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SEPPO GRANLUND, PETER LINDQVIST AND OLLI MARTIO

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S. GRANLUND, P. LINDQVIST AND O. MARTIO

The Perron-Wiener-Brelot (PWB) method is applied to an important nonlinear situation. Unbounded subsolutions, their approximation and a counterpart of the harmonic measure are considered.

Introduction. The Perron-Wiener-Brelot (PWB-) method as introduced by O. Perron [P] and refined by several mathematicians is well-known in Potential Theory and it is mainly used in the theory of harmonic functions although it has a wider scope of applications [CC]. The PWB-method was generalized by E. Beckenbach and L. Jackson [BJ, J] to the non-linear situation. Their approach used the strong maximum principle for the difference of two solutions [BJ, Postulate 2]. The purpose of this note is to show that the PWB-method can be employed without this assumption in certain important non-linear cases. We are also able to deal with unbounded subsolutions.

We consider weak solutions, called F -extremals, of an Euler equation

$$(1.1) \quad \nabla \cdot \nabla_h F(x, \nabla u) = 0,$$

where the variational kernel $F: G \times R^n \rightarrow R$ satisfies the assumptions:

(a) For each $\varepsilon > 0$ there is a closed set C in the domain $G \subset R^n$ such that $m(G \setminus C) < \varepsilon$ and $F|_{C \times R^n}$ is continuous.

(b) For a.e. $x \in G$ the function $h \mapsto F(x, h)$ is strictly convex and differentiable in R^n .

(c) There are $0 < \alpha \leq \beta < \infty$ such that for a.e. $x \in G$

$$\alpha|h|^n \leq F(x, h) \leq \beta|h|^n,$$

$h \in R^n$.

(d) For a.e. $x \in G$

$$F(x, \lambda h) = |\lambda|^n F(x, h),$$

$\lambda \in R, h \in R^n$.

For a thorough analysis of the above assumptions see [GLM1]. Some of the assumptions are not necessary for the constructions. The exponent n in (c) is essential for applications in conformal geometry, cf. [GLM1-2].

A function $u \in C(G) \cap \text{loc } W_n^1(G)$, i.e. u is ACL^n , is called an F -extremal in G if for all domains $D \subset \subset G$

$$I_F(u, D) = \inf_{v \in \mathcal{F}_u} I_F(v, D),$$

where

$$I_F(v, D) = \int_D F(x, \nabla v(x)) \, dm(x)$$

is the variational integral with the kernel F and

$$\mathcal{F}_u = \{v \in C(\overline{D}) \cap W_n^1(D) \mid v = u \text{ in } \partial D\}.$$

A function u is an F -extremal if and only if $u \in C(G) \cap \text{loc } W_n^1(G)$ is a solution of (1.1) in the weak sense.

An upper semi-continuous function $u: G \rightarrow R \cup \{-\infty\}$ is called a sub- F -extremal if u satisfies the F -comparison principle in G , i.e. if $D \subset \subset G$ is a domain and $h \in C(\overline{D})$ is an F -extremal in D , then $h \geq u$ in ∂D implies $h \geq u$ in D . A function $u: G \rightarrow R \cup \{\infty\}$ is a super- F -extremal if $-u$ is a sub- F -extremal. For the basic properties of F -extremals and sub- F -extremals we refer to [GLM1].

Let $G \subset R^n$ be a domain and let $f: \partial G \rightarrow R \cup \{\pm \infty\}$ be any function. The fundamental concepts in the PWB-method are the upper and lower classes \mathcal{U}_f and \mathcal{L}_f determined by f . Our first theorem, Theorem 2.2, states in the complete analogy with the PWB-method that the function

$$\underline{H}_f(x) = \sup\{u(x) \mid u \in \mathcal{L}_f\}$$

is either $+\infty$, $-\infty$ or an F -extremal in G . The proof differs in several aspects from the classical proof, cf. e.g. [H]. First, a method like the Poisson modification of a sub- F -extremal is needed and since no Poisson formula is available in the non-linear case, our modification is based on approximation and on the solvability of the Dirichlet's problem in balls. The crucial step in the proof is to show that the function \underline{H}_f is continuous if it is not $+\infty$ or $-\infty$. The proof is based on a uniform Hölder-estimate, see [GLM1, Theorem 4.7], which is quite similar to Harnack's inequality. Moreover, the proof for Theorem 2.2 uses a uniform approximation argument, Lemma 2.14, for the function \underline{H}_f and Harnack's principle several times.

The rest of the paper is devoted to applications of the PWB-method and to byproducts of the method. In Chapter 3 we develop the barrier method for the non-linear case. Here the method works as in the linear case. Moreover, this method gives a necessary and sufficient condition for the solvability of the Dirichlet problem with continuous boundary values. In the non-linear case the best condition for solvability has been the

celebrated Wiener criterion, see [Maz], [GZ]. In the harmonic case these conditions are equivalent but this is not known in the non-linear situation.

Approximation of sub- F -extremals by means of regular sub- F -extremals is studied in Chapter 4. These results which are usually proved using a simple convolution argument, cf. [R], are more difficult to obtain in the non-linear case and we need a solution to an obstacle problem in the calculus of variations, cf. [GLM1, Theorem 5.15]. As a consequence we especially show that a bounded subharmonic function in a plane domain belongs to the Sobolev space $\text{loc } W_2^1$. These results are needed in the variational interpretation of subharmonicity and, more generally, sub- F -extremality.

Chapter 5 is devoted to the construction of the F -harmonic measure in general domains. This concept has turned out useful in studying the boundary behavior of F -extremals, see [GLM2], [GLM3].

Our notation is standard and generally as in [GLM1].

2. The Perron-Wiener-Brelot method. Suppose that $G \subset R^n$ is a domain and that $f: \partial G \rightarrow R \cup \{\pm \infty\}$ is a function.

2.1. DEFINITION. The lower class \mathcal{L}_f consists of the functions $u: G \rightarrow R \cup \{-\infty\}$ for which

- (a) u is a sub- F -extremal in G ,
- (b) u is bounded above,
- (c) $\lim_{x \rightarrow y} u(x) \leq f(y)$ for all $y \in \partial G$,
- (d) there is a compact set $K_u \subset R^n$ such that $u \leq 0$ in $G \setminus K_u$.

The upper class \mathcal{U}_f is defined analogously via super- F -extremals.

Let $\bar{H}_f = \inf\{u | u \in \mathcal{U}_f\}$ and $\underline{H}_f = \sup\{u | u \in \mathcal{L}_f\}$. The next theorem is fundamental for the PWB-method.

2.2. THEOREM. *The function \underline{H}_f satisfies one of the following conditions:*

- (i) \underline{H}_f is an F -extremal in G ,
- (ii) $\underline{H}_f = \infty$ in G ,
- (iii) $\underline{H}_f = -\infty$ in G .

The same is true for the function \bar{H}_f .

Some auxiliary results are needed in the proof of the above theorem. The so-called F -comparison principle is a basic tool.

2.3. LEMMA. *Let $G \subset R^n$ be a bounded open set, u a sub- F -extremal and v a super- F -extremal in G . Suppose*

$$(2.4) \quad \overline{\lim}_{x \rightarrow y} u(x) \leq \underline{\lim}_{x \rightarrow y} v(x)$$

for all $y \in \partial G$. If the left and right-hand sides are neither ∞ nor $-\infty$ at the same time, then $u \leq v$ in G .

Proof. Fix any $x \in G$. We will show that $u(x) \leq v(x)$. Let $\varepsilon > 0$ and consider the open set $H = \{y \in G \mid u(y) < v(y) + \varepsilon\}$. There exists a regular domain $D_\varepsilon, \bar{D}_\varepsilon \subset G$, such that $x \in D_\varepsilon$ and $\partial D_\varepsilon \subset H$. Choose a decreasing sequence $\varphi_i \in C^\infty(G)$ and an increasing sequence $\psi_i \in C^\infty(G)$ such that $\varphi_i \rightarrow u$ and $\psi_i \rightarrow v + \varepsilon$. Since ∂D_ε is compact we have $\varphi_i < \psi_i$ on ∂D_ε for some $i \in \mathbb{N}$. Let h_i^1 and h_i^2 be F -extremals such that $h_i^1|_{\partial D_\varepsilon} = \varphi_i|_{\partial D_\varepsilon}$, $h_i^2|_{\partial D_\varepsilon} = \psi_i|_{\partial D_\varepsilon}$. It follows from [GLM1, Definition 5.1] that

$$u \leq h_i^1 \leq h_i^2 \leq v + \varepsilon \quad \text{in } D_\varepsilon.$$

Since $x \in D_\varepsilon$ and $\varepsilon > 0$ was arbitrary, we obtain the desired inequality $u(x) \leq v(x)$.

The F -comparison principle yields:

2.5. LEMMA. $\underline{H}_f \leq \bar{H}_f$.

The Poisson modification of a subharmonic function so as to be harmonic over part of its domain is a basic operation in the classical potential theory. In the proof of Theorem 2.2 we employ a similar modification method for sub- F -extremals, cf. [R].

2.6. *Modification of sub- F -extremals.* Suppose that $G \subset \mathbb{R}^n$ is a domain and that $u: G \rightarrow \mathbb{R} \cup \{-\infty\}$ is a sub- F -extremal. Let $\bar{B} \subset G$ be a ball. We modify the sub- F -extremal by an approximation argument. Since u is upper semicontinuous in G , there exists a sequence $\varphi_i \in C^\infty(G)$ such that $\varphi_1 \geq \varphi_2 \geq \dots \geq u$ in \bar{B} and $\lim_{i \rightarrow \infty} \varphi_i = u$ in \bar{B} .

Choose F -extremals h_i in \bar{B} such that $h_i|_{\partial B} = \varphi_i|_{\partial B}$ and $h_i \in C(\bar{B}) \cap W_n^1(B)$. By Lemma 2.3, $h_1 \geq h_2 \geq \dots \geq u$ in \bar{B} . The function $h = \lim_{i \rightarrow \infty} h_i$ is an F -extremal or identically $-\infty$ in B , see [GLM1, Theorem 4.22]. For any $\zeta \in \partial B$

$$\overline{\lim}_{\substack{x \rightarrow \zeta \\ x \in B}} h(x) \leq \overline{\lim}_{\substack{x \rightarrow \zeta \\ x \in B}} h_i(x) = \varphi_i(\zeta), \quad i = 1, 2, 3, \dots,$$

and thus

$$(2.7) \quad \overline{\lim}_{\substack{x \rightarrow \zeta \\ x \in B}} h(x) \leq u(\zeta).$$

Write

$$P(u, B)(x) = \begin{cases} u(x), & x \in G \setminus B \\ h(x), & x \in B. \end{cases}$$

It is easy to see that $P(u, B)$ is independent of the sequence φ_i , although this fact is not needed in the sequel.

Now $P(u, B) \geq u$ in G and we shall prove that the function $P(u, B)$ is a sub- F -extremal. For that purpose an auxiliary result is needed.

2.8. LEMMA. *A sub- F -extremal u is identically $-\infty$ if and only if it is $-\infty$ in some nonempty open subset of G .*

PROOF. Write $H = \inf\{x \in G \mid u(x) = -\infty\}$ and suppose $H \neq G$. Let $x_0 \in \partial H \cap G$ and choose $\delta > 0$ such that $\bar{B}^n(x_0, \delta) \subset G$ and $S^{n-1}(x_0, \delta) \cap H \neq \emptyset$. Pick a closed cap $K \subset S^{n-1}(x_0, \delta) \cap H$ and denote the F -harmonic measure $\omega(K, B^n(x_0, \delta); F)$ by h . It follows from [GLM2, Theorem 4.10] that h is not identically zero. Let $\alpha > 0$ and consider the F -extremal $v = M - \alpha h$ where $M = \sup_{B^n(x_0, \delta)} u$. Now

$$\lim_{x \rightarrow y} v(x) \geq \overline{\lim}_{x \rightarrow y} u(x)$$

for all $y \in S^{n-1}(x_0, \delta)$ and the left-hand side is finite. Hence by Lemma 2.3, $v \geq u$ in $B^n(x_0, \delta)$. Letting $\alpha \rightarrow \infty$ we obtain $u(x) = -\infty$ for all $x \in B^n(x_0, \delta)$, a contradiction since $B^n(x_0, \delta)$ contains points not in H . The lemma follows.

We are ready to prove

2.9. LEMMA. *The function $\mathcal{U} = P(u, B)$ is a sub- F -extremal in G .*

Proof. If h is identically $-\infty$, so is u by Lemma 2.8 and there is nothing to prove. Otherwise, h is an F -extremal and we first show that \mathcal{U} is upper semicontinuous. We need only consider points $\zeta \in \partial B$. Now

$$\overline{\lim}_{\substack{x \rightarrow \zeta \\ x \in G \setminus B}} \mathcal{U}(x) = \overline{\lim}_{\substack{x \rightarrow \zeta \\ x \in G \setminus B}} u(x) \leq u(\zeta) = \mathcal{U}(\zeta).$$

By combining (2.7) and the inequality above it readily follows that \mathcal{U} is upper semicontinuous.

Next we prove that \mathcal{U} satisfies the F -comparison principle in G . Suppose that $D \subset \subset G$ is a domain and that $H \in C(\bar{D})$ is an F -extremal in D with $H|_{\partial D} \geq \mathcal{U}|_{\partial D}$. We will show that $H \geq \mathcal{U}$ in D . Now $H \geq u$ in D , since $\mathcal{U} \geq u$ in G and u is a sub- F -extremal. Let $\zeta \in \partial(D \cap B)$. Now

$$H(\zeta) \geq u(\zeta) \geq \overline{\lim}_{\substack{x \rightarrow \zeta \\ x \in D \cap B}} h(x),$$

and hence

$$\lim_{\substack{x \rightarrow \xi \\ x \in D \cap B}} H(x) \geq \overline{\lim}_{\substack{x \rightarrow \xi \\ x \in D \cap B}} h(x).$$

Lemma 2.3 implies $H \geq h = \mathcal{U}$ in $D \cap B$. Consequently $H \geq \mathcal{U}$ in the whole D as desired.

2.10 *Three lemmas for \underline{H}_f .* In what follows we shall only consider the function \underline{H}_f . The lemmas and proofs for \overline{H}_f are similar. First we prove that it is possible to replace \mathcal{L}_f by a subfamily, which is bounded from below in compact subsets of G . This new family gives the same \underline{H}_f .

2.11. LEMMA. *Suppose $K \subset G$ is compact and \underline{H}_f is not identically $-\infty$. Then there exists $\mathcal{L}_K \subset \mathcal{L}_f$ such that \mathcal{L}_K is bounded from below in K and $\underline{H}_f = \sup \mathcal{L}_K$.*

Proof. There exists $u_0 \in \mathcal{L}_f$ such that u_0 is not identically $-\infty$. Choose a finite cover $\{B^n(x_1, R_{x_1}), \dots, B^n(x_k, R_{x_k})\}$ of K such that $\overline{B}^n(x_i, 2R_{x_i}) \subset G$ for $i = 1, \dots, k$, and let $\mathcal{U}_{x_i} = P(u_0, B^n(x_i, 2R_{x_i}))$, $i = 1, \dots, k$. Lemma 2.9 yields $\mathcal{U}_{x_i} \in \mathcal{L}_f$ and Lemma 2.8 shows that $\mathcal{U}_{x_i} > -\infty$ in $B^n(x_i, 2R_{x_i})$, $i = 1, \dots, k$. Since \mathcal{U}_{x_i} is an F -extremal in $B^n(x_i, 2R_{x_i})$ the continuity of \mathcal{U}_{x_i} gives $M_{x_i} < \infty$ such that $\mathcal{U}_{x_i} > -M_{x_i}$ in $B^n(x_i, R_{x_i})$. Choose $\mathcal{L}_K = \{\max\{u, \mathcal{U}_{x_1}, \dots, \mathcal{U}_{x_k}\} \mid u \in \mathcal{L}_f\}$. Observe that for all $u \in \mathcal{L}_K$ we have $u \geq \min\{-M_{x_1}, \dots, -M_{x_k}\}$ in K . Let $u \in \mathcal{L}_f$. Then there exists $u^* \in \mathcal{L}_K$ such that $u^* \geq u$ in G . It follows that $\sup \mathcal{L}_K = \underline{H}_f$.

The next lemma is the basic step for the proof of Theorem 2.2.

2.12. LEMMA. *If \underline{H}_f is locally bounded above, then \underline{H}_f is continuous or identically $-\infty$.*

Proof. Let $\varepsilon > 0$ and assume that \underline{H}_f is not identically $-\infty$. We will show that there is $r > 0$ such that

$$(2.13) \quad |\underline{H}_f(x_1) - \underline{H}_f(x_2)| < \varepsilon \quad \text{for } x_1, x_2 \in B^n(x_0, r).$$

Fix a ball $\overline{B}^n(x_0, R) \subset G$ and let $K = \overline{B}^n(x_0, R)$. By Lemma 2.11 we can restrict to the class \mathcal{L}_K and there is a constant $0 < M_K < \infty$ such that $|u(x)| < M_K$, $x \in K$, for all $u \in \mathcal{L}_K$. Suppose $x_1, x_2 \in B^n(x_0, r) \subset B^n(x_0, R)$, where $r > 0$ will be fixed later. Assume, for example, that $\underline{H}_f(x_2) \geq \underline{H}_f(x_1)$. We can choose a sequence of functions $u_i \in \mathcal{L}_K$

such that $\lim_{i \rightarrow \infty} u_i(x_2) = \underline{H}_f(x_2)$. Consider the functions $\mathcal{U}_i = P(u_i, B^n(x_0, R))$. Again we have

$$\lim_{i \rightarrow \infty} \mathcal{U}_i(x_2) = \lim_{i \rightarrow \infty} P(u_i, B^n(x_0, R))(x_2).$$

Choose $i_0 \in N$ so large that $\underline{H}_f(x_2) - \mathcal{U}_i(x_2) \leq \varepsilon/2$ for $i > i_0$. Now

$$\begin{aligned} 0 &\leq \underline{H}_f(x_2) - \underline{H}_f(x_1) < \mathcal{U}_i(x_2) + \frac{\varepsilon}{2} - \mathcal{U}_i(x_1) \\ &\leq \text{osc}(\mathcal{U}_i, B^n(x_0, r)) + \frac{\varepsilon}{2}. \end{aligned}$$

It is possible to choose r independent of i such that $\text{osc}(\mathcal{U}_i, B^n(x_0, r)) < \varepsilon/2$. Since \mathcal{U}_i is an F -extremal in $B^n(x_0, R)$, this follows from the Hölder-estimate of F -extremals

$$\text{osc}(\mathcal{U}_i, B^n(x_0, r)) \leq c \left(\frac{r}{R} \right)^\kappa \text{osc}(\mathcal{U}_i, B^n(x_0, R)) \leq 2c \left(\frac{r}{R} \right)^\kappa M_K,$$

cf. [GLM1, Theorem 4.7]. In the same way (2.13) can be proved if $\underline{H}_f(x_2) \leq \underline{H}_f(x_1)$.

2.14. LEMMA. *Suppose that $C \subset G$ is compact and that \underline{H}_f is locally bounded from above in G . Then for arbitrary $\varepsilon > 0$ there exists $v_\varepsilon \in \mathcal{L}_f$ such that*

$$(2.15) \quad \underline{H}_f(x) \leq v_\varepsilon(x) + \varepsilon \quad \text{for } x \in C.$$

Proof. By Lemma 2.12 there are two possibilities: either \underline{H}_f is identically $-\infty$ or continuous. In the first case choose $v_\varepsilon \equiv -\infty$. In the second case for each $x \in C$ choose $\bar{B}^n(x, 2R_x) \subset G$. Let

$$K = \overline{\bigcup_{x \in C} B^n(x, R_x)},$$

and replace \mathcal{L}_f by the class \mathcal{L}_K . Now \mathcal{L}_K is uniformly bounded on the compact subset K of G .

For all $x \in C$ there exists $u_x \in \mathcal{L}_K$ such that $u_x(x) \geq \underline{H}_f(x) - \varepsilon/3$. Consider $\mathcal{U}_x = P(u_x, B^n(x, R_x))$. There exists $M_K > 0$ such that $\text{osc}(\mathcal{U}_x, B^n(x, R_x)) \leq M_K$. Let $x \in K$. Since \underline{H}_f is continuous and the Hölder-estimate [GLM1, Theorem 4.7] is valid for \mathcal{U}_x in $B^n(x, R_x)$ we can choose a ball $B^n(x, r_x)$, $0 < r_x < R_x$, such that

$$\text{osc}(\mathcal{U}_x, B^n(x, r_x)) < \frac{\varepsilon}{3}, \quad \text{osc}(\underline{H}_f, B^n(x, r_x)) < \frac{\varepsilon}{3}.$$

Now $\{B^n(x, r_x) | x \in C\}$ is an open cover for C and there is a finite subcover $\{B^n(x_1, r_{x_1}), \dots, B^n(x_k, r_{x_k})\}$ of C . For $y \in B^n(x_i, r_{x_i})$, $i \in$

$\{1, \dots, k\}$, we have

$$\begin{aligned} \underline{H}_f(y) - \mathcal{U}_{x_i}(y) &= (\underline{H}_f(y) - \underline{H}_f(x_i)) + (\underline{H}_f(x_i) - \mathcal{U}_{x_i}(x_i)) \\ &\quad + (\mathcal{U}_{x_i}(x_i) - \mathcal{U}_{x_i}(y)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

The function $v_\varepsilon = \max\{\mathcal{U}_{x_1}, \dots, \mathcal{U}_{x_k}\}$ is a sub- F -extremal [GLM1, Lemma 5.2] and has the desired property.

Proof of Theorem 2.2. Assume first that \underline{H}_f is not locally bounded from above. Then there are a sequence of functions $u_i \in \mathcal{L}_f$ and a sequence of points $x_i \in G$ such that $\lim_{i \rightarrow \infty} u_i(x_i) = \infty$, $\lim_{i \rightarrow \infty} x_i = x_0 \in G$.

Suppose that $y \in G$. We will prove that $\underline{H}_f(y) = \infty$. There is a domain D such that \bar{D} is compact in G , $y \in D$ and $x_i, x_0 \in D$. By Lemma 2.11 we can restrict to a subclass $\mathcal{L}_{\bar{D}} \subset \mathcal{L}_f$, which is uniformly bounded from below in \bar{D} . For that reason we may assume that the functions in $\mathcal{L}_{\bar{D}}$ are non-negative in D . Choose balls $\bar{B}^n(z_j, 2r_j) \subset D$, $j = 1, \dots, k$, with the following properties.

- (i) $x_i, x_0 \in B^n(z_1, r_1)$, for $i > i_0$,
- (ii) $y \in B^n(z_k, r_k)$,
- (iii) $B^n(z_j, r_j) \cap B^n(z_{j+1}, r_{j+1}) \neq \emptyset$, $j = 1, \dots, k-1$.

Let $i > i_0$ and define the functions \mathcal{U}_i^j as follows: $\mathcal{U}_i^1 = u_i$, $\mathcal{U}_i^{j+1} = P(\mathcal{U}_i^j, B^n(z_j, 2r_j))$, $j = 1, \dots, k-1$. By iterated use of Harnack's inequality for the functions \mathcal{U}_i^j , $j = 1, \dots, k$, it is easy to see that there is a constant $c > 0$ independent of i such that

$$u_i(x_i) \leq \frac{1}{c} \mathcal{U}_i^k(y) \leq \frac{1}{c} \underline{H}_f(y).$$

By letting $i \rightarrow \infty$ we obtain $\underline{H}_f(y) = \infty$.

Next assume that \underline{H}_f is locally bounded from above. According to Lemma 2.12 either \underline{H}_f is continuous or identically $-\infty$. In the latter case the proof is complete. Suppose that \underline{H}_f is continuous. Let $\bar{B}^n(x_0, r) \subset G$ and choose $C = \bar{B}^n(x_0, r)$ in Lemma 2.14. Lemma 2.14 shows that there is a sequence $v_i \in \mathcal{L}_{\bar{D}}$ such that $v_i > \underline{H}_f - 1/i$ in $\bar{B}^n(x_0, r)$. Consider the functions $V_i = P(v_i, B^n(x_0, r))$. Again we have $\lim_{i \rightarrow \infty} V_i = \underline{H}_f$ uniformly in $\bar{B}^n(x_0, r)$. It follows from Harnack's principle [GLM1, Theorem 4.21], that \underline{H}_f is an F -extremal in $B^n(x_0, r)$ and thus in G .

3. Regular boundary points. As in the classical harmonic case it is possible to define a barrier function for the boundary value problem of F -extremals. In this chapter we show that it gives a necessary and

sufficient condition for the regularity of boundary points. The proof for necessity differs considerably from the linear situation. Our variational principle also gives a new proof for Bouligand's theorem [H, p. 169].

Let $G \subset R^n$ be a bounded domain. A point $x_0 \in \partial G$ has an F -barrier if there exists a sub- F -extremal $w: G \rightarrow R$ such that

- (a) $\lim_{x \rightarrow y} w(x) < 0$ for all $y \in \partial G$, $y \neq x_0$,
- (b) $\lim_{x \rightarrow x_0} w(x) = 0$.

3.1. THEOREM. *Suppose that $f: \partial G \rightarrow R$ is bounded and continuous at $x_0 \in \partial G$. If x_0 has an F -barrier, then*

$$\lim_{x \rightarrow x_0} \underline{H}_f(x) = f(x_0).$$

Proof. The proof is completely analogous to the classical proof. Let $\varepsilon > 0$ and $M = \sup|f|$. By virtue of the assumptions there are constants $\delta > 0$ and $k < 0$ such that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$ and $kw(x) \geq 2M$ if $|x - x_0| \geq \delta$. Note that the functions $f(x_0) + \varepsilon + kw$ and $f(x_0) - \varepsilon - kw$ belong to the classes \mathcal{U}_f and \mathcal{L}_f respectively. Observe that kw is a super- F -extremal and $-kw$ is a sub- F -extremal. Then

$$f(x_0) - \varepsilon - kw(x) \leq \underline{H}_f(x) \leq \overline{H}_f(x) \leq f(x_0) + \varepsilon + kw(x)$$

or

$$|\underline{H}_f(x) - f(x_0)| \leq \varepsilon + kw(x).$$

Since $w(x) \rightarrow 0$ as $x \rightarrow x_0$ we obtain $\underline{H}_f(x) \rightarrow f(x_0)$ as $x \rightarrow x_0$.

3.2. DEFINITION. A bounded domain $G \subset R^n$ is called F -regular, if for all continuous $f: \partial G \rightarrow R$ there is an F -extremal $u \in C(\overline{G})$ with $u|_{\partial G} = f$.

3.3. LEMMA. *A domain $G \subset R^n$ is F -regular if and only if $\lim_{x \rightarrow y} \underline{H}_f(x) = f(y)$ for all $y \in \partial G$.*

Proof. Suppose that G is F -regular. Then for $f \in C(\partial G)$ there is u as in Definition 3.2. Since $u \in \mathcal{L}_f$ it follows that $\underline{H}_f \geq u$ in G . Let $v \in \mathcal{L}_f$ and $y \in \partial G$. Now

$$\overline{\lim}_{x \rightarrow y} v(x) \leq f(y) = \lim_{x \rightarrow y} u(x)$$

and the F -comparison principle implies $v \leq u$ in G . Then $\underline{H}_f = \sup\{v|v \in \mathcal{L}_f\} \leq u$ in G . We have proved that $\underline{H}_f = u$ and thus $\lim_{x \rightarrow y} \underline{H}_f(x) = \lim_{x \rightarrow y} u(x) = f(y)$ for all $y \in \partial G$. The converse is trivial.

3.4. DEFINITION. A boundary point x_0 of a bounded domain $G \subset \mathbb{R}^n$ is called *F-regular*, if for all continuous $f: \partial G \rightarrow \mathbb{R}$

$$\lim_{x \rightarrow x_0} \underline{H}_f(x) = f(x_0).$$

Lemma 3.3 implies

3.5. COROLLARY. *A bounded domain G is F-regular if and only if each boundary point of G is F-regular.*

Theorem 3.1 gives

3.6. COROLLARY. *A point $x_0 \in \partial G$ is F-regular if it has an F-barrier.*

The converse of Corollary 3.6 is also true.

3.7. THEOREM. *A point $x_0 \in \partial G$ is F-regular if and only if x_0 has an F-barrier.*

Proof. Suppose that $x_0 \in \partial G$ has an *F*-barrier. It follows from Corollary 3.6 that x_0 is *F*-regular. To show the converse assume that $x_0 \in \partial G$ is *F*-regular. Let $\bar{G} \subset B^n(x_0, R)$. We shall construct a barrier function w at x_0 . For this purpose we need a continuous sub-*F*-extremal u in $B^n(x_0, R)$ such that $u(x_0) = 0$ and $u(x) > 0$ for $x \in B^n(x_0, R)$. The function u is constructed as a solution of an obstacle problem.

We will use the function $\varphi = |x - x_0|$ as an obstacle. Let $B = B^n(x_0, R)$ and

$$\mathcal{F}(\varphi) = \{v \in C(\bar{B}) \cap W_n^1(B) \mid v \leq \varphi \text{ in } B, v = \varphi \text{ in } \partial B\}.$$

There exists $u \in \mathcal{F}(\varphi)$ such that $I_F(u, B) = \inf\{I_F(v, B) \mid v \in \mathcal{F}(\varphi)\}$, see [GLM1, Theorem 5.15]. Now [GLM1, Theorem 5.17(ii)] implies that the function u is a sub-*F*-extremal.

In what follows we will show that $u(x_0) = 0$ and $u(x) > 0$ for $x \in \bar{B}$, $x \neq x_0$. The function $h = \max\{u, 0\}$ belongs to the class $\mathcal{F}(\varphi)$, hence $u(x_0) = 0$ and $u \geq 0$ in $B^n(x_0, R)$. Suppose that there is $x_1 \in B^n(x_0, R)$ such that $u(x_1) = 0$ and $x_1 \neq x_0$. Now $x_1 \in \tilde{A}$, where \tilde{A} is a component of the open set $\{x \in B^n(x_0, R) \mid u(x) < \varphi(x)\}$. Observe that u is an *F*-extremal in the set \tilde{A} [GLM1, pp. 39–40]. Harnack's inequality implies that $u(x) = 0$ for $x \in \tilde{A}$. This is a contradiction. Hence $u(x) > 0$ for $x \in B^n(x_0, R) \setminus \{x_0\}$.

We are ready to construct the barrier at $x_0 \in \partial G$. Consider the function \underline{H}_u . Now $u \in \mathcal{L}_u$ and hence $\underline{H}_u \geq u$ in G . This yields

$$\lim_{x \rightarrow y \in \partial G} \underline{H}_u(x) \geq \lim_{x \rightarrow y \in \partial G} u(x) = u(y) > 0 \quad \text{for } y \neq x_0.$$

Since x_0 is F -regular it follows from Definition 3.4 that $\lim_{x \rightarrow x_0} \underline{H}_u(x) = u(x_0) = 0$.

For the barrier we choose the function $-\underline{H}_u$.

3.8. REMARK. The function \underline{H}_u is the barrier sought in Bouligand's theorem.

4. Approximation of sub- F -extremals. In the classical potential theory it is well-known that subharmonic functions can be approximated by regular subharmonic functions. The following theorem gives a corresponding approximation result for sub- F -extremals. In particular, it follows that a general sub- F -extremal which is locally bounded from below is in the Sobolev-space $\text{loc } W_n^1(G)$.

4.1. THEOREM. *Suppose $u: G \rightarrow R \cup \{-\infty\}$ is a sub- F -extremal and $D \subset\subset G$ a domain. Then there exists a decreasing sequence of sub- F -extremals $u_i \in C(\bar{D}) \cap W_n^1(D)$ such that $\lim_{i \rightarrow \infty} u_i = u$ in D . If u is locally bounded from below then u is in $\text{loc } W_n^1(G)$ and*

$$(4.2) \quad \int_{\text{spt } \eta} F(x, \nabla u) \, dm \leq \int_{\text{spt } \eta} F(x, \nabla (u - \eta)) \, dm,$$

for all non-negative $\eta \in C_0^\infty(G)$.

Proof. Since u is upper semicontinuous there exists a decreasing sequence $\varphi_i \in C^\infty(D) \cap C(\bar{D})$ such that $\lim_{i \rightarrow \infty} \varphi_i = u$ in \bar{D} . We may assume that the domain D is regular. We shall again employ the solutions of an obstacle problem. Choose functions u_i which minimize the integral

$$(4.3) \quad I_F(u, D) = \int_D F(x, \nabla u) \, dm$$

in the class $\mathcal{F}(\varphi_i) = \{u \in C(\bar{D}) \cap W_n^1(D) | u \leq \varphi_i \text{ in } D, u = \varphi_i \text{ in } \partial D\}$, see [GLM1, Theorem 5.15]. The functions u_i are sub- F -extremals.

Next we show that $u \leq u_i \leq \varphi_i$ in \bar{D} . Consider the set $A_i = \{x \in D | u_i(x) < \varphi_i(x)\}$. Let \tilde{A}_i be a component of A_i . Then u_i is an F -extremal in \tilde{A}_i , see [GLM1, the proof of Theorem 5.17], and $u_i|_{\partial \tilde{A}_i} = \varphi_i|_{\partial \tilde{A}_i} \geq u|_{\partial \tilde{A}_i}$. By the F -comparison principle $u_i \geq u$ in \tilde{A}_i and clearly in the whole \bar{D} . Thus $u = \lim_{i \rightarrow \infty} \varphi_i \geq \lim_{i \rightarrow \infty} u_i \geq u$ in \bar{D} .

Next we prove that the sequence u_i is decreasing. Assume the contrary. Then the open set $A = \{x \in D \mid u_{i+1}(x) > u_i(x)\}$ is non-empty for some i . The function $\min(u_i, u_{i+1})$ belongs to the class $\mathcal{F}(\varphi_{i+1})$. Now

$$\begin{aligned} I_F(u_{i+1}, D) &= I_F(u_{i+1}, A) + I_F(u_{i+1}, D \setminus A) \\ &\leq I_F(u_i, A) + I_F(u_{i+1}, D \setminus A) \end{aligned}$$

and hence $I_F(u_{i+1}, A) \leq I_F(u_i, A)$. In the same way we obtain

$$\begin{aligned} I_F(u_i, D) &= I_F(u_i, A) + I_F(u_i, D \setminus A) \\ &\leq I_F(\max(u_i, u_{i+1}), D) \\ &= I_F(u_{i+1}, A) + I_F(u_i, D \setminus A), \end{aligned}$$

i.e. $I_F(u_i, A) \leq I_F(u_{i+1}, A)$. Thus $I_F(u_i, A) = I_F(u_{i+1}, A)$ and it follows from the strict convexity of the kernel F that the set A is empty and $u_{i+1} \leq u_i$ in D .

Suppose u is locally bounded from below. We prove that u is in $\text{loc } W_n^1(D)$. Since $u_i \in C(\bar{D}) \cap W_n^1(D)$, [GLM1, Theorem 5.17] implies that

$$\int_{\text{spt } \eta} F(x, \nabla u_i) \, dm \leq \int_{\text{spt } \eta} F(x, \nabla(u_i - \eta)) \, dm$$

for all non-negative $\eta \in C_0^\infty(D)$. Since u is locally bounded from below we may assume that it is non-negative in D . Then also the functions u_i are non-negative. Let $\bar{B}^n(x_0, r) \subset D$ and consider the condenser $(D, \bar{B}^n(x_0, r))$. Analogously to the proof of [GLM1, Lemma 4.2] it can be shown that

$$\begin{aligned} (4.4) \quad \int_{B^n(x_0, r)} |\nabla u_i|^n \, dm &\leq c \, \text{osc}(u_i, D)^n \, \text{cap}_n(D, B^n(x_0, r)) \\ &\leq L \, \text{cap}_n(D, B^n(x_0, r)), \end{aligned}$$

where the constant L does not depend on i . This shows that the L^n -norms of ∇u_i are uniformly bounded. Hence there is a subsequence of ∇u_i converging weakly in $L^n(B^n(x_0, r))$ to the generalized gradient ∇u of u , which is in $L^n(B^n(x_0, r))$. Since the ball $B^n(x_0, r)$ was arbitrary, u belongs to $\text{loc } W_n^1(D)$.

In order to prove the inequality (4.2) we show that there is a subsequence of ∇u_i such that $\nabla u_i \rightarrow \nabla u$ a.e. in compact subsets of D . The expression

$$(4.5) \quad (\nabla_h F(x, h_1) - \nabla_h F(x, h_2)) \cdot (h_1 - h_2),$$

is strictly positive for a.e. $x \in G$, and all $h_1, h_2 \in R^n$, $h_1 \neq h_2$. Since the functions u_i are sub- F -extremals in D and belong to $C(\bar{D}) \cap W_n^1(D)$, they satisfy the inequality

$$(4.6) \quad \int_{\text{spt } \eta} \nabla_h F(x, \nabla u_i) \cdot \nabla \eta \, dm \leq 0,$$

for all non-negative $\eta \in C_0^\infty(D)$.

Let $\bar{B} = \bar{B}^n(x_0, r) \subset D$, $0 < r' < r$, $\zeta \in C_0^\infty(B)$, $0 \leq \zeta \leq 1$, and $\zeta(x) = 1$ for $x \in B^n(x_0, r')$. Put $\eta = \zeta(u_i - u)$ and use (4.6) to obtain

$$\begin{aligned} & \int_{\text{spt } \zeta} (\nabla_h F(x, \nabla u_i) - \nabla_h F(x, \nabla u)) \cdot \nabla (\zeta(u_i - u)) \, dm \\ &= \int_{\text{spt } \zeta} \zeta (\nabla_h F(x, \nabla u_i) - \nabla_h F(x, \nabla u)) \cdot (\nabla u_i - \nabla u) \, dm \\ & \quad + \int_{\text{spt } \zeta} (u_i - u) (\nabla_h F(x, \nabla u_i) - \nabla_h F(x, \nabla u)) \cdot \nabla \zeta \, dm \\ &= I_i^1 + I_i^2 \leq - \int_{\text{spt } \zeta} \nabla_h F(x, \nabla u) \cdot \nabla (\zeta(u_i - u)) \, dm \\ &= - \int_{\text{spt } \zeta} (u_i - u) \nabla_h F(x, \nabla u) \cdot \nabla \zeta \, dm \\ & \quad - \int_{\text{spt } \zeta} \zeta \nabla_h F(x, \nabla u) \cdot (\nabla u_i - \nabla u) \, dm. \end{aligned}$$

Because of the inequality (4.4) we can choose a subsequence of u_i such that $u_i \rightarrow u$ in $L^n(B^n(x_0, r))$ and $\nabla u_i \rightarrow \nabla u$ weakly in $L^n(B^n(x_0, r))$, see [M, p. 75, Theorem 3.4.4]. Then the last two integrals and the integral I_i^2 tend to zero for $i \rightarrow \infty$. Now (4.5) yields $I_i^1 \geq 0$ and hence $\lim_{i \rightarrow \infty} I_i^1 = 0$. Then we employ the condition (4.5) to show that there is a subsequence of ∇u_i such that $\nabla u_i \rightarrow \nabla u$ for a.e. $x \in B^n(x_0, r')$. Write

$$g_i(x) = (\nabla_h F(x, \nabla u_i(x)) - \nabla_h F(x, \nabla u(x))) \cdot (\nabla u_i(x) - \nabla u(x)).$$

Then $g_i \rightarrow 0$ in $L^1(B^n(x_0, r'))$ and hence there is a subsequence such that $g_i(x) \rightarrow 0$ for a.e. $x \in B^n(x_0, r')$. It follows from (4.5) that $\nabla u_i(x) \rightarrow \nabla u(x)$ for a.e. $x \in B^n(x_0, r')$.

Finally choose a non-negative $\eta \in C_0^\infty(B^n(x_0, r'))$ in (4.6). Since the integrals

$$\int_{B^n(x_0, r')} |\nabla_h F(x, \nabla u_i)|^{n/(n-1)} \, dm$$

are uniformly bounded and $\nabla_h F(x, \nabla u_i(x)) \rightarrow \nabla_h F(x, \nabla u(x))$ a.e. in $B^n(x_0, r')$, the inequality (4.6) yields via weak convergence

$$\int_{B^n(x_0, r')} \nabla_h F(x, \nabla u) \cdot \nabla \eta \, dm \leq 0.$$

Thus the above inequality holds in D and (4.2) follows from [GLM1, Theorem 5.17].

5. F -harmonic measure. The PWB-method can be used in the definition of the F -harmonic measure. In [GLM2] the F -harmonic measure was constructed via generating sequences. This method can only be used in regular domains.

Suppose $G \subset R^n$ is a bounded open set. Let $C \subset \partial G$ be a closed set and let $f: \partial G \rightarrow R$ be the characteristic function of C . The function \bar{H}_f , which is an F -extremal, is the F -harmonic measure of C with respect to G . The next theorem shows that in regular domains this concept gives the same F -harmonic measure.

5.1. THEOREM. *Suppose that $G \subset R^n$ is a regular domain, and that $C \subset \partial G$ is a closed set. If f is the characteristic function of C , then $\bar{H}_f = \omega(C, G; F)$, where $\omega(C, G; F)$ is the F -harmonic measure as in [GLM2, Definition 2.16].*

Proof. Let φ_i be a (C, G) -boundary sequence, see [GLM2, pp. 235–236]. Consider the F -extremals $u_i \in C(\bar{G}) \cap W_n^1(G)$ with $u_i|_{\partial G} = \varphi_i|_{\partial G}$. It was shown in [GLM2, pp. 3–4] that $\lim_{i \rightarrow \infty} u_i = \omega(C, G; F)$ locally uniformly in G . Now $u_i \in \mathcal{U}_f$ and hence $u_i \geq \bar{H}_f$. Thus $\omega(C, G; F) = \lim_{i \rightarrow \infty} u_i \geq \bar{H}_f$. On the other hand, for $u \in \mathcal{U}_f$, $\lim_{x \rightarrow y \in \partial G} u(x) \geq f(y) \geq \lim_{x \rightarrow y \in \partial G} \omega(C, G; F)$, see [GLM2, Remark 2.20]. Lemma 2.3 implies that $u \geq \omega(C, G; F)$, in G and thus $\bar{H}_f \geq \omega(C, G; F)$, which together with the previous inequality completes the proof.

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HELSINKI UNIVERSITY OF TECHNOLOGY
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