APPROXIMATE GREEN FUNCTIONS AS A TOOL TO PROVE CORRECTNESS OF A FORMAL APPROXIMATION IN A MODEL OF COMPETING AND DIFFUSING SPECIES

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The purpose of this paper is two-fold; firstly, we shall derive some new results concerning a singular perturbation problem describing the stationary distribution of two competing and diffusing species; secondly, we want to demonstrate the power of a technique using approximate Green functions to prove the validity of a constructed formal approximation in a singular perturbation problem.

A mathematical model for the spatial distribution of two species consists of two coupled 2nd order O.D.E. on the interval $[-1, 1]$ with Neumann boundary conditions:

(1.1) $\varepsilon^2 u'' = f(x, u, v), \qquad v'' = g(x, u, v)$

(1.2) $u'(-1) = v'(-1) = 0, \quad u'(1) = v'(1) = 0.$

Here, ε is a small parameter > 0 and $'$ denotes derivation w.r.t. the x-variable.

This model (or rather its time-dependent version) has been proposed by several authors to explain the coexistence of competing and diffusing species in some subdomain and not elsewhere, cf. [4], [9], [5], [6], [10]. For certain non-linearities f, g there is the possibility of a solution with a sharp transition phenomenon in the u-variable at an internal point $y \in (-1, 1)$. The domain $(-1, 1)$ is then subdivided in two subdomains $(-1, y)$ and $(y, 1)$ where u behaves essentially different. At y the jump in the u-behaviour is smoothed by an internal layer.

These results are derived by constructing an asymptotic approximation for $\varepsilon \downarrow 0$ of the solution of (1.1–2). In [4], only a first approximation without detailed information on the internal layer is used. In [9], all over the domain higher order terms are included in the construction. This is done by dealing with two separate problems on $[-1, y + \delta(\varepsilon)]$ and $[y + \delta(\varepsilon), 1]$ with Dirichlet boundary conditions at the transition point. Free constants introduced in this way, such as $\delta(\varepsilon)$, are determined later on by requiring a smooth connection at the transition point. In this work a restriction has to be made about the non-linearity g (see [9], assumption

(ii)) just to ensure the solvability of each of the artificial problems on the subintervals $[-1, y + \delta]$ and $[y + \delta, 1]$. Though this assumption is sufficient for the construction, it is far from being sharp.

Once in [1], [9] a formal approximation has been obtained, its validity is proved. However, a purely proof-technical assumption has to be made, namely a small interaction between one of the species and the other. (For example: g_u is required to be sufficiently small). Such an assumption does not play a role in the construction, nor does it have a biological background. In our opinion this was a somewhat unsatisfactory situation and at the same time it was for us a challenge motivating this work.

In this work we shall construct a global formal approximation of arbitrary algebraic order in ε for the solution of (1.1–2), see §2. This is done under less restrictive conditions than in [9]. For the construction we use a method of matched asymptotic expansions (cf. [6]), in which the internal layer is dealt with in a direct way (without interval subdivision as in [9]). It appears, that the problem for the higher order terms in the internal layer exhibits a *resonance* phenomenon. This is due to an eigenvalue 0 in the spectrum of the linearized operator governing the local expansion, for the homogeneous equation has a non-trivial solution decaying exponentially towards $\pm\infty$. In order to be able to construct higher order terms a *global solvability condition* concerning terms in $[-1, y]$, $[y, 1]$ as well as in the internal layer in a coupled way has to be satisfied. This solvability condition can nicely be interpreted in terms of the solvability of the reduced ($\varepsilon = 0$) problem. Namely, it is equivalent to requiring that the solution of the reduced problem is stable for small model-perturbations, i.e. small perturbations of the non-linearities f and g.

Furthermore, we shall prove the *validity* of the constructed approximated *under precisely the same conditions* necessary to be able to do the construction. Smallness of the interaction of one of the species with the other is not necessary in this proof. The proof is based on linearisation at the constructed approximation and a fixed point argument. In itself this idea is rather standard, cf. [7], [8], [13], [1]. A central point in this type of proof is the analysis of the inverted, linearised operator in order to get an estimate for its norm. Several techniques have been used, such as maximum principles, a-prior estimates, variational principles, but in this case under the conditions we want to impose, neither of these methods seem to work. Of course, another method to analyse the inverse, linearised operator would be to use its Green kernel. This method is rather unpopular and for a good reason: exact Green functions are hard to obtain (cf. [12], [11]). However, it seems to have been overlooked, that instead of an exact Green

function a sufficiently accurate, *approximate Green function* does the job just as well. Such an approximate Green function can be found in a constructive way, actually its construction is quite analogous to the construction of higher order terms in the formal approximation of the solution, see §3. This explains that one can find such an approximate Green function under exactly the same conditions, which have to be introduced to make the construction of the formal approximation including higher order terms possible.

For these reasons the technique to use approximate Green functions to invert the linearised operator approximately is very powerful in our opinion: it is potentially applicable to a wide range of problems and it works in rather complicate situations.

2. Construction of a formal approximation. We shall consider the problem as given in (1.1–2) with f and g elements of $C^\infty([-1,1] \times \mathbf{R}^2)$ in the situation, where the reduced problem has a solution with a jump in the u-variable in some internal point of the interval. Thus we start the construction with:

2.1. *The 0th order approximation.* Let us call the internal transition point y and the 0th order regular approximation at the left and right of $y U_0$, V_0 and $\overline{U}_0$, $\overline{V}_0$, respectively. Then we suppose:

Assumption 0:
The reduced regular problem for U_0, V_0, $y, \overline{U}_0$ and $\overline{V}_0$:

$$(2.1.1) \qquad f(x, U_0, V_0) = 0 \qquad f(x, \overline{U}_0, \overline{V}_0) = 0$$
$$V_0'' = g(x, U_0, V_0) \qquad \overline{V}_0'' = g(x, \overline{U}_0, \overline{V}_0)$$
$$\textit{for } x \in [-1, y] \qquad \textit{for } x \in [y, 1]$$

with the boundary conditions

$$(2.1.2) \qquad\qquad V_0'(-1) = 0, \quad \overline{V}_0'(1) = 0$$

and the continuity conditions in the transition point

$$(2.1.3) \qquad\qquad V_0(y) = \overline{V}_0(y), \quad V_0'(y) = \overline{V}_0'(y)$$

plus an extra condition to determine the transition point namely

$$(2.1.4) \qquad\qquad \int_{U_0(y)}^{\overline{U}_0(y)} f(y, u, V_0(y))\, du = 0$$

has a solution such that $y \in (-1, 1)$ and

$$U_0, V_0 \in C^\infty[-1, y], \quad \overline{U}_0, \overline{V}_0 \in C^\infty[y, 1]$$

with a jump in the u-variable at the transition point y:

(2.1.5) $U_0(y) \neq \bar{U}_0(y).$

By rescaling the u variable as $(u - U_0(y))/(\bar{U}_0(y) - U_0(y))$ and the v-variable as $v - V_0(y)$ and next denoting the transformed problem just as before in (1.1–2) we see, that it is no restriction to put

(2.1.6) $U_0(y) = 0, \quad \bar{U}_0(y) = 1, \quad V_0(y) = 0.$

We observe, that because of (2.1.1): $f(y, 0, 0) = f(y, 1, 0) = 0$. Furthermore, we introduce some notation:

(2.1.7) $F(u) = \int_0^u f(y, \bar{u}, 0) \, d\bar{u}$

$f_u^0(x) = \dfrac{\partial f}{\partial u}(x, U_0(x), V_0(x)) \quad \text{for } x \in [-1, y]$

$\bar{f}_u^0(x) = \dfrac{\partial f}{\partial u}(x, \bar{U}_0(x), \bar{V}_0(x)) \quad \text{for } x \in [y, 1]$

and g_u^0, $\bar{g}_u^0$ are defined analgously.
Note, that $F(0) = 0$ and that (2.1.4) implies: $F(1) = 0$.

In addition to assumption 0 we require:

Assumption 0':

(2.1.9) $f_u^0 > 0 \quad on \ [-1, y], \qquad \bar{f}_u^0 > 0 \quad on \ [y, 1]$

and:

(2.1.10) $F(u) > 0 \quad on \ (0, 1).$

The condition in (2.1.9) is assumed because of the construction of higher order terms, see (2.2.2). As a consequence of (2.1.10) we can calculate the transition layer in the u-variable near the transition point in 0th order:

(2.1.11) $u \simeq T_0(\xi) + \cdots \quad \text{with } \xi = (x - y)/\varepsilon$

where T_0 satisfies

(2.1.12) $\dfrac{d^2 T_0}{d\xi^2} = f(y, T_0, 0), \quad \lim_{\xi \to -\infty} T_0(\xi) = 0, \quad \lim_{\xi \to \infty} T_0(\xi) = 1.$

The solution is given implicitly by

(2.1.13) $T_0(\xi) = \tau(\xi - \xi_0), \quad \int_{1/2}^{\tau(\zeta)} \dfrac{du}{\sqrt{2F(u)}} = \zeta.$

Here ξ_0 is an undetermined constant, due to translation invariancy in (2.1.12). The study of higher order terms will reveal the value of ξ_0, as we shall see.

It is a nice exercise to show, that T_0 approaches its limit values at ∞ in an exponential way:

$$(2.1.14) \quad \tau(\zeta) = A \exp(\mu\zeta) \cdot (1 + O(\exp(\mu\zeta))) \qquad \text{for } \zeta \to -\infty$$

$$\frac{d^k\tau}{d\zeta^k} = A'_k \exp(\mu\zeta) \cdot (1 + O(\exp(\mu\zeta))) \qquad \text{for } \zeta \to -\infty$$

$$\tau(\zeta) - 1 = B \exp(-\nu\zeta) \cdot (1 + O(\exp(-\nu\zeta))) \qquad \text{for } \zeta \to \infty$$

$$\frac{d^k\tau}{d\zeta^k} = B'_k \exp(-\nu\zeta) \cdot (1 + O(\exp(-\nu\zeta))) \qquad \text{for } \zeta \to \infty$$

with $\mu = f_u^0(y) > 0$, $\nu = \bar{f}_y^0(y) > 0$ and $k \geq 1$.
Furthermore, T_0 is strictly increasing on $(-\infty, \infty)$.

2.2. *On the effect of model perturbations.* As we saw, the 0th order terms in the approximation have been found with the proviso that ξ_0 still has to be determined. Our next concern will be the construction of higher order terms. For this construction we need another assumption. It is interesting, that this assumption can be understood in terms of model perturbations. Namely, suppose that the model is slightly perturbed, such that the non-linearities change into: $f + \delta f_p$, $g + \delta g_p$. Here δ measures the size of the perturbations and f_p, $g_p \in C^\infty([-1,1] \times \mathbf{R}^2)$. Now, we want the solution of the reduced problem (2.1–4) to be "stable" under such model perturbations, i.e. the perturbed reduced model has a solution for small δ of the form:

$$U_0 + \delta u_p + O(\delta^2), \quad V_0 + \delta v_p + O(\delta^2),$$
$$\bar{U}_0 + \delta \bar{u}_p + O(\delta^2), \quad \bar{V}_0 + \delta \bar{v}_p + O(\delta^2), \quad y + \delta y_p + O(\delta^2).$$

In order to make this condition more concrete we shall derive the problem for the perturbed quantities $u_p, v_p, \bar{u}_p, \bar{v}_p, y_p$. It is convenient to introduce the following shorthand notation

$$(2.2.1) \qquad \mathscr{L}(U, V, \bar{U}, \bar{V}, C) = \left(r_1, r_2, \bar{r}_1, \bar{r}_2; \alpha, \bar{\alpha}; j, j'; \beta\right)$$

for the following problem

$$(2.2.2) \qquad f_u^0 U + f_v^0 V + r_1 = 0 \qquad \bar{f}_u^0 \bar{U} + \bar{f}_v^0 \bar{V} + \bar{r}_1 = 0$$
$$V'' = g_u^0 U + g_v^0 V + r_2 \qquad \bar{V}'' = \bar{g}_u^0 \bar{U} + \bar{g}_v^0 \bar{V} + \bar{r}_2$$

$$(2.2.3) \qquad\qquad U'(-1) = \alpha, \quad \bar{U}'(1) = \bar{\alpha}$$

(2.2.4) $U'(y) = U(y) + j,$

$$\overline{U}'(y) + \overline{V}_0''(y)C = U'(y) + V_0''(y)C + j'$$

(2.2.5) $C \cdot \displaystyle\int_0^1 f_x(y, u, 0)\, du$

$$+ \left(V(y) + CV_0'(y) \right) \cdot \int_0^1 f_v(y, u, 0)\, du + \beta = 0.$$

Then the perturbed quantities have to satisfy

(2.2.6) $\mathscr{L}\left(u_p, v_p, \overline{u}_p, \overline{v}_p, y_p \right) = \left(f_p^0, g_p^0, \overline{f}_p^0, \overline{g}_p^0; 0, 0; 0, 0; 0 \right)$

with

$$f_p^0 = f_p\left(x, U_0(x), V_0(x) \right), \quad \overline{f}_p^0 = f_p\left(x, \overline{U}_0(x), \overline{V}_0(x) \right),$$

etc. Note, that $\mathscr{L}$ plays the role of a linearized operator of the reduced problem. Let us now formulate the condition in a precise way:

Assumption 1:
The homogeneous problem

(2.2.7) $\mathscr{L}(U, V, \overline{U}, \overline{V}, C) = (0, 0, 0, 0; 0, 0; 0, 0; 0)$

has only the trivial solution.

Using a Fredholm alternative it is then easily seen, that inhomogeneous problems such as (2.2.1) or (2.2.6) can be solved in a unique way. Further, it is not difficult to verify, that if the inhomogeneous term in (2.2.1) is in $(C^\infty[-1, y])^2 \times (C^\infty[y, 1])^2 \times \mathbf{R}^5$, the solution is an element of $(C^\infty[-1, y])^2 \times (C^\infty[y, 1])^2 \times \mathbf{R}$. Moreover, the solution of (2.2.1) depends continuously on the right hand side in an obvious way.

It will be clear, that assumption 1 implies the previously mentioned "stability" property, because of the implicit function theorem.

In §2.3 we shall see, that inhomogeneous problems such as (2.2.1) are the heart of the construction of higher order terms.

2.3. *A formal approximation of order* ε^{N+1}. The structure of this formal approximation of the solution of (1.1–2) is as follows:

(2.3.1) $u \simeq Z_1^n \underset{\text{def}}{=} (U^N + G^{N+1})H + (\overline{U}^N + \overline{G}^{N+1})\overline{H}$

$$+ (1 - H - \overline{H})T^N$$

$$v \simeq Z_2^N \underset{\text{def}}{=} (V^N + K^{N+2})H + (\overline{V}^N + \overline{K}^{N+2})H$$

$$+ (1 - H - \overline{H})Y^{N+2} + \varepsilon^{N+1}\left(k_{N+1} + \overline{k}_{N+1} \right).$$

Here U^N, V^N and $\bar{U}^N$, $\bar{V}^N$ denote the regular expansion at the left and right of the transition point

$$(2.3.2) \qquad U^N = \sum_{n=0}^{N} \varepsilon^n U_n(x) \qquad \bar{U} = \sum_{n=0}^{N} \varepsilon^n \bar{U}_n(x)$$

$$V^N = \sum_{n=0}^{N} \varepsilon^n V_n(x) \qquad \bar{V}^N = \sum_{n=0}^{N} \varepsilon^n \bar{V}_n(x).$$

Further G^{N+1}, K^{N+2} and $\bar{G}^{N+1}$, $\bar{K}^{N+2}$ describe boundary layer corrections at $x = -1$ and $x = 1$, respectively

$$(2.3.3) \qquad G^{N+1} = \sum_{n=1}^{N+1} \varepsilon^n G_n(\eta) \qquad \bar{G}^{N+1} = \sum_{n=1}^{N+1} \varepsilon^n \bar{G}_n(\bar{\eta})$$

$$K^{N+2} = \sum_{n=3}^{N+2} \varepsilon^n K_n(\eta) \qquad \bar{K}^{N+2} = \sum_{n=3}^{N+2} \varepsilon^n \bar{K}_n(\bar{\eta})$$

with layer variables $\eta = (x + 1)/\varepsilon$ and $\bar{\eta} = (1 - x)/\varepsilon$.

The other terms T^N, Y^{N+2} describe the behaviour in the internal layer

$$(2.3.4) \qquad T^N = \sum_{n=0}^{N} \varepsilon^n T_n(\xi), \quad Y^{N+2} = \sum_{n=0}^{N+2} \varepsilon^n Y_n(\xi)$$

where the layer variable is $\xi = (x - y)/\varepsilon$, as before in (2.1.11).

The functions H and $\bar{H}$ are suitably chosen cut-off functions. We take

$$(2.3.5) \qquad H(x, \varepsilon) = \chi\left(\frac{y - x}{\sigma(\varepsilon)}\right), \quad \bar{H}(x, \varepsilon) = \chi\left(\frac{x - y}{\bar{\sigma}(\varepsilon)}\right)$$

where χ is a smooth function on $\mathbf{R}$ with the properties $\chi(t) \equiv 0$ for $t \leq 1$, $\chi' \geq 0$, $\chi(t) \equiv 1$ for $t \geq 2$. The order functions $\sigma(\varepsilon)$, $\bar{\sigma}(\varepsilon)$ will be specified in §2.3.3.

Finally, k_{N+1} and $\bar{k}_{N+1}$ are introduced to take care that the approximation will exactly satisfy the boundary conditons:

$$(2.3.6) \qquad k_{N+1} = -\frac{dK_{N+2}}{d\eta}(0) \cdot (1 + x) \cdot \chi\left(\frac{y - x}{4(1 + y)}\right)$$

$$\bar{k}_{N+1} = \frac{d\bar{K}_{N+2}}{d\bar{\eta}}(0) \cdot (x - 1) \cdot \chi\left(\frac{x - y}{4(1 - y)}\right).$$

2.3.1. *On the internal layer.* Substitution of (2.3.4) into (1.1–2) and collecting terms of order ε^n provides us with the equations governing the internal layer expansion:

$$(2.3.1.1) \qquad \frac{d^2 T_n}{d\xi^2} - \hat{f}_u^0(\xi) T_n = F_n^1(\xi) \quad \text{for } n \geq 1$$

with

(2.3.1.2) (i) $F_1^1 = \xi \hat{f}_x^0 + Y_1 \hat{f}_v^0$

(ii) $F_n^1 = \left(\xi \hat{f}_{xu}^0 + T_1 \hat{f}_{uu}^0 + Y_1 \hat{f}_{uv}^0 \right) T_{n-1} + Y_n \hat{f}_v^0 + \tilde{F}_n^1,$

$$\tilde{F}_n^1 = \frac{1}{n!} \frac{d^n}{d\varepsilon^n} \left\{ f\left(y + \varepsilon \xi, \sum_{k=0}^{n-2} \varepsilon^k T_k, \sum_{k=0}^{n-1} \varepsilon^k Y_k \right) \right\}_{\varepsilon=0}$$

for $n \geq 2$

and

(2.3.1.3) $\dfrac{d^2 Y_{n+2}}{d\xi^2} = F_n^2(\xi)$ for $n + 2 \geq 1$

with

(2.3.1.4) (i) $F_{-1}^2 = 0, \quad F_0^2 = \hat{g}^0$

(ii) $F_n^2 = T_n \hat{g}_u^0 + \tilde{F}_n^2$

$$\tilde{F}_n^2 = \frac{1}{n!} \left(\frac{d}{d\varepsilon} \right)^n \left\{ g\left(y + \varepsilon \xi, \sum_{k=0}^{n-1} \varepsilon^k T_k, \sum_{k=0}^{n} \varepsilon^k Y_k \right) \right\}_{\varepsilon=0}$$

for $n \geq 1$.

Here the superscript $\hat{}^0$ means: evaluation with the argument $(y, T_0(\xi), 0)$.

Because of the matching for $\xi \to \pm \infty$ we are only interested in solutions of (2.3.1.1), which together with their derivatives are polynomially bounded. Now, T_0', the derivative of T_0 w.r.t. ξ, is a solution of the homogeneous equation corresponding to (2.3.1.1) and T_0' decays exponentially for $\xi \to \pm \infty$, see (2.1.14). Multiplying both sides in (2.3.1.1) with T_0' and integrating from $-\infty$ to ∞ it is clear, that this equation can only have a solution, which together with its derivatives is polynomially bounded, if:

(2.3.1.5) $\displaystyle\int_{-\infty}^{\infty} F_n^1(\xi) T_0'(\xi) \, d\xi = 0.$

On the other hand, if $F_n^1(\xi)$ and its derivatives are polynomially bounded and the orthogonality condition (2.3.1.5) is satisfied, then (2.3.1.1) has solutions of the desired type and these solutions are given by

(2.3.1.6) $T_n = \hat{T}_n - A_n T_0'$

$$\hat{T}_n(\xi) = T_0'(\xi) \int_0^{\xi} \left[T_0'(\bar{\xi}) \right]^{-2} \int_{-\infty}^{\bar{\xi}} T_0'(\tilde{\xi}) F_n^1(\tilde{\xi}) \, d\tilde{\xi} \, d\bar{\xi}$$

with a free constant A_n, which will be determined later on. Therefore, the problem for the T_n's is characterised by a non-zero kernel of the operator. Thus, it turns out that the expansion of the u-variable in the internal layer is of *resonant type*.

Of course, the solution of (2.3.1.3) is given by

$$(2.3.1.7) \quad Y_1 = \gamma_1 + \gamma_1'\xi, \quad Y_n = \gamma_n + \gamma_n'\xi + \hat{Y}_n(\xi), \quad \text{for } n \geq 2$$

with $\hat{Y}_n(\xi) = \int_0^\xi (\xi - \bar{\xi}) F_{n-2}^2(\bar{\xi}) \, d\bar{\xi}$ and certain constants γ_n, γ_n', which will follow later on by matching with the regular expansion.

Note, that $\hat{T}_n$, $\hat{Y}_n$ and $\hat{Y}_{n+1}$ follow in a unique way once $T_0, \ldots, T_{n-1}$ and $Y_1, \ldots, Y_{n-1}$ have been determined.

Using induction w.r.t. n it is now easy to check, that $\hat{T}_n$ and $\hat{Y}_n$ are smooth functions, which behave as a polynomial of degree n plus an exponentially vanishing term both for $\xi \to -\infty$ and $\xi \to \infty$.

$$(2.3.1.8) \quad \hat{T}_n \underset{\xi \to -\infty}{\sim} P_n(\xi) = \sum_{k=0}^n p_k^n \xi^k, \quad \hat{Y}_n \underset{\xi \to -\infty}{\sim} Q_n(\xi) = \sum_{k=0}^n q_k^n \xi^k$$

$$\hat{T}_n \underset{\xi \to \infty}{\sim} \bar{P}_n(\xi) = \sum_{k=0}^n \bar{p}_k^n \xi^k, \quad \hat{Y}_n \underset{\xi \to \infty}{\sim} \bar{Q}_n(\xi) = \sum_{k=0}^N \bar{q}_k^n \xi^k.$$

Moreover, the derivatives behave in analogous way. After these preparations we are in the position to deduce some useful relations. In the first place we shall exploit the matching of the v-component in the layer and the regular expansion. This leads us to certain jump conditions for the regular expansion at the free surface in the following way.

A. Comparing constants terms in the expansion of Y^N for $\xi \to \pm\infty$ and V^N, $\bar{V}^N$ for $x \to y$ we find that we have to impose the following matching relations:

$$(2.3.1.9) \qquad\qquad \bar{V}_1(y) = V_1(y) = \gamma_1$$

and for $n \geq 1$:

$$(2.3.1.10) \quad V_{n+1}(y) = \gamma_{n+1} + q_0^{n+1}, \quad \bar{V}_{n+1}(y) = \gamma_{n+1} + \bar{q}_0^{n+1}.$$

Hence:

$$(2.3.1.11) \qquad\qquad \bar{V}_{n+1}(y) = V_{n+1}(y) + j_n,$$

where $j_n =_{\text{def}} q_0^{n+1} - \bar{q}_0^{n+1}$ is completely determined by $T_0, \ldots, T_{n-1}$, $Y_1, \ldots, Y_{n-1}$. Note, that the jump condition in (2.3.1.11) is of the form given in (2.2.4).

B. Comparing linear terms in the expansion of Y^N for $\xi \to \pm\infty$ with those arising by expansion of V^N, $\overline{V}^N$ in the variable $y + \varepsilon\xi$ we are led to the following matching relations:

$$(2.3.1.12) \qquad\qquad \gamma_1' = V_0'(y)$$

and for $n \geq 0$:

$$(2.3.1.13) \quad V_{n+1}'(y) = \gamma_{n+2}' + q_1^{n+2}, \quad \overline{V}_{n+1}'(y) = \gamma_{n+2}' + \overline{q}_1^{n+2}.$$

Hence:

$$(2.3.1.14) \qquad\qquad \overline{V}_{n+1}'(y) = V_{n+1}'(y) + q_1^{n+2} - \overline{q}_1^{n+2}.$$

We shall now demonstrate, that (2.3.1.14) is of the form given in (2.2.4) where the role of C is played by ξ_0 if $n = 0$ and by A_n if $n \geq 1$.

In the case $n = 0$ we use that

$$\frac{d\hat{Y}_{n+2}}{d\xi} = \int_0^{\xi} g(y, T_0(\bar{\xi}), 0)\, d\bar{\xi}$$

$$= g(y, 1, 0)\xi + \int_{-\xi_0}^{\infty} [g(y, \tau(\zeta), 0) - g(y, 1, 0)]\, d\zeta, \quad \text{for } \xi \to \infty$$

$$= g(y, 0, 0)\xi - \int_{-\infty}^{-\xi_0} [g(y, \tau(\zeta), 0) - g(y, 0, 0)]\, d\zeta, \quad \text{for } \xi \to -\infty$$

with $\tau(\zeta)$ as in (2.1.13–14). As a consequence we obtain

$$q_1^2 - \overline{q}_1^2 = j_0' + \xi_0[g(y, 0, 0) - g(y, 1, 0)],$$

with

$$j_0' = -\int_{-\infty}^0 [g(y, \tau(\zeta), 0) - g(y, 0, 0)]\, d\zeta$$

$$-\int_0^{\infty} [g(y, \tau(\zeta), 0) - g(y, 1, 0)]\, d\zeta.$$

Since $V_0''(y) = g(y, 0, 0)$ and $\overline{V}_0''(y) = g(y, 1, 0)$, (2.3.1.14) yields:

$$(2.3.1.15) \quad \overline{V}_1'(y) + \overline{V}_0''(y)\xi_0 = V_1'(y) + V_0''(y)\xi_0 + j_0'.$$

This relation is indeed of the form given in (2.2.4) and j_0' is completely determined.

In the case $n \geq 1$ one has

$$\frac{d\hat{Y}_{n+2}}{d\xi} = -A_n \int_0^{\infty} g(y, T_0(\bar{\xi}), 0) T_0'(\bar{\xi})\, d\bar{\xi} + \left(\frac{d\hat{Y}_{n+2}}{d\xi}\right)_r,$$

where the latter term is completely determined by $T_0, \ldots, T_{n-1}, Y_1, \ldots, Y_n$. It is now easy to verify that (2.3.1.14) yields:

$$(2.3.1.16) \quad \overline{V}'_{n+1}(y) + \overline{V}''_0(y)A_n = V'_{n+1}(y) + V''_0(y)A + j'_n.$$

This is again of the form specified in (2.2.4) and j'_n depends only on $T_0, \ldots, T_{n-1}, Y_1, \ldots, Y_n$.

Another useful relation is found by analyzing the orthogonality condition (2.3.1.5) somewhat further.

C. In the first place substitution of (2.3.1.2–(i)) with

$$(2.3.1.17) \qquad\qquad Y_1 = V_1(y) + V'_0(y)\xi$$

and a change of variables $\zeta = \xi + \xi_0$ yields

$$
(2.3.1.18) \quad
\begin{aligned}
&\xi_0 \cdot \int_0^1 f_x(y, u, 0)\, du \\
&+ \left(V_1(y) + \xi_0 V'_0(y) \right) \cdot \int_0^1 f_v(y, u, 0)\, du + \beta_0 = 0
\end{aligned}
$$

with

$$\beta_0 = \int_{-\infty}^{\infty} \left\{ f_x(y, \tau(\zeta), 0) + V'_0(y) f_v(y, \tau(\zeta), 0) \right\} \zeta \tau'(\zeta)\, d\zeta.$$

Note, that (2.3.1.18) is of the form given in (2.2.5) with ξ_0 playing the role of C. The constant β_0 is known.

Next, we substitute F^1_{n+1}, as in (2.3.1.2–(ii)) in the orthogonality condition for the equation T_{n+1} and we use the following expressions for T_n and Y_{n+1}: $T_n = \hat{T}_n - A_n T'_0$ and

$$(2.3.1.19) \quad Y_{n+1} = \left(V_{n+1}(y) + q_0^{n+1} \right) + \left(V'_n(y) - q_1^{n+1} \right)\xi + \hat{Y}_{n+1}.$$

Then several integrations by parts in combination with the equation for T_1 lead us to

$$
(2.3.1.20) \quad
\begin{aligned}
&A_n \cdot \int_0^1 f_x(y, u, 0)\, du \\
&+ \left(V_{n+1}(y) + A_n V'_0(y) \right) \cdot \int_0^1 f_v(y, u, 0)\, du + \beta_n = 0
\end{aligned}
$$

with

$$
\begin{aligned}
\beta_n = \int_{-\infty}^{\infty} T'_0 \Big[\hat{T}_n \{ \xi \hat{f}^0_{xu} + Y_1 \hat{f}^0_{uv} \} \\
+ \hat{f}^0_v \{ -q_0^{n+1} + \left(V'_n(y) - q_1^{n+1} \right)\xi + \hat{Y}_{n+1} \} + \tilde{F}^1_{n+1} \Big]\, d\xi.
\end{aligned}
$$

The important thing is, that (2.3.1.20) is again of the form given in (2.2.5), while β_n depends only on $T_0, \ldots, T_{n-1}, Y_1, \ldots, Y_n, V_n$. The rather surprising conclusion of this section is, that the relations derived by analysis of

the internal layer have a lot to do with the reduced problem after a model perturbation, though no internal layer is present in that problem, only a transition point.

2.3.2. *A scheme to determine all higher order terms uniquely.* The construction of the higher order terms is organised as an iterative procedure. At the start U_0, V_0, y, $\overline{U}_0$, $\overline{V}_0$ are known and T_0 is known except for the value of ξ_0. Further, G_1 and $\overline{G}_1$ can now in the usual way be determined as the solutions of the homogeneous boundary layer equations vanishing at infinity and correcting the error in the b.c. due to U_0, $\overline{U}_0$.

$$(2.3.2.1) \qquad G_1(\eta) = \frac{U_0'(-1)}{a} e^{-a\eta}, \quad \overline{G}_1(\overline{\eta}) = \frac{-U_0'(1)}{\overline{a}} e^{-\overline{a}\overline{\eta}}$$

with

$$a = \sqrt{f_u^0(-1)} > 0, \quad \overline{a} = \sqrt{\overline{f}_u^0(1)} > 0.$$

Then we proceed to determine the other terms and constants according to the following scheme:

step 1.   ✓   a. U_1, V_1, ξ_0, $\overline{U}_1$, $\overline{V}_1$
                                          b. G_2, $\overline{G}_2$; K_3, $\overline{K}_3$
                                          c. T_0, γ_1, Y_1, $\hat{T}_1$, $\hat{Y}_2$, γ_2'.

          ✓
          ⋮
          ✓

step $(n + 1)$, $n \geq 1$ a. U_{n+1}, V_{n+1}, A_n, $\overline{U}_{n+1}$, $\overline{V}_{n+1}$
                                          b. G_{n+2}, $\overline{G}_{n+2}$, K_{n+3}, $\overline{K}_{n+3}$
                                          c. T_n, γ_{n+1}, Y_{n+1}, $\hat{T}_{n+1}$, $\hat{Y}_{n+2}$, γ_{n+2}'.

          ✓

The problem for U_{n+1}, V_{n+1}, A_n, $\overline{U}_{n+1}$, $\overline{V}_{n+1}$ is of the form (2.2.1):

$$(2.3.2.2) \qquad \mathscr{L}\big(U_{n+1}, V_{n+1}, A_n, \overline{U}_{n+1}, \overline{V}_{n+1}\big)$$
$$= \big(r_n^1, r_n^2, \overline{r}_n^1, \overline{r}_n^2; \alpha_n, \overline{\alpha}_n; j_n, j_n'; \beta_n\big)$$

also in the case $n = 0$, if we identify A_0 with ξ_0.

Substitution of the regular expansion in (1.1–2) yields:

$$(2.3.2.3) \quad r_n^1 = \frac{1}{(n+1)!}\left[\left(\frac{d}{d\varepsilon}\right)^{n+1} f\left(x, \sum_{k=0}^{n} \varepsilon^k U_k, \sum_{k=0}^{n} \varepsilon^k V_k\right)\right]_{\varepsilon=0} + U_{n-1}''$$

$$r_n^2 = \frac{1}{(n+1)!}\left[\left(\frac{d}{d\varepsilon}\right)^{n+1} g\left(x, \sum_{k=0}^{n} \varepsilon^k U_k, \sum_{k=0}^{n} \varepsilon^k V_k\right)\right]_{\varepsilon=0},$$

where for $n = 0$ we interprete U_{n-1}'' as 0.

For $\bar{r}_n^1$, $\bar{r}_n^2$ analogous expressions hold.

The boundary values α_n, $\bar{\alpha}_n$ are taken in such a way that the effect of K_{n+2}, $\overline{K}_{n+2}$ is annihilated

$$(2.3.2.4) \qquad \alpha_n = -\frac{dK_{n+2}}{d\xi}(0), \quad \bar{\alpha}_n = \frac{d\overline{K}_{n+2}}{d\xi}(0).$$

In the case $n = 0$ we take $\alpha_n = \bar{\alpha}_n = 0$, of course. The definition of j_n was given in (2.3.1.11), j_n' was introduced in (2.3.1.15) for $n = 0$ and in (2.3.1.16) for $n \geq 1$. It is now clear that at the end of nth iteration step all information to determine U_{n+1}, V_{n+1}, A_n, $\overline{U}_{n+1}$, $\overline{V}_{n+1}$ is available and part a of the $(n + 1)$th iteration step consists of solving (2.3.2.2). This can be done because of assumption 1.

Part b of the $(n + 1)$th iteration step is a matter of more or less standard boundary layer theory. G_{n+2} is defined as the solution of:

$$(2.3.2.5) \qquad \frac{d^2 G_{n+2}}{d\eta^2} - a^2 G_{n+2} = w_{n+2}^1$$

$$\frac{dG_{n+2}}{d\eta}(0) = -U_{n+1}'(-1), \quad \lim_{\eta \to \infty} G_{n+2}(\eta) = 0$$

with

$$w_{n+2}^1 = \frac{1}{(n + 2)!} \left[\left(\frac{d}{d\varepsilon} \right)^{n+2} \{ f(x, U^{n+1} + G^{n+1}, V^{n+1} + K^{n+2}) \right.$$

$$\left. -f(x, U^{n+1}, V^{n+1}) \} \right]_{\varepsilon=0}$$

where $x = -1 + \varepsilon\eta$ and U^{n+1}, etc. are defined as before in (2.3.2–3). Using induction one finds, that G_{n+2} is uniquely determined and equal to a polynomial of degree $(n + 1)$ in ξ multiplied with $\exp(-a\xi)$. The second component is found from

$$(2.3.2.6) \qquad \frac{d^2 K_{n+3}}{d\eta^2} = w_{n+1}^2, \quad \lim_{\eta \to \infty} K_{n+3}(\eta) = 0$$

with

$$w_{n+1}^2 = \frac{1}{(n + 1)!} \left[\left(\frac{d}{d\varepsilon} \right)^{n+1} \{ g(x, U^{n+1} + G^{n+1}, V^{n+1} + K^{n+1}) \right.$$

$$\left. -g(x, U^{n+1}, V^{n+1}) \} \right]_{\varepsilon=0}.$$

It is easy to check, that K_{n+3} is uniquely determined and equal to a polynomial of degree n in ξ multiplied with $\exp(-a\xi)$. Of course the construction of $\overline{G}_{n+2}$ and $\overline{K}_{n+3}$ is completely analogous.

So, only the verification of part c of the $(n + 1)$th induction step now remains, but this is an immediate consequence of the results in §2.3.1, namely (2.3.1.6) provides us with T_n and $\hat{T}_{n+1}$, (2.3.1.10) yields γ_{n+1}; Y_{n+1} and $\hat{Y}_{n+1}$ follow from (2.3.1.7) and finally (2.3.1.13) combined with our knowledge of $\hat{Y}_{n+2}$ gives us the value of γ'_{n+2}.

This completes the calculation of the formal approximation as defined in (2.3.1).

2.4. *Estimation of the remainder in the equation.* The result of the construction is, that the formal approximation has the following properties:

$$(2.4.1)\quad \varepsilon^2 \frac{d^2 Z_1^N}{dx^2} = f\left(x, Z_1^N, Z_2^N\right) + R_1^N, \quad \frac{d^2 Z_2^N}{dx^2} = g\left(x, Z_1^N, Z_2^N\right) + R_2^N$$

$$(2.4.2)\quad \left(Z_1^N\right)'(-1) = \left(Z_2^N\right)'(-1) = 0, \quad \left(Z_1^N\right)'(1) = \left(Z_2^N\right)'(1) = 0.$$

Note, that the formal approximation satisfies the boundary conditions exactly, but some remainders arise in the equation. If N is sufficiently large, these errors are small. In the maximum norm on $[-1, 1]$ we get an estimate

$$(2.4.3)\qquad\qquad \left|R_1^N\right|_{\max} + \left|R_2^N\right|_{\max} \leq \tilde{R}_N \left[\varepsilon \ln\left(\frac{1}{\varepsilon}\right)\right]^{N+1},$$

if the order functions σ and $\bar{\sigma}$ in the cut-off near the transition point (see (2.3.5)) are suitably chosen:

$$(2.4.4)\qquad\qquad \sigma(\varepsilon) = \sigma_0 \varepsilon \ln\left(\frac{1}{\varepsilon}\right), \quad \bar{\sigma}(\varepsilon) = \bar{\sigma}_0 \varepsilon \ln\left(\frac{1}{\varepsilon}\right)$$

with σ_0 and $\bar{\sigma}_0$ sufficiently large, ε-independent constants.

The verification of (2.4.3) consists of two elements. The first element is a piece of algebraic reasoning to show, that the internal layer expansion and the regular expansion have a good overlap in the regions $\sigma \leq y - x \leq 2\sigma$ and $\bar{\sigma} \leq x - y \leq 2\bar{\sigma}$, compare [9], [2]. Consider on one hand T^N for $\xi \to -\infty$, $T^N \sim \sum_{n=0}^N \varepsilon^k P_n(\xi)$ with P_n as in (2.3.1.8) and on the other hand: the Taylor series up to $O(\varepsilon^N)$ of $\sum_{n=0}^N \varepsilon^n U_n(y + \varepsilon\xi)$ being $\sum_{n=0}^N \varepsilon^n \tilde{P}_n(\xi)$. Then both P_N and $\tilde{P}_n$ are polynomials in ξ of degree n and using the matching it is clear that these polynomials have the same constant term and for $n \geq 1$ also the same linear term. By the same reasoning $\sum_{n=1}^{N+2} \varepsilon^n \check{Q}_n$ with $\check{Q}_n = Q_n + \gamma_n + \gamma'_n\xi$, Q_n as in (2.3.1.8) and the

Taylor series up to $O(\varepsilon^{N+2})$ of $\sum_{n=1}^{N+2} \varepsilon^n V_n(y + \varepsilon\xi) = \sum_{n=1}^{N+2} \varepsilon^n \tilde{Q}_n$ are such that $\check{Q}_n$ and $\tilde{Q}_n$ are polynomials in ξ of degree n with the same constant and linear term. Moreover, by construction both $\sum_{n=0}^{N} \varepsilon^n P_n$, $\sum_{n+1}^{N+2} \varepsilon^n \check{Q}_n$ and $\sum_{n=0}^{N} \varepsilon^n \tilde{P}_n$, $\sum_{n=0}^{N+2} \varepsilon^n \tilde{Q}_n$ satisfy equation 1.1–2 with an error $O(\varepsilon^{N+1})$. It is now easy to verify with just some algebra on the coefficients of the polynomials, that $P_n = \tilde{P}_n$, $n = 0, \ldots, N$ and $\check{Q}_n = \tilde{Q}_n$, $n = 1, \ldots, N + 2$. In this way we can prove the overlap property:

(2.4.5)
$$|U^N - T^N| \le C|\varepsilon\xi|^{N+1}$$
$$|V^N - Y^{N+2}| \le C|\varepsilon\xi|^{N+3}$$

on the regions $\sigma \le y - x \le 2\sigma$ and $\bar{\sigma} \le x - y \le 2\bar{\sigma}$ with an ε-independent constant C.

The second element is a rather lengthy, but in principal elementary calculation using the well-known properties of Taylor series expansion to check (2.4.3) in the various subregions of $[-1, 1]$. For example, in the region $y - \sigma \le x \le y + \bar{\sigma}$ containing the internal layer we obtain for the error in the 2nd component the formula

(2.4.6)
$$R_2^{N,i} = \frac{1}{N!} \int_0^1 (1 - t)^N \left(\frac{d}{dt}\right)^{N+1}$$
$$\times g\left(y + \varepsilon t\xi, \sum_{n=0}^{N} (\varepsilon t)^n T_n, \sum_{n=1}^{N+2} (\varepsilon t)^n Y_n\right) \cdot dt$$

and it is easy to check, that

(2.4.7)
$$|R_2^{N,i}| \le C\varepsilon^{N+1}(1 + |\xi|)^{N+1}$$

on that region.

In the region $y - 2\sigma \le x \le y - \sigma$ one splits the remainder as

(2.4.8) $R_2^N = R_2^{N,i} - \dfrac{d^2}{dx^2}\{(V^N - Y^{N+2})H\}$
$$+ g(X, T^N + (U^N - T^N)H, Y^{N+2} + (V^N - Y^{N+2})H)$$
$$- g(X, T^N, Y^{N+2}).$$

Using (2.4.6) and the overlap properties as in (2.4.5) together with analogous estimates for the derivatives, such as

$$\left|\frac{d}{dx}(V^N - Y^{N+2})\right| \le C_1 \varepsilon^{N+1}\xi^N \quad \text{for } y - 2\sigma \le x \le y - \sigma$$

we then find in this region an estimate for R_2^N of the same type as (2.4.7).

Further details of the proof of (2.4.3) are left to the reader.

3. Proof of the validity of the formal approximation. In this section we shall prove the existence of a solution u, v of (1.1–2), such that

(3.1) $$\left| u - Z_1^N \right|_{\max} + \left| v - Z_2^N \right|_{\max} = O(\varepsilon^{N+1})$$

where $|\ \ |_{\max}$ is the maximum norm of $[-1, 1]$.

The proof will be based on a contraction principle, which we shall first discuss in general.

3.1. *The general idea of the proof.* Suppose, that one is in the situation where for a singular perturbation problem

(3.1.1) $$F_\varepsilon(u) = 0$$

a formal approximation Z has been constructed, for which

(3.1.2) $$F_\varepsilon(Z) = R$$

where the remainder term is small, i.e. $R \in B$ with B a Banach-space with norm, $|\ \ |$, say:

(3.1.3) $$|R| = \mu(\varepsilon).$$

Let us denote the linearised operator $DF_\varepsilon(Z)$ by L_ε and the strict non-linear terms by N_ε, i.e.

(3.1.4) $$F_\varepsilon(Z + v) \underset{\text{def}}{=} F_\varepsilon(Z) + L_\varepsilon v + N_\varepsilon(v).$$

The problem for the remainder term

(3.1.5) $$v = u - Z$$

can then be written as

(3.1.6) $$L_\varepsilon v = -R - N_\varepsilon(v).$$

Now, suppose that an *approximate right inverse* operator $(L_\varepsilon^{-1})'$ of L_ε can be found, such that

(3.1.7) $$L_\varepsilon\left(L_\varepsilon^{-1}\right)' = I + \delta(\varepsilon) K_\varepsilon$$

with I the identity on B, K_ε a bounded operator from B into itself and the operator norm satisfies:

(3.18) $$\left| \delta(\varepsilon) K_\varepsilon \right| \leq \frac{1}{2} \quad \text{for } 0 < \varepsilon \leq \varepsilon_1.$$

Let us put

(3.1.9) $$v = \left(L_\varepsilon^{-1}\right)' w$$

then w has to satisfy:

(3.1.10) $$\left(I + \delta K_\varepsilon\right) w = -R - N_\varepsilon\!\left(\left(L_\varepsilon^{-1}\right)' w\right).$$

This is an equation which can easily be solved by a fixed point argument.

Suppose the non-linearity satisfies a Lipschitz condition

$$(3.1.11) \qquad \left| N_\varepsilon\left(\left(L_\varepsilon^{-1}\right)' w_1\right) + N_\varepsilon\left(\left(L_\varepsilon^{-1}\right)' w_2\right) \right| \le l(\varepsilon, \rho) |w_1 - w_2|$$

for all $w_1, w_2 \in B(\rho) = \{ w \in B | \ |w| \le \rho \}$ with a Lipschitz constant

$$(3.1.12) \qquad l(\varepsilon, \rho) = \frac{\rho}{\nu(\varepsilon)} \quad \text{for } 0 < \rho \le \rho_2(\varepsilon), 0 < \varepsilon \le \varepsilon_2$$

with $\nu(\varepsilon) > 0$.

If the remainder is sufficiently small

$$(3.1.13) \qquad 4\mu(\varepsilon) \le \rho_0 \underset{\text{def}}{=} \min\left\{ \rho_2(\varepsilon), \tfrac{1}{16}\nu(\varepsilon) \right\}$$

for $0 < \varepsilon \le \varepsilon_0 \le \min(\varepsilon_1, \varepsilon_2)$, then there is a unique solution w of (3.1.10) in $B(\rho_0)$. This solution satisfies the estimate

$$(3.1.14) \qquad |w| \le 4\mu(\varepsilon).$$

In terms of a solution of (3.1.1) *this means, that if*

$$(3.1.15) \qquad |r| \le \tfrac{1}{4}\rho_0$$

then there is a unique solution of (3.1.1) *in the set* $Z + (L_\varepsilon^{-1})'B(\rho_0)$ *and the following error estimate holds*

$$(3.1.16) \qquad |u - Z| \le 4\left|\left(L_\varepsilon^{-1}\right)'\right| \cdot |R|.$$

In comparison with the proofs in [7], [8] or [13] the new point is that in this approach one works with an approximate right inverse of the linearised operator. In cases where L is a (partial or ordinary) differential operator (or an integro-differential operator, etc.) one can try to find such approximate inverses by using an approximate function of Green satisfying

$$(3.1.17) \qquad L_\varepsilon \, \mathrm{Gr}(x, t) = \delta(x - t) + h$$

where h is $o(1)$ for $\varepsilon \downarrow 0$ in a suitable sense. Of course, for systems of differential equations we can proceed analogously using an approximate Green matrix. The operator $(L_\varepsilon^{-1})'$ defined by

$$(3.1.18) \qquad \left[\left(L_\varepsilon^{-1}\right)' u\right](x) = \int_\Omega \mathrm{Gr}(x, t) u(t) \, dt$$

with Ω the domain of the functions under consideration is then a good candidate to be used as an approximate inverse with the desired properties.

3.2. *Construction of an approximate Green matrix.* In our case L_ε is an ordinary differential operator with as its domain the pairs of C^2 functions

on $[-1, 1]$ satisfying homogeneous boundary conditions of Neumann type:

$$(3.2.1) \qquad L_\varepsilon = \begin{pmatrix} \varepsilon^2 \dfrac{d^2}{dx^2} & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \check{f}_u & \check{f}_v \\ \check{g}_u & \check{g}_u \end{pmatrix}.$$

Here $\check{\ }$ means evaluation with the argument (x, Z_1^N, Z_2^N). We shall now construct an approximate Green matrix

$$(3.2.2) \qquad \mathrm{Gr} = \begin{pmatrix} I^1 & J^1 \\ I^2 & J^2 \end{pmatrix}$$

where $\mathrm{Gr}(x, t)$ has the following properties:
(i) $\mathrm{Gr} \in C([-1, 1]^2)$ and the restrictions of Gr to the triangles $\Delta_1 = \{(x, t) \mid -1 \le x \le t \le 1\}$ and $\Delta_2 = \{(x, t) \mid -1 \le t \le x \le 1\}$ are smooth up to the boundary of these triangles.
(ii) The jump in the derivative $\partial \mathrm{Gr}/\partial x$ at $x = t$, $[\partial \mathrm{Gr}/\partial x]_{x=t}$, is of the following form

$$(3.2.3) \qquad \left[\frac{\partial \mathrm{Gr}}{\partial x} \right]_{x=t} = 1 + \varepsilon \mathbf{A}(x, \varepsilon)$$

with 1 the identity and $\mathbf{A}$ uniformly bounded on $[-1, 1]$ for $\varepsilon \downarrow 0$.
(iii) The matrix function "$L_\varepsilon \mathrm{Gr}$", defined as $(L_\varepsilon \mathrm{Gr})(x, t)$ for $x \ne t$ with a jump at $x = t$, but without the distributional terms for $x = t$ in $L_\varepsilon \mathrm{Gr}$, satisfies:

$$(3.2.4) \qquad \int_{-1}^{1} \| \text{“}L_\varepsilon \mathrm{Gr}(x, t)\text{”} \| \, dt \le K \varepsilon^\nu$$

with constants $K, \nu > 0$ independent of x and ε and $\| \ \|$ some ε-independent matrix norm.
   (iv) Gr satisfies the homogeneous Neumann boundary conditions

$$(3.25) \qquad \frac{\partial \mathrm{Gr}}{\partial x} \bigg|_{x=-1} = \frac{\partial \mathrm{Gr}}{\partial x} \bigg|_{x=1} = 0.$$

Of course Gr will be dependent on ε, but in the notation we have suppressed this dependence.

   3.2.1. *The 2nd column of* Gr. In order to construct the 2nd column of Gr we put

$$(3.2.1.1) \qquad \begin{pmatrix} J^1 \\ J^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2}|t - x| \end{pmatrix} + \begin{pmatrix} \tilde{J}^1 \\ \tilde{J}^2 \end{pmatrix}.$$

Note, that the first term has the correct singularity at $x = t$. $\tilde{J}_1$ and $\tilde{J}^2$ will be continuously differentiable at $x = t$ and they are of the following form

(3.2.1.2) $\qquad \tilde{J}^1 = (J_0^1 + \varepsilon D_1)H + (\bar{J}_0^1 + \varepsilon \bar{D}_1)\bar{H}$

$$+ (1 - H - \bar{H}) \cdot \{\varepsilon^{-1}E_{-1} + E_0\}$$

(3.2.1.3) $\quad \tilde{J}^2 = J_0^2 H + \bar{J}_0^2 \bar{H} + (1 - H - \bar{H})\{J_0^2(y) + \varepsilon \hat{E}_1 + \varepsilon^2 \hat{E}_2\}.$

Here J_0^1, J_0^2, $\bar{J}_0^1$, $\bar{J}_0^2$ play the role of the regular expansion which together with some constant c will be found from a problem of type (2.2.1). The constant c comes from the term $\varepsilon^{-1}E_{-1}$ in the internal layer, where E_{-1} is just a solution of the homogeneous layer equation in (2.3.1.1).

(3.2.1.4) $\qquad\qquad\qquad E_{-1} = -cT_0'.$

Note, that the presence of the term E_{-1} is typically a resonance phenomenon. The next term has to satisfy the equation

(3.2.1.5) $\quad \left[\dfrac{d^2}{d\xi^2} - \hat{f}_u^0(\xi)\right]E_0 = -c\left(\xi \hat{f}_{ux}^0 + T_1 \hat{f}_{uu} + Y_1 \hat{f}_{uv}^0\right)T_0' + J_0^2(y) \cdot \hat{f}_v^0$

where the constant c is chosen, such that the r.h.s. is orthogonal to T_0' in L_2-sense. The solution E_0 is then given by the expression in (2.3.1.6). In this case the value of the free constant multiplying T_0' in this solution is unimportant, we shall take that constant $= 0$. Now J_0^1, J_0^2, $\bar{J}_0^1$, $\bar{J}_0^2$ and c follow from

(3.2.1.6) $\quad \mathscr{L}\left(J_0^1, J_0^2, \bar{J}_0^1, \bar{J}_0^2, c\right)$

$$= \left(\dfrac{1}{2}|t - x|f_v^0, \dfrac{1}{2}|t - x|g_v^0, \dfrac{1}{2}|t - x|\bar{f}_v^0,\right.$$

$$\left.\dfrac{1}{2}|t - x|\bar{g}_v^0; 0, 0; 0, 0; 0\right).$$

It is easy to check, that

(3.2.1.7) $\qquad E_0(\xi) \underset{\xi \to \infty}{\to} J_0^1(y), \quad E_0(\xi) \underset{\xi \to \infty}{\to} \bar{J}_0^1(y).$

where the convergence is of exponential type.

For the second component in the internal layer we obtain

(3.2.1.8) $\qquad\qquad\qquad \dfrac{d^2 \hat{E}_1}{d\xi^2} = -cT_0' \cdot \hat{g}_u^0$

$$(3.2.19) \quad \frac{d^2\hat{E}_2}{d\xi^2} = \hat{g}_u^0 E_0 + \hat{g}_v^0 J_0^1(y) - cT_0'\{\hat{g}_{uu}^0 T_1 + \hat{g}_{uv}^0 y_1 + \hat{g}_{ux}^0 \xi\} \underset{\text{def}}{=} \hat{w}_0.$$

Using the matching conditions and choosing constants which are still free in a convenient way this leads to

$$(3.2.1.10) \quad \hat{E}_1 = (J_0^2)'(y) \cdot \xi - c\int_{-\infty}^{\xi} [g(y, T_0(\xi), 0) - g(y, 0, 0)] \, d\xi$$

$$(3.2.1.11) \qquad \hat{E}_2 = \int_0^\infty (\xi - \bar{\xi})\hat{w}_0(\bar{\xi}) \, d\bar{\xi}.$$

A simple calculation shows, that the following overlap property holds:

$$(3.2.1.12) \quad |J_0^2(x) - \{J_0^2(y) + \varepsilon\hat{E}_1 + \varepsilon^2\hat{E}_2\}|$$

$$\le C(|x - y|^3 + \varepsilon + \varepsilon^2|\xi|) \quad \text{for } y - 2\sigma \le x \le y - \sigma.$$

Such an overlap is also present at the other side of the transition point. Moreover analogous estimates hold for the derivatives. Finally, D_1 and $\bar{D}_1$ are just corrections to satisfy the boundary codnitions

$$(3.2.1.13) \qquad D_1(\eta) = \frac{1}{a}(J_0^1)'(-1) \cdot \exp(-a\eta)$$

$$\bar{D}_1(\eta) = \frac{-1}{\bar{a}}(J_0^1)'(1) \cdot \exp(-\bar{a}\bar{\eta}),$$

compare (2.3.2.1).

We can now immediately verify, that the 2nd column of Gr as defined above has properties as necessary in view of (i)–(iv). We find, that the 2nd column of **A** is identically $\binom{0}{0}$, since the jump in the derivative is exactly correct, and we obtain

$$(3.2.1.14) \qquad \int_{-1}^1 |\text{``}L_\varepsilon \text{Gr''}_{\cdot,2}| \, dt \le K\varepsilon \ln\left(\frac{1}{\varepsilon}\right)$$

in an obvious notation.

The verification of (3.2.1.14) is analogous to §(2.4) and we leave the details to the reader.

To conclude this section we want to point out, how remarkable it is, that in the construction of the 2nd column of Gr we don't have to distinguish between various cases for the location of t. For example, t in the internal layer is no special case here, it simply introduces some ε-dependence in the "regular part" J_0^1, J_0^2, etc. but that is all. This will be different in the next section.

3.2.2. *The 1st column of* Gr. Here we distinguish between 3 regions for the location of t.

    (a) $t \in [-1, y - \sigma]$,   (b) $t \in [y + \bar{\sigma}, 1]$,   (c) $t \in [y - 2\sigma, y + 2\bar{\sigma}]$.

Let us call this column I^a, I^b and I^c in these respective cases. Then the 1st column I for $(x, t) \in [-1, 1]^2$ will be given by

$$(3.2.2.1) \qquad I = H_2 I^a + \bar{H}_2 I^b + \left(1 - H_2 - \bar{H}_2\right) I^c$$

with $H_2 = \chi((y - t)/\sigma)$, $\bar{H}_2 = \chi((t - y)/\bar{\sigma})$ with χ the cut-off function introduced in (2.3.5).

   ad a: I^a is introduced in the following way:

$$(3.2.2.2) \quad I^a = (1 - \bar{H}) \begin{pmatrix} -\dfrac{1}{2\varepsilon\lambda} \exp(-\lambda|\zeta|) + \varepsilon^{-1} M_{-1} \\ S_0 + \varepsilon N_1 \end{pmatrix} + \begin{pmatrix} \tilde{I}^{1,a} \\ \tilde{I}^{2,a} \end{pmatrix}.$$

Here $\lambda = \sqrt{f_u^0(t)}$ and $\zeta = (x - t)/\varepsilon$. It is clear, that the first term of the first component has exactly the correct jump in the derivative at $x = t$. To correct the error in the b.c. we take M_{-1} as the following solution of the homogeneous boundary layer equation at $x = -1$:

$$(3.2.2.3) \qquad M_{-1}(\eta) = -\frac{1}{2} \exp(-\lambda(1 + t)/\varepsilon) \cdot a^{-1} \exp(-a\eta)$$

with $a = \sqrt{f_u^0(-1)}$, compare (2.3.2.1).
   S_0 is introduced to compensate the effect of $-(2\varepsilon\lambda)^{-1}\exp(-\lambda|\zeta|)$ in the second equation

$$(3.2.2.4) \qquad S_0 = \int_x^y (x - x') g_u^0 \frac{e^{-\lambda|x' - t|/\varepsilon}}{2\varepsilon\lambda} \, dx'.$$

Note, that S_0 decays exponentially in ζ for $\zeta \to \infty$ and behaves as $(x - t)g_u^0(t) +$ exponentially small terms in ζ for $\zeta \to -\infty$.
   In a similar way N_1 compensates the effect of $\varepsilon^{-1}M_{-1}$:

$$(3.2.2.5) \qquad N_1 = g_u^0(-1) a^{-2} M_{-1}$$

   The other terms in (3.2.2.2) will be of the following form

$$(3.2.2.6) \qquad \tilde{I}^{1,a} = \left[I_0^1 + \varepsilon M_0\right] H + \left[\bar{I}_0^1 + \varepsilon \bar{M}_0\right] \bar{H}$$

$$+ (1 - H - \bar{H})\left\{-\varepsilon^{-1} A T_0' + \Lambda_0\right\}$$

$$\tilde{I}^{2,a} = I_0^2 H + \bar{I}_0^2 \bar{H} + (1 - H - \bar{H})\left\{I_0^2(y) + \varepsilon\Omega_1 + \varepsilon^2\Omega_2\right\}.$$

Since the process to construct $I^{1,a}$, $I^{2,a}$ is so similar to §(3.2.1), we only give a brief sketch.
   Having the calculations of §3.2.1 in mind it will not be surprising, that I_0^1, I_0^2, $\bar{I}_0^1$, $\bar{I}_0^2$ and A are the solution of an inhomogeneous problem

as in 2.2.1:

(3.2.2.7) $\quad \mathscr{L}\left(I_0^1, I_0^2, \bar{I}_0^1, \bar{I}_0^2, A\right)$

$$= \left(f_v^0 S_0, g_v^0 S_0, 0, 0; -S_0'(-1) - \frac{dN_1}{d\eta}(0), 0; 0, 0; 0\right).$$

Consequently the r.h.s. in the equation for Λ_0 is orthogonal to T_0' and Λ_0 is determined in the same way as E_0 in (3.2.1.5), but with c replaced by A and $J_0^2(y)$ replaced by $I_0^2(y)$. Of course, Ω_1, Ω_2 are found analogous to $\hat{E}_1$, $\hat{E}_2$ in (3.2.1.8–11) with the same changes $c \to A$, $J_0^2 \to I_0^2$. Finally, the boundary layer corrections M_1, $\overline{M}_1$ are as in (3.2.1.13), but with I_0^1 instead of J_0^1. It will be clear, that estimates analogous to (3.2.1.8) and (3.2.1.12) hold in this case. Therefore we find for this part of the first column the correct properties in view of (i)–(iv), specifically

(3.2.2.8) $\qquad \displaystyle\int_{-1}^{y} |\ ``L_\varepsilon \mathrm{Gr}_{0,1}"\ |\, dt \le K\varepsilon \ln\!\left(\frac{1}{\varepsilon}\right).$

ad b: The construction of I^b is completely analogous to the construction of I^a.

ad c: I^c will be of the following form:

(3.2.2.9) $\qquad I^c = \begin{pmatrix} -\dfrac{1}{2\varepsilon}\exp(-|\zeta|) \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{I}^{1,c} \\ \tilde{I}^{2,c} \end{pmatrix}$

with $\zeta = (x - t)/\varepsilon$ as before and with $t = y + \varepsilon\tau$, $\zeta = \xi - \tau$. Further:

(3.2.2.10) $\quad \tilde{I}^{1,c} = \left(\varepsilon^{-1}I_{-1}^1 + N_0\right)H + \left(\varepsilon^{-1}\bar{I}_{-1}^1 + \overline{N}_0\right)\overline{H}$

$$+ (1 - H - \overline{H})\left\{-\varepsilon^{-2}BT_0' + \varepsilon^{-1}\Lambda_{-1}\right\}$$

$$\tilde{I}^{2,c} = \varepsilon^{-1}I_{-1}^2 H + \varepsilon^{-1}\bar{I}_{-1}^2 \overline{H}$$

$$+ (1 - H - \overline{H})\left\{\varepsilon^{-1}I_{-1}^2(y) + \Omega_0 + \varepsilon\Omega_1\right\}.$$

It is not necessary to include more terms in the expansions here, since I^c is only used for t in an $O(\varepsilon \ln(1/\varepsilon))$ interval, which yields an extra small factor in the integration in (3.2.4).

Furthermore, the structure of the first column of Gr with t in the internal layer shows the resonance even more drastically then before, because of the term $-\varepsilon^{-2}BT_0'$.

In this case Λ_{-1} has to satisfy the equation

$$(3.2.2.11) \quad \left(\frac{d^2}{d\xi^2} - \hat{f}_u^0\right)\Lambda_{-1} = -\left(\xi \hat{f}_{xu}^0 + T_1 \hat{f}_{uv}^0\right)BT_0' + \hat{f}_v^0 I_{-1}^2(y)$$

$$+ \frac{1}{2}\left(1 - \hat{f}_u^0\right)e^{-|\xi|}$$

hence an extra forcing term arises in the equation. We introduce the notation

$$(3.2.2.12) \quad \delta_0(\tau) = \frac{1}{2}\int_{-\infty}^{\infty}\left(1 - \hat{f}_u^0(\xi)\right)e^{-|\xi - \tau|}T_0'(\xi)\,d\xi.$$

Note, that $\delta_0(\tau)$ vanishes exponentially for $\tau \to -\infty$ and $\tau\infty + \infty$. Now, we find I_{-1}^1, I_{-1}^2, $\bar{I}_{-1}^1$, $\bar{I}_{-1}^2$, B as the solution of the following problem

$$(3.2.2.13) \quad \mathscr{L}\left(I_{-1}^1, I_{-1}^2, \bar{I}_{-1}^1, \bar{I}_{-1}^2, B\right) = (0,0,0,0;0,0;0,0;\delta(\tau)).$$

Hence, for large $|\tau|$ this solution vanishes exponentially.

The r.h.s. of (3.2.2.11) is now orthogonal to T_0' and the solution Λ_{-1} is found as in (2.3.1.6), where the free constant can be taken to be 0. The construction of the other terms in (3.2.2.10) is again analogous to section (3.2.1). The result is that Ω_0, Ω_1 and N_0, $\bar{N}_0$ are given by the expressions in (3.2.1.10, 11) and (3.2.1.13), respectively, but with the obvious changes $c \to B$, $J_0^2 \to I_{-1}^2$. Further, mutatis mutandis (3.2.1.7) still holds and as an analogue of the overlap property in (3.2.1.12) we now have:

$$(3.2.2.14) \quad \left|\varepsilon^{-1}I_{-1}^2(x) - \left\{\varepsilon^{-1}I_{-1}^2(y) + \Omega_0 + \varepsilon\Omega_1\right\}\right|$$

$$\leq C\delta(\tau)\cdot\left\{\varepsilon^{-1}(x - y)^3 + 1 + \varepsilon^2|\xi|\right\}$$

in the region $y - 2\sigma \leq x \leq y - \sigma$.

Putting all things together we find that the contribution of I^c has the correct properties in view of (i)–(iv). Especially we mention that

$$(3.2.2.15) \quad \int_{y-2\bar{\sigma}}^{y+2\bar{\sigma}}\left|\text{``}L_\varepsilon\,\text{Gr}_{\cdot,1}\text{''}\right|dt \leq K\varepsilon\ln\left(\frac{1}{\varepsilon}\right).$$

Herewith the construction of the approximate Green matrix is complete.

3.3. *The approximate inverse operator and the derivation of the correctness result.* Using the properties of Gr given in (i)–(iv) it follows, that $(L_\varepsilon^{-1})'$ as defined in (3.1.18) is an operator with domain $\{C[-1,1]\}^2$ and range contained in the domain of L_ε. Moreover

$$(3.3.1) \quad L_\varepsilon\left(L_\varepsilon^{-1}\right)' = I + \varepsilon''K_\varepsilon$$

where actually K_ε is purely an integral operator with kernel "L_ε Gr", since we succeeded to do the construction with $\mathbf{A}$ in (3.2.3) $\equiv 0$. We take as the Banach space B in §3.1, $B = \{C[-1,1]\}^2$ provided with the maximum

norm $|\ \ |_0$ being the sum of the norms $|\ \ |_{\max}$ for each of the compo-
nents.

Of course (3.2.4) implies, that the operator $\varepsilon^\nu K_\varepsilon$ satisfies the estimate
in (3.1.8):

$$(3.3.2) \qquad\qquad |\varepsilon^\nu K_\varepsilon|_0 \le K\varepsilon^\nu.$$

Using explicitly the form of Gr given in §3.2 it is possible to estimate the
norm of $(L_\varepsilon^{-1})'$ as

$$(3.3.3) \qquad\qquad \left|(L_\varepsilon^{-1})'\right|_0 \le m\varepsilon^{-1}$$

With an ε-independent constant $m > 0$. The factor ε^{-1} in this norm is due
to a resonance effect in the internal layer, which is only present in the first
component. It is left to the reader to check, that this resonance effect is
precisely what one should expect on the basis of the construction of a
formal approximation for the linearised problem with smooth inhomoge-
neous terms.

In this case the non-linear operator N_ε is given by

$$(3.3.4) \quad N_\varepsilon\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \int_0^1 (1-z)\frac{d^2}{dz^2}\left\{\begin{pmatrix} f \\ g \end{pmatrix}(x, Z_1^N + zv_1, Z_2^N + zv_2)\right\} \cdot dz.$$

Using a method as in [7], [8] it is easy to show, that

$$(3.3.5) \qquad \left|N\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - N\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right|_0 \le l_0\rho\left|\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right|_0$$

for $0 \le \rho \le \rho_2$ and $0 < \varepsilon \le \varepsilon_2$ with ε-independent positive constants l_0
and ρ_2. In combination with (3.3.3) this leads to the following value for
$\nu(\varepsilon)$ in (3.1.12).

$$(3.3.6) \qquad\qquad \nu(\varepsilon) = \varepsilon(l_0 m)^{-1}.$$

Now an application of the results in (3.1.15–16) show that there is a
unique solution of (1.1–2) satisfying the estimate

$$(3.3.7) \qquad \left|\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} Z_1^N \\ Z_2^N \end{pmatrix}\right|_0 \le 4m\varepsilon^{-1} \cdot \tilde{R}_N\left[\varepsilon\ln\left(\frac{1}{\varepsilon}\right)\right]^{N+1}$$

for $N \ge 2$.

The main result given in (3.1) is then an easy consequence of (3.3.7)
and the triangle inequality

$$\left|\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} Z_1^N \\ Z_2^N \end{pmatrix}\right|_0 \le \left|\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} Z_1^{N+2} \\ Z_2^{N+2} \end{pmatrix}\right|_0 + \left|\begin{pmatrix} Z_1^{N+2} \\ Z_2^{N+2} \end{pmatrix} - \begin{pmatrix} Z_1^N \\ Z_2^N \end{pmatrix}\right|_0.$$

The latter term is $O(\varepsilon^{N+1})$ because of the structure of the formal ap-
proximation.

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