumann [N] and Bredon [B].

LEMMA [0.1]. *If M is a compact Mostert manifold, then the orbit space M^* is the circle S^1, or the interval I. The only exceptional orbits of the action map to boundary points of I.*

Proof. The lemma follows immediately from compactness of M and the slice theorem.

LEMMA [0.2]. *If M is a Mostert G-manifold, with principal isotropy H and orbit space S^1, then M fibers as*

$$G/H \to M \to S^1,$$

with structure group $N(H)/H$, where $N(H)$ is the normalizer in G of H.

LEMMA [0.3]. *If M is a Mostert G-manifold with orbit space $I = [-1, 1]$, and exceptional orbits G/K, G/K', then M splits equivariantly into two tubular neighborhoods A and A' of the exceptional orbits.*

Each half is a mapping cylinder, defined by the natural projection. Thus A is the mapping cylinder defined by the projection

$$f\colon G/H \to G/K.$$

The two halves are sewn together along a principal orbit. The sewings are classified up to equivariance by components of the double coset space $N \backslash N(H)/N'$, where $N = N(K) \cap N(H)$ and $N' = N(K') \cap N(H)$.

LEMMA [0.4]. *If $M^* = I$, then a linear tube about G/K has the form $G \times_K V$ for some vector space V. Since the orbit space is one dimensional, K acts transitively on the spheres in V about the origin. Each such sphere is an orbit H/K of the K action on V. Thus H/K must be a sphere.*

In summation, a Mostert action is equivariantly classified by four items:

(1) A group G,

(2) an orbit space S^1 or I,

(3) a representation of a principal isotropy, (if $M^* = I$, we also need two representations of exceptional isotropy satisfying [0.4]),

(4) a coset representative, to define the sewing map.

Further, given any such four objects, Mostert's construction gives a unique G-manifold.

We let $I(G; H; K, K')$ represent the G-manifold with principal isotropy H, and exceptional isotropies K and K', and the identity coset representative. We let $I(G; H; K, K'; g)$ represent the G-manifold with the same orbit types, but with $g \in N(H)$ as a coset representative.

We let $S(G; H)$ and $S(G; H; g)$ represent G-manifolds with the single orbit type G/H.

EXAMPLE [0.5]. The Klein bottle supports a circle action, described by $I(T^1; 1; Z_2, Z_2)$.

EXAMPLE [0.6]. The torus supports the circle action $S(T^1; 1)$.

1. Lie groups and subgroups. To provide an exhaustive list of 4-dimensional Mostert manifolds, we list all Lie groups G that can act effectively, and all subgroups H that may be a principal isotropy. For each pair (G, H), we must then list all possible exceptional isotropies K.

An elegant classical results narrows our search.

THEOREM [1.1] (*Eisenhart* [**E**]). *If G acts effectively on an orbit of dimension n, then*

$$n \le \dim(G) \le n(n + 1)/2,$$

and if $\dim(G) = n(n + 1)/2$, *then* $G = \mathrm{SO}(n + 1)$, *and* $H = \mathrm{SO}(n)$ *or* $\mathrm{O}(n)$.

In our case, this implies that

$$3 \le \dim(G) \le 6.$$

The compact connected Lie groups in this range are well known. (See e.g., Hochschild [**Ho**].) Some groups with a torus, such as T^4, T^5, or $\mathrm{SO}(3) \times T^2$, do not give us effective actions on three-dimensional orbits. In these groups, the principal isotropy contains a torus as a normal subgroup, which acts trivially on any point.

We are left with the following groups: $\{T^3,\ SO(4),\ SU(2),\ SO(3),\ SO(3) \times T^1\}$. We consider each group in turn.

$$G = T^3.$$

Any subgroup of T^3 is normal. Without loss of generality, we may assume that the principal isotropy is the identity. Exceptional isotropies must be spheres, by Lemma [0.4], thus an embedding of Z_2 or T^1 in T^3. The embeddings of T^1 in T^3 are given by subgroups of the form

$$U(p,q,r) = \{(e^{2\pi ipt},\ e^{2\pi iqt},\ e^{2\pi irt}) | t \in R\},$$

for relatively prime integers p, q and r.

Up to weak equivariance, we may assume that $K \subset U(1,0,0)$, and $K' \subset U(p,q,0)$. Thus there is a free action of $T^1 = U(0,0,1)$ on M^4. Since $N(H) = T^3$ is connected, there is a unique manifold, so M^4 is a product $N^3 \times S^1$, where N^3 is a Mostert manifold with a T^2 action and the appropriate invariants. These 3-dimensional Mostert manifolds have been classified by Mostert [**Mo**] and Neumann [**N**].

$$G = SO(4).$$

If $G = SO(4)$, then H must be $SO(3)$ or $O(3)$. If $H = SO(3)$, then we may have $K \in \{O(3), SO(3)\}$. If $H = O(3)$, then there is no subgroup K between G and H.

$$G = SU(2).$$

If $G = SU(2)$, H must be discrete. Any group H of even order contains the central Z_2 in $SU(2)$. Thus the only effective actions of G have principal isotropy $H = Z_{2n+1}$. However, we will choose to view some actions as almost effective actions of $SU(2)$, rather than as effective actions of $SO(3)$, to unify the classification. In particular, actions with principal isotropy $H = Z_n$, and $K \in \{Z_{2n}, T^1, SU(2)\}$, for n even or odd, are listed as $SU(2)$ actions, whatever the parity of n. All other almost effective actions of $SU(2)$ are listed as $SO(3)$ actions.

$$G = SO(3).$$

If $G = SO(3)$, H must be discrete. Our choices are cyclic (Z_n), dihedral of order $2n$ (D_n), tetrahedral (**T**), octahedral (**O**), or icosahedral (**I**).

If $H = Z_n$, then any $K \in \{Z_{2n}, D_n, T^1\}$ satisfies [0.4]. If $H = D_n$, then we may pick $K \in \{D_{2n}, O(2)\}$. If $H = \mathbf{T}$, then $K = \mathbf{O}$. Neither of

the other candidates give actions with exceptional isotropy, though all yield actions with $M^* = S^1$.

$$G = SO(3) \times T^1.$$

If $G = SO(3) \times T^1$, then H must be one dimensional, and thus contains a torus. If H contains the subgroup $1 \times T^2$, then the action is not effective. Thus H must be $SO(2) \times 1$ or $O(2) \times 1$. These both admit extensions to exceptional isotropy subgroups. If $H = SO(2)$, then $K \in \{O(2), SO(3), H \times Z_2, H \times T^1\}$, and if $H = O(2)$, $K \in \{H \times Z_2, H \times T^1\}$.

Normalizers

If $G = T^3$, then G is abelian, so $N(H) = G$. If $G = SO(4)$, then $H = SO(3)$ or $O(3)$, and $N(H) = O(3)$. If $G = SO(3)$, then the normalizers are:

$N(SO(2)) = N(O(2)) = N(Z_k) = O(2)$,
$N(D_2) = N(T) = N(O) = O$, $N(I) = I$,
$N(D_k) = D_{2k}$, for $k > 2$. (See [**Hu**], [**R**], or [**P**].)

The example of greatest interest is the case $G = SO(3)$, $H = D_2$. Then $N(H)/H = S_3$, the symmetric group on three symbols. We will graph the fundamental groups of these actions in §3.

2. Equivariant classification. Mostert's results allow us to take the list of groups and subgroups from the previous section, and to construct the following list of 4-dimensional Mostert manifolds. Many of the actions are easy to identify as actions on the direct product, join, or connected sum of actions on two G-manifolds.

There are also actions of S^4 and $\mathbf{RP}^4$ that have been studied previously ([**B**], [**Ma**]). All of these actions are labeled in the list.

Most of the manifolds unlabeled in the list may be classified by their fundamental group, if it is infinite. With a few exceptions, those with finite fundamental group support $SU(2)$ actions. To dispose of the exceptions that do not fall into either of the classes mentioned we first take a closer look at the equivariant covers of an action.

If a Mostert manifold with orbit space I has two identical exceptional orbits, the trivial sewing map, and a principal orbit that supports a free involution a, then we may define a free involution α on M.

We split M into two halves along a principal orbit P. The action of α on $x \in P$ is defined by $\alpha(x) = a(x)$. To extend this action to all of M, we exchange halves of M, and extend the action of a on the principal

orbit radially inward. As an example, take the two-sphere S^2. The standard circle action spins the sphere, fixing north and south poles. For the involution a on the principal orbit, pick rotation by 180 degrees. The involution α exchanges north and south poles, and $M/\alpha = \mathbf{RP}^2$. As the involution a commutes with the circle action, we obtain an equivariant circle action on $\mathbf{RP}^2$. Thus we may study the circle action on $\mathbf{RP}^2$ by looking at the circle action on S^2.

Consider the manifold $S^3 \times S^1$, which supports several Mostert actions. The group SO(3) acts on the equatorial S^2 in S^3, giving an action of $SO(3) \times T^1$ on $S^3 \times S^1$, with isotropy SO(2), SO(3), and SO(3). We take a to be an element of G, $a = (I, -1)$, which acts on the principal orbit $S^2 \times S^1$. We then define the orientation reversing free involution α as above. The manifold M/α is the non-orientable S^3 bundle over S^1. The exceptional orbits are $G/SO(3)$ and $G/(SO(2) \times Z_2)$.

There is another equivariant action on the non-orientable S^3 bundle over S^1. Let $G = SO(4)$, acting freely on S^3. We let G act trivially on S^1, giving a Mostert action on $S^3 \times S^1$. Our involution α is an element of $G \times T^1$, $(-I, -1)$. The involution clearly commutes with G, and is orientation reversing. The quotient $N = M/\alpha$ is the same non-orientable bundle, but with a new action. In this case, $N^* = S^1$, and the sewing map for the torus construction is the non-identity element of $N(H)/H = O(3)/SO(3) = Z_2$.

This identifies the manifolds numbered 35 and 18. We will see that the manifold numbered 31 ($I(SO(3) \times T^1; SO(2); O(2), T^2)$) supports a Mostert action of SO(3). Thus we can say that all manifolds not labeled in the following chart have infinite fundamental group, or support actions of SU(2) or SO(3).

$$M^* = S^1$$

	G	H	$\pi_0(N(H)/H)$	Element	M^4	Action Type
1	T^3	1	1		T^4	Product
2	SU(2)	1	1		$S^3 \times S^1$	Product
3	—	Z_n	Z_2	e	$L(n,1) \times S^1$	Product
4	—	—	—	1	$L(n,1) \to M \to S^1$	
5	SO(3)	D_2	S_3	e	$G/H \times S^1$	Product
6	—	—	—	(1 2)		
7	—	—	—	(1 2 3)		
8	—	D_n	Z_2	e	$G/H \times S^1$	Product
9	—	—	—	1		
10	—	T	Z_2	e	$G/H \times S^1$	Product
11	—	—	—	1		
12	—	0	1		$G/H \times S^1$	Product
13	—	I	1		$G/H \times S^1$	Product
14	$SO(3) \times T^1$	SO(2)	Z_2	e	$T^2 \times S^2$	Product
15	—	—	—	1		
16	—	O(2)	1		$T^2 \times RP^2$	Product
17	SO(4)	SO(3)	Z_2	e	$S^3 \times S^1$	Product
18	—	—	—	1	$S^3 \to M \to S^1$	
19	—	O(3)	1		$RP^3 \times S^1$	Product

$M^* = I$

	G	H	K	K'	$\pi_1(N \backslash N(H)/N')$	M^4	Action Type
20	T^3	1	$Z_2 \times I$	$Z_2 \times 1$		$T^2 \times K^2$	Product
21	—	—	—	$1 \times Z_2$		$A^3 \times S^1$ [N]	Product
22	—	—	—	$U(1,0,0)$		$T^2 \times RP^2$	Product
23	—	—	—	$U(0,1,0)$		$B^3 \times S^1$ [N]	Product
24	—	—	$U(1,0,0)$	$U(0,1,0)$		$T^2 \times S^2$	Product
25	—	—	—	$U(q,p,0)$		$L(p,q) \times S^1$	Product
26	SO(4)	SO(3)	O(3)	O(3)		$RP^4 \# RP^4$	Sum
27	—	—	—	SO(4)		RP^4	Equivariant
28	—	—	SO(4)	SO(4)		S^4	Join
29	$SO(3) \times T^1$	SO(2)	O(2)	O(2)		$(RP^3 \# RP^3) \times S^1$	Product
30	—	—	—	$SO(2) \times Z_2$			
31	—	—	—	T^2			
32	—	—	—	SO(3)		$RP^3 \times S^1$	Equivariant
33	—	—	$SO(2) \times Z_2$	$SO(2) \times Z_2$		$S^2 \times K^2$	Product
34	—	—	—	T^2		$S^2 \times RP^2$	Product
35	—	—	—	SO(3)			
36	—	—	T^2	T^2		$S^2 \times S^2$	Product
37	—	—	—	SO(3)		S^4	Join
38	—	—	SO(3)	SO(3)		$S^3 \times S^1$	Product
39	—	O(2)	$H \times Z_2$	$H \times Z_2$		$RP^2 \times K^2$	Product
40	—	—	—	$O(2) \times T^1$		$RP^2 \times RP^2$	Product
41	—	—	$O(2) \times T^1$	$O(2) \times T^1$		$S^2 \times RP^2$	Product
42	SU(2)	1	Z_2	Z_2		$RP^4 \# RP^4$	Sum
43	—	—	—	T^1		$RP^4 \# CP^2$	Sum
44	—	—	—	SU(2)		RP^4	Sum
45	—	—	T^1	T^1		$CP^2 \# -CP^2$	Sum
46	—	—	—	SU(2)		CP^2	Sum
47	—	—	SU(2)	SU(2)		S^4	Suspension
48	—	Z_n	Z_{2n}	Z_{2n}			
49	—	—	—	T^1			
50	—	—	T^1	T^1			
51	SO(3)	Z_n	Z_{2n}	D_n			
52	—	—	Z_{2n}	SO(2)			
53	—	—	D_n	D_n			
54	—	—	D_n	SO(2)			
55	—	—	SO(2)	SO(2)			
56	—	T	O	O			
57	—	D_2	D_4	D_4	Z_2		
58	—	—	—	—			
59	—	—	D_4	O(2)	Z_2		
60	—	—	—	—		RP^4	[B]
61	—	—	O(2)	O(2)	Z_2		
62	—	—	—	—		S^4	[Ma]
63	—	D_n	D_{2n}	D_{2n} $(n > 2)$			
64	—	—	D_{2n}	O(2)			
65	—	—	O(2)	O(2)			

3. The fundamental group. We will use several invariants to classify our remaining manifolds. The fundamental group, it if is infinite, is enough to classify the remaining manifolds.

If the action has exceptional orbits that are simply connected, then the fundamental group will be finite. In this case, we show in §§4 and 5 that

(1) The manifolds are fibrations, where base and fiber are either S^2 or RP^2.

(2) Such fibrations are classified by
    (i) fundamental group (1, Z_2, or $Z_2 \times Z_2$),
    (ii) orientability,
    (iii) intersection form in $H_2(M, Z_2)$ and
    (iv) twofold cover.

We appeal to the constructions in §0 to make the following calculations.

LEMMA [3.1]. *Let M be a Mostert G manifold, with $M^* = S^1$, principal isotropy H, and monodromy f. Then*

$$\pi_1(M) = \pi_1(G/H) *_{f_\#}.$$

Proof. Apply Van Kampen's Theorem.

LEMMA [3.2]. *Let M be a Mostert G manifold with orbit space I, principal isotropy H, and exceptional isotropy K and K'. Then*

$$\pi_1(M) = \pi_1(G/K) *_{\pi_1(G/H)} \pi_1(G/K').$$

Proof. Apply Van Kampen's Theorem.

Invariants of π_1

We introduce two technical lemmas from [**MKS**], §§4.2, 4.4.5, and 4.5. They are useful when $G =$ SU(2), and when K and K' are finite. In this case, $\pi_1(M) = K *_H K'$, by Lemma [3.2].

LEMMA [3.3]. *Let H, K and K' be finite, and let g be an element of finite order in $K *_H K'$. Then g is conjugate to an element of K or K'.*

LEMMA [3.4]. *Assume that H, K, and K' are finite. The center of $K *_H K'$ lies in conjugates of H.*

Graphs of the Infinite Fundamental Group

We give below the graphs of the fundamental groups. We present only the interesting and beautiful graphs.

Let M^4 be a Mostert manifold with an action of $G =$ SU(2) or SO(3) and infinite fundamental group. The principal isotropy H must be a finite subgroup of G.

With these assumptions, the following observations hold. The universal cover of M^4 is $S^3 \times R^1$ with a transitive action of $P = \pi_1(M)$. P contains H as a normal subgroup. The group P/H is Z if $M^* = S^1$, and $Z_2 * Z_2$ if M^* is I. The graph of the group P is formed by stacking

together a countable number of copies of the graph of H, and lacing them together. Each copy of H represents a coset in P.

$$M^* = S^1.$$

Our first graphs represent the fundamental groups of manifolds 3 and 4. These are $L(n,1) \times S^1$ and a non-trivial S^1 bundle over $L(n,1)$. The product manifold has abelian fundamental group, while in the second group, commutators are not closed loops.
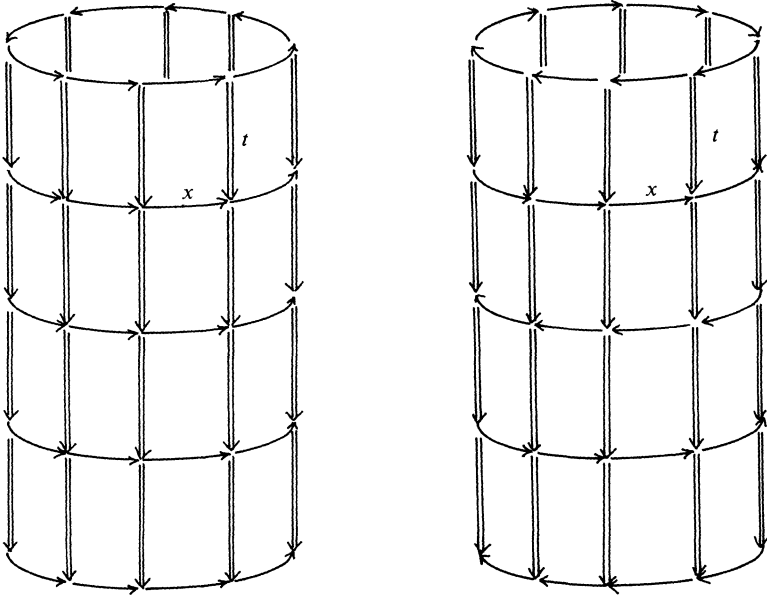
FIGURES 3.5.1, 3.5.2

$$P_3 = \langle x, t | x^p, [x,t] \rangle \qquad P_4 = \langle x, t | x^p, txt^{-1} = x^{-1} \rangle$$

Next, consider examples of the form $S(SO(3); D_n; f)$, manifolds 5 through 9. The most complex case occurs when $H = D_2$, as $N(D_2)/D_2 = S_3$, the symmetric group on three symbols. We obtain, up to weak equivalence, three distinct conjugate classes of elements in S_3. The coset representatives may be taken to be the identity e, the two cycle $(1,2)$, or the three cycle $(1,2,3)$. The three manifolds have different first homology. Their graphs are given below.
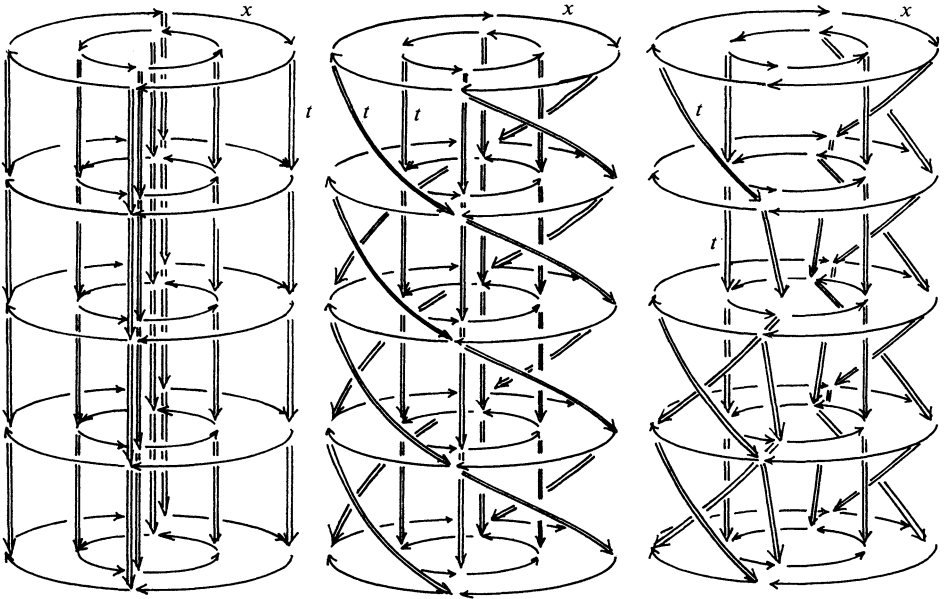
FIGURES 3.6.1, 3.6.2, 3.6.3

$$P_5 = Q_8 \times Z = \left\langle x, y, t | x^2 y^2, x^4, yxy^{-1}x, [x, t], [y, t] \right\rangle$$

$$P_6 = \left\langle x, y, t | x^2 y^2, x^4, yxy^{-1}x, [x, t], tyt^{-1} = xy \right\rangle$$

$$P_7 = \left\langle x, y, t | x^2 y^2, x^4, yxy^{-1}x, txt^{-1} = y, tyt^{-1} = xy \right\rangle$$

These groups have distinct abelianization: $Z + Z_2 + Z_2$, $Z + Z_2$, and Z respectively. The element x, of order 4, has the maximal finite order.

If n is greater than 2, $N(D_n) = D_{2n}$, so $N(H)/H = Z_2$ We get two manifolds, and graphs similar to 3.6.1, 3.6.2. They have an element of order n, and they differ from each other in the index of their center.

$$M^* = I.$$

Again, P is a stack of a countable number of cosets of H. However, in the previous case, M fibered over S^1, so there was a generator t of infinite order. In this case, H, K and K' are finite. The group $P = K *_H K'$ contains elements of infinite order of the form xy, where x and y are generators of K and K' that do not lie in H.

Consider manifolds 48, $I(SU(2); Z_n; Z_{2n}, Z_{2n})$, manifolds 51, $I(SO(3); Z_n; Z_{2n}, D_n)$, manifolds 53, $I(SO(3); Z_n; D_n, D_n)$, and manifolds 63 and 57, $I(SO(3); D_n; D_{2n}; D_{2n})$. These form a contrasting set of graphs, if we pick n so that a principal orbit has fundamental group of a uniform size.

In the representations below, x is a generator in K, y a generator in K', and z is an element of H. The group Q_{4n} is the quaternionic group in $SU(2)$, of size $4n$, that is the twofold cover of D_n.
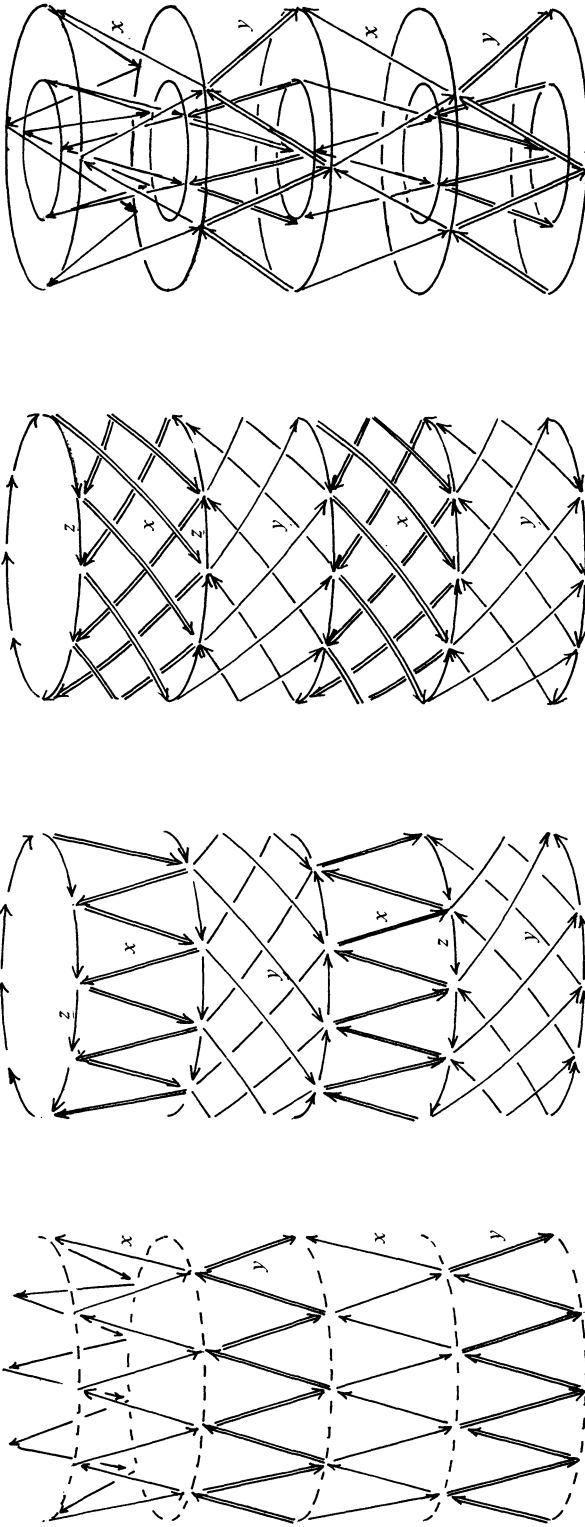
FIGURES 3.7.1, 3.7.2, 3.7.3, 3.7.4

$$P_{48} = Z_{16} * _{Z_8} Z_{16} = \langle x, y | x^{16}, x^2 = y^2 \rangle$$

$$P_{51} = Q_{16} * _{Z_8} Z_{16} = \langle x, y, z | z^8, y^2 = z^4, x^2 = z, yzy^{-1} = z^{-1} \rangle$$

$$P_{53} = Q_{16} * _{Z_8} Q_{16} = \langle x, y, z | z^8, y^2, x^2 = z^4, yzy^{-1} = xzx^{-1} = z^{-1} \rangle$$

$$P_{57} = Q_{16} * _{Q_8} Q_{16} = \langle x, y, z | z^8, y^4 = x^4 = z^2, zxz^{-1} = x^{-1}, zyz^{-1} = y^{-1} \rangle$$

The above manifolds have very similar structure. They are all covered by $S^3 \times R^1$, so they have the same higher homotopy. However, the center, the size of an element g of maximal finite order, and the structure of the group $P/\langle g \rangle$ obtained by killing an element of maximal torsion, distinguishes these manifolds. For example, compare the fundamental groups of manifolds P_{57} and of manifold P_{58}, not illustrated above:

$$P_{58} = \langle x, y, z | x^8, x^4 = z^2, zxz^{-1} = x^{-1}, y^2 = z, x^2xz^{-2} = y^{-1} \rangle.$$

The two manifolds have the same first homology (Z_2^3). However, $P_{57}/\langle x \rangle$ $= D_8$, while $P_{58}/\langle x \rangle = Z_2$.

4. Equivariant fibrations.

The remaining manifolds support actions of SU(2) or SO(3), and have finite fundamental group. The following theorem applies to most of them. We may generalize the observation to all of our remaining examples.

THEOREM [4.1]. *Let M^n be a Mostert manifold, with isotropy $K' \supseteq K$ $\supset H$. Then M is a fiber bundle over G/K'. The fiber is a Mostert manifold supporting a K' action, with isotropy H, K and K'.*

Proof. Recall that M is the union of a disc bundle over K and a disc bundle over K'. Since $K' \supseteq K$, we are able to extend the disc transverse to the exceptional orbit G/K' into the bundle over K. Due to the low codimension, we may construct a section s to the projection $p: M \to M^*$, $s[-1, 1] \to M$. Note that if we act on the section with an isotropy subgroup, such as $D = K'(s[0, 1])$, we obtain a disc transverse to G/K'. The translations of D under coset representatives of G/K' are identical copies of the Mostert manifold $I(K'; H; K, K')$. The local product structure of the disc bundle extends to a local product structure for the manifold.

EXAMPLE [4.2]. Consider the circle action on $\mathbf{RP}^2$. The action has orbit invariants $I(T^1; 1; Z_2, T^1)$.

EXAMPLE [4.3]. Let $M^4 = I(SU(2); Z_n; Z_{2n}, T^1)$ manifold number 49. Then [4.1] implies that M fibers over $G/K' = S^2$, with fiber $I(K'; H; K, K) = I(T^1; 1; Z_2, T^1)$. Thus M is an $\mathbf{RP}^2$ bundle over S^2.

We are not always presented with situations where K includes K'. Consider the manifold number 54, $I(SO(3); Z_n; D_n, SO(2))$. The isotropy subgroups are distinct, and SO(2) does not include D_n. However, we may produce a fibration over $\mathbf{RP}^2 = SO(3)/O(2)$. Clearly O(2) contains both

exceptional isotropy subgroups. Both halves of M fiber over $\mathbf{RP}^2$ equivariantly, and we sew the two halves together.

To visualize the fiber, consider the two halves. The neighborhood of $S^2 = G/K'$ is a D^2 bundle over S^2. The exceptional orbit S^2 is a Z_2 bundle over $\mathbf{RP}^2$. The neighborhood of S^2 is a $D^2 \cup D^2$ bundle over $\mathbf{RP}^2$. We can view this fiber as two polar caps. The neighborhood of the orbit G/D_n is a quaternionic space, an I bundle over the exceptional orbit. The orbit itself is an S^1 bundle over $\mathbf{RP}^2$. The neighborhood is a $S^1 \times I$ bundle over $\mathbf{RP}^2$. View $S^1 \times I$ as the equatorial band from the Tropic of Capricorn to the Tropic of Cancer. The sum of the two fibers is an S^2.
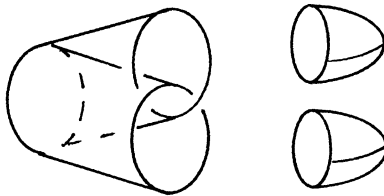
FIGURE 4.5

The same construction works for the manifold number 51, $I(SO(3); Z_n; Z_n, D_n)$, yielding a fibration of the Klein bottle K^2 over $\mathbf{RP}^2 = G/O(2)$, and for the manifold number 31, $I(SO(3) \times T^1;$ $SO(2); O(2), T^2)$, an S^2 bundle over $G/O(2) \times T^1 = \mathbf{RP}^2$.

5. Classification of fibrations. Although we may apply our results to obtain fibrations with fiber K^2 or T^2, these fibers give manifolds with infinite fundamental group. We are concerned with the simpler case, when fiber and base are both S^2 or $\mathbf{RP}^2$.

Base of Fibration is S^2

This case is well known: Steenrod [S] shows that if the fiber is S^2, the total space is $S^2 \times S^2$ or $\mathbf{CP}^2 \# -\mathbf{CP}^2$. This class of manifolds is classified by intersection form in $H_2(M; Z_2)$.

If the fiber is $\mathbf{RP}^2$, the same analysis is possible. There are two bundles in this class: the product bundle $\mathbf{RP}^2 \times S^2$, and a twisted $\mathbf{RP}^2$ bundle over S^2. Neither bundle is orientable, as both contain (two copies of) $\mathbf{RP}^2 \times D^2$. Again, the intersection form in $H_2(M; Z_2)$ classifies the total space.

Base of the Fibration is $\mathbf{RP}^2$

We may extend Steenrod's analysis in this case. Decompose $\mathbf{RP}^2$ into the union of the disk D and Moebius band Mb. Over the disc, the fibration is trivial. Over Mb, the bundle is classified by an element of $\pi_0(U)$, where U is the group of rigid motions of the space. If the fiber is S^2, $U = O(3)$, giving two bundles over Mb: the orientable and the non-orientable S^2 bundle. If the fiber is $\mathbf{RP}^2$, then $U = SO(3)$, and there is only a product bundle, $\mathbf{RP}^2 \times Mb$.

Next we need to sew the bundles over D and Mb together. These maps are classified by elements of $\pi_1(SO(3)) = Z_2$, and the intersection form classifies them.

This gives us two $\mathbf{RP}^2$ bundles over $\mathbf{RP}^2$, and four S^2 bundles over $\mathbf{RP}^2$.

We have three new classes: $\mathbf{RP}^2$ bundles over $\mathbf{RP}^2$, and two classes of S^2 bundles over $\mathbf{RP}^2$. The S^2 bundles are classified by orientability and intersection form.

If we consider fibrations over S^2 and $\mathbf{RP}^2$, we have built ten manifolds. Two are identical, since the product bundle $\mathbf{RP}^2 \times S^2$ is also the product $S^2 \times \mathbf{RP}^2$. In all, this leaves us with at most nine manifolds: all are distinct. Orientability and fundamental group are enough to distinguish all of the classes except those with fundamental group Z_2: the $\mathbf{RP}^2$ bundles over S^2 and the S^2 bundles over $\mathbf{RP}^2$. We have observed that the product bundles are the same. To show that the remaining two manifolds are not the same, we observe that their twofold covers are distinct.

Steenrod's construction may be extended to build the universal covers of both manifolds. In the case of the $\mathbf{RP}^2$ bundle over S^2, the fundamental group is generated by a curve in the fiber. The twofold cover is an S^2 bundle over S^2, with skew intersection form.

In building the twofold cover of the skew S^2 bundle over $\mathbf{RP}^2$, the fundmental group is generated by the essential curve in the base. This is the curve that we sew along: as a result, the map used to join the two S^2 bundles over D^2 is twice the non-trivial element of $\pi_1(SO(3))$, or the identity.

Another way to construct the universal covers is to consider equivariant covers. The two classes of bundles are covered by two distinct actions on S^2 bundles over S^2. The results of §6 allow us to identify the covers as $S^2 \times S^2$, or $\mathbf{CP}^2 \# - \mathbf{CP}^2$.

6. Topological classification of actions. Our results on equivariant fibrations describe the 4-dimensional Mostert manifolds with finite funda-

mental group as fibrations. We can calculate the fiber, base, and orientation from the isotropy information. For most of the actions, it will also be easy to compute the intersection form from the orbit data. Finally, we are able to construct equivariant covers of the manifolds with fundamental group Z_2.

Intersection forms

A spectral sequence argument shows that $H_2(M^4; Z_2)$ has dimension 2. We observe that the fiber and the base are two-dimensional manifolds that generate $H_2(M; Z_2)$. In general, we will be able to find a section to the fibration. It is clear that the fiber has no self-intersection, and intersects the base once. This gives the following intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$$

where n is the self-intersection of the base. Our calculation of the intersection form reduces to finding a section, and computing its self intersection.

Let us take each of the actions in turn.

Action 50

If $M = I(\mathrm{SU}(2); Z_n; T^1, T^1)$, then M fibers over $G/T^1 = S^2$, with fiber the T^1 manifold $I(T^1; Z_n; T^1, T^1) = I(T^1; 1; T^1, T^1) = S^2$. The base is the exceptional orbit $S^2 = G/K'$. The tubular neighborhood of this orbit is a disc bundle with boundary the principle orbit $G/H = L(n,1)$, a lens space. This disc bundle has Euler class n, so the self-intersection of the exceptional orbit is n. The bundle is the product $S^2 \times S^2$ if n is even, and $\mathbf{CP}^2 \# -\mathbf{CP}^2$ if n is odd.

Action 49

If $M = I(\mathrm{SU}(2); Z_n; Z_{2n}, T^1)$, then M fibers over $G/T^1 = S^2$, with fiber the T^1 manifold $I(T^1; Z_n; Z_{2n}, T^1) = \mathbf{RP}^2$. the base is the exceptional orbit $P = G/K'$. The tubular neighborhood of P is a disc bundlle with boundary a principal orbit $G/H = L(n,1)$. This disc bundle has Euler class n, so the self-intersection of P is n. The bundle is the product $S^2 \times \mathbf{RP}^2$ if n is even, and a twisted $\mathbf{RP}^2$ bundle over S^2 if n is odd.

Action 59 and 64

If $M = I(\mathrm{SO}(3); D_n; D_{2n}, \mathrm{O}(2))$, it will fiber over the orbit $P = G/K' = \mathbf{RP}^2$. The fiber is also $\mathbf{RP}^2$. The neighborhood of P is a quaternionic space, with Euler class n. If n is even, $M = \mathbf{RP}^2 \times \mathbf{RP}^2$. If n is add, M is

the twisted $\mathbf{RP}^2$ bundle over $\mathbf{RP}^2$. Note that M is covered by the manifold $N = I(SO(3); Z_n; Z_{2n}, T^1) = I(SU(2); Z_{2n}; Z_{4n}, T^1)$. The covering translation on N is right multiplication by an element of $D_n - Z_n$. The manifold N is a special case of manifold 49: it is $S^2 \times \mathbf{RP}^2$.

Action 61 and 65

Consider $M = I(SO(3); D_n; O(2), O(2))$. The manifold fibers over $G/K = \mathbf{RP}^2$, with fiber S^2. We found two classes of S^2 bundles over $\mathbf{RP}^2$: the orientable and the non-orientable. The manifold M is the quotient of the orientation preserving involution described above on manifold 49, acting on the manifold $N = I(SO(3); Z_n; T^1, T^1)$. Since N is orientable, M is also orientable. the intersection form is again skew if and only if n is odd.

Action 31

As noted in §4, the manifold $O(SO(3) \times T^1; SO(2); O(2), T^2)$ fibers over $G/(O(2) \times T^1) = \mathbf{RP}^2$, with fiber S^2. The exceptional orbit $G/O(2) = \mathbf{RP}^2 \times T^1$ has a one-sided embedding, so the manifold is orientable. Thus we need only calculate the intersection form to classify M. We may construct a section to the bundle. A geometric argument shows that it has even self-intersection. Thus M is $S^2 \times \mathbf{RP}^2$.

The following abbreviations are used.
$P = \pi_1(M^4)$,
m is the order of an element g in P of maximal finite torsion,
$Q = P/\langle g \rangle$,
C is the center of P,
$o = H_4(M^4)$ measures the orientability of the manifold, and
$r = \text{rank}(H_2(M))$.
Invariants are omitted if $H_1(M)$ classifies the manifold.

Neumann [N] mentions two 3-manifolds that lack standard names: $A^3 = Mb \times S^1 \cup S^1 \times Mb$, and the non-trivial bundle $S^2 \to B^3 \to S^1$. These arise as terms in the actions 21 and 23.

We also list a manifold M^4 that has no simple description.

$H_1(M)$	M^4	G	Action	Invariants
Z^4	T^4	T^3	1	
Z^3	$A^3 \times S^1$ [N]	T^3	21	
$Z^3 + Z_2$	$T^2 \times K^2$	T^3	20	
Z^2	$T^2 \times S^2$	$SO(3) \times T^1$	14	$o = Z$
		T^3	24	
	$B^3 \times S^1$ [N]	$SO(3) \times T^1$	15	$o = 0$
		T^3	23	

$H_1(M)$	M^4	G	Action	Invariants
$Z^2 + Z_2$	$T^2 \times \mathbf{RP}^2$	$SO(3) \times T^1$	16	
		T^3	22	
Z	$S^3 \times S^1$	$SU(2)$	2	$o = Z$, $m = 1$
		$SO(4)$	17	
		T^3	25 ($p = 1$, $q = 0$)	
		$SO(3) \times T^1$	38	
	Twisted $S^3 \to M \to S^1$	$SO(4)$	18	$o = 0$, $m = 1$
		$SO(3) \times T^1$	35	
	Twisted $L(2k+1,1) \to M \to S^1$	$SU(2)$	4 ($n = 2k + 1$)	$m = 2k + 1$
	$S^3/T \to M \to S^1$	$SO(3)$	11	$m = 6$
	$S^3/Q_8 \to M \to S^1$	$SO(3)$	7	$m = 4$
$Z + Z_2$	$\mathbf{RP}^3 \times S^1$	$SO(3)$	3 ($n = 2$)	$m = 2$, $o = Z$
		$SO(4)$	19	
		T^3	25 ($p = 1$, $q = 2$)	
		$SO(3) \times T^1$	32	
	$(S^3/\mathbf{I}) \times S^1$	$SO(3)$	13	$m = 10$
	$(S^3/\mathbf{O}) \times S^1$	$SO(3)$	12	$m = 8$
	Twisted $(S^3/Q_8) \to M \to S^1$	$SO(3)$	6	$m = 4$
	$S^2 \times K^2$	$SO(3) \times T^1$	33	$m = 1$
	Twisted $K^2 \to M \to \mathbf{RP}^2$	$SO(3) \times T^1$	30	$m = 2$, $o = 0$
$Z + Z_3$	$L(3,1) \times S^1$	$SU(2)$	3 ($n = 3$)	$m = 3$
		T^3	25 ($p = 3$, $q = 1$)	
	$(S^3/T) \times S^1$	$SO(3)$	10	$m = 6$
$Z + Z_4$	$L(4,1) \times S^1$	$SU(2)$	3 ($n = 4$)	$m = 4$
		T^3	25 ($p = 4$, $q = 1$)	
	$S^3/Q_{8k} \times S^1$	$SO(3)$	8	$m = 4k$
$Z + Z_k$	$L(k,1) \times S^1$	$SU(2)$	3 ($n = k$)	Reidemeister torsion
		T^3	25 ($p = k$, $q = 1$)	
	$L(k,q) \times S^1$	T^3	25 ($p = k$)	Reidemeister torsion
	Twisted $L(2k,1) \to M \to S^1$	$SU(2)$	4 ($n = 2k$)	$m = 2k$, $o = Z$, $Q = Z$
$Z + Z_2^2$	$\mathbf{RP}^2 \times K^2$	$SO(3) \times T^1$	39	$m = 2$, $o = 0$
	$(\mathbf{RP}^3 \# \mathbf{RP}^3) \times S^1$	$SO(3) \times T^1$	29	$m = 2$, $o = Z$
	$S^3/Q_{8n} \to M \to S^1$	$SO(3)$	9	$m = 4n$
	$(S^3/Q_8) \times S^1$	$SO(3)$	5	$m = 4$
1	S^4	$SO(4)$	28	$r = 0$
		$SO(3) \times T^1$	37	
		$SU(2)$	47	
		$SO(3)$	62	
	$\mathbf{CP}^2$	$SU(2)$	46	$r = 1$
	$\mathbf{CP}^2 \# -\mathbf{CP}^2$	$SU(2)$	45	$r = 2$, skew
		$SU(2)$	50 (n odd)	
	$S^2 \times S^2$	$SO(3) \times T^1$	36	$r = 2$, product
		$SU(2)$	50 (n even), or 55	
Z_2	$\mathbf{RP}^4$	$SO(4)$	27	$r = 1$, $o = 0$
		$SU(2)$	44	
		$SO(3)$	60	
	$S^2 \times \mathbf{RP}^2$	$SO(3) \times T^1$	31	$r = 2$, $o = 0$, product
		$SO(3) \times T^1$	34	
		$SO(3) \times T^1$	41	
		$SU(2)$	49 (n even), or 52	
		$SO(3)$	54 (n even)	
		$SO(3)$	61	
		$SO(3)$	65 (n even)	
	$\mathbf{CP}^2 \# \mathbf{RP}^4$	$SU(2)$	43	$r = 2$, $o = 0$, skew; Covered by $\mathbf{CP}^2 \# \mathbf{CP}^2$
	Twisted, non-orientable $S^2 \to M \to \mathbf{RP}^2$	$SO(3)$	54 (n odd)	$r = 2$, $o = 0$, skew; Covered by $S^2 \times S^2$

$H_1(M)$	M^4	G	Action	Invariants
	Twisted $S^2 \to M \to \mathbf{RP}^2$	$SO(3) \times T^1$ $SO(3)$	31 65 (n odd)	$r = 2$, o $= Z$, skew
	Twisted $\mathbf{RP}^2 \to M \to S^2$	$SO(3)$	49 (n odd)	$r = 2$, o $= 0$, skew Covered by $\mathbf{CP}^2 \# \mathbf{CP}^2$
Z_2^2	$\mathbf{RP}^4 \# \mathbf{RP}^4$	$SO(4)$ $SU(2)$	26 42	Rank(P) $= 1$, $m = 2$
	$\mathbf{RP}^2 \times \mathbf{RP}^2$	$SO(3) \times T^1$ $SO(3)$	40 59, 64 (n even)	$m = 2$, product, rank $= 0$
	Twisted $\mathbf{RP}^2 \to M \to \mathbf{RP}^2$	$SO(3)$	64 (n odd)	$m = 2$, skew rank $= 0$
	$S^1 \to M \to S^3/O$	$SO(3)$	56	$m = 8$, rank $= 1$ P has 3-torsion
	M^4	$SO(3)$	58	$m = 8$, rank $= 1$ P has no 3-torsion
Z_2^3	A class of fibrations $T^2 \to M \to \mathbf{RP}^2$	$SO(3)$	53 (n even)	$|P| = 4n$, $Q = Z_2 * Z_2$
	A class of fibrations $K^2 \to M \to \mathbf{RP}^2$	$SO(3)$	57, 63	$|P| = 2n$, $Q = Z_2 + Z_2$
$Z_2 + Z_p$	A class of fibrations $T^2 \to M \to \mathbf{RP}^2$	$SO(3)$	53 (n odd)	$|P| = 2n$, $C = Z_2$, $Q = Z_2 * Z_2$
	A class of fibrations $K^2 \to M \to S^2$	$SU(2)$	48	$|P| = 2n$, $C = Z_p$
	A class of fibrations $K^2 \to M \to \mathbf{RP}^2$	$SO(3)$	51	$|P| = 4n$, $C = Z_2$, $Q = Z_2$

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