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## **ON EXISTENCE CRITERIA FOR CAPILLARY FREE SURFACES WITHOUT GRAVITY**

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Consider a cylinder of homogenous material closed at one end by a base of general cross section  $\Omega$  and partly filled with liquid. We want to find conditions under which in the absence of gravity the liquid can cover  $\Omega$  and is in mechanical equilibrium.

If the liquid can cover  $\Omega$ , then the liquid surface is a graph over the base. In general, the surface has constant mean curvature and makes constant angle with the bounding wall. Even if  $\Omega$  is convex analytic, such a surface may not exist. However, it is the case when  $\Omega$  is piecewise smooth that interests us. In this case, the interior angles at the corners play an important role. It turns out that the existence of the liquid surface as a graph over the base can be characterized by the nonexistence of a certain subsidiary variational problem.

**1. Introduction and preliminary results.** Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$ ,  $n \geq 2$ . The problem of minimizing the following functional has received much attention:

$$(1.1) \quad \Phi(u; \gamma) \equiv \int_{\Omega} \sqrt{1 + |Du|^2} + \int_{\Omega} H_{\gamma} u \, dx - \cos \gamma \int_{\partial\Omega} u \, dH_{n-1}$$

for  $u \in BV(\Omega)$ , the space of functions of bounded variation in  $\Omega$ , where  $\pi/2 > \gamma \geq 0$  is a constant and  $H_{\gamma} = (|\partial\Omega|/|\Omega|) \cos \gamma$ .

In order to study the problem, we also consider the subsidiary functional:

$$(1.2) \quad G(A; \gamma) \equiv \int_{\Omega} |D\phi_A| + \int_{\Omega} H_{\gamma} \phi_A \, dH_n - \cos \gamma \int_{\partial\Omega} \phi_A \, dH_{n-1}$$

where  $A \subset \Omega$  is a Caccioppoli set (or set of locally finite perimeter), and  $\phi_A$  is the characteristic function of  $A$ . The following necessary condition for the existence of (variational) solution of  $\Phi(u; \gamma)$  was obtained by Concus-Finn [2]:

$$(1.3) \quad G(A; \gamma) > 0 \quad \text{for all } A \subset \Omega, A \neq \emptyset \text{ or } \Omega.$$

As for sufficiency, Giusti [9] gave the following general criterion: if there exists  $\varepsilon > 0$ , such that

$$(1.3)' \quad (1 - \varepsilon) \int_{\Omega} |D\phi_A| + \int_{\Omega} H_{\gamma} \phi_A \, dH_n - \cos \gamma \int_{\partial\Omega} \phi_A \, dH_{n-1} > 0$$

for all  $A \subset \Omega$ ,  $A \neq \emptyset$  or  $\Omega$ , then  $\Phi(u; \gamma)$  has a solution. Finn [5] proved that in some cases we need only the weaker condition (1.3). To be more precise, Finn proved that if  $n = 2$ ,  $\Omega$  is a piecewise smooth domain,  $2\alpha =$  smallest interior angle and  $\pi/2 > \alpha > \pi/2 - \gamma$  at each corner, then (1.3) is also sufficient for the existence of solution of  $\Phi(u; \gamma)$ . In fact, he proved that in this case (1.3) implies (1.3)'.

The case  $\alpha + \gamma = \pi/2$  is more delicate, because in that case there exists no  $\varepsilon > 0$  so that (1.3)' is true. In this case, Finn [6] introduced some additional conditions on  $\partial\Omega$ .

Specifically, if  $\Omega$  satisfies the following hypothesis, then (1.3) still guarantees the existence of a solution:

*Hypothesis  $\alpha(\gamma)$ .* At each vertex  $P$  with interior angle  $2\alpha$ , it is possible to place a lower hemisphere  $v(x; \gamma)$  of radius  $R_\gamma = 2H_\gamma^{-1}$ , with equatorial circle  $Q$  passing through  $P$  in such a way that at each point of  $\partial\Omega$  interior to  $Q$  and to some neighborhood  $N_P$  of  $P$  there holds  $Tv \cdot \nu \geq \cos \gamma$ , where  $\nu$  is the outward normal of  $\partial\Omega$ .

The results of Finn are restricted to  $n = 2$ . The methods of proof do not extend readily to higher dimensions. In this paper, it is our aim to generalize those results to higher dimensions by using another method, and at the same time give a new proof of Finn's result. Moreover we shall also show that even if  $\Omega$  does not satisfy hypothesis  $\alpha(\gamma)$  (but satisfies (1.3)), we still can find a solution of

$$(1.4) \quad \begin{cases} \operatorname{div} Tw = H_\gamma & \text{in } \Omega \\ Tw = \frac{Du}{\sqrt{1 + |Dw|^2}} \\ Tw \cdot \nu = \cos \gamma & \text{weakly on } \partial\Omega. \end{cases}$$

We shall use the idea of generalized solution introduced by M. Miranda [16]. This idea has been employed to deal with different problems, for example see [10], [11], [16] and [17].

The functional  $\Phi(u; \gamma)$  is related to the following functional:

$$(1.5) \quad F(U; \gamma) \equiv \int_{\Omega \times \mathbf{R}} |D\phi_U| + \int_{\Omega \times \mathbf{R}} H_\gamma \phi_U dx dt - \cos \gamma \int_{\partial\Omega \times \mathbf{R}} \phi_U dH_n$$

where  $U \subset \Omega \times \mathbf{R}$  is a Caccioppoli set.

**DEFINITION 1.1.** A Caccioppoli set  $U \subset \Omega \times \mathbf{R}$  is said to be a solution of  $\Phi(u; \gamma)$  if for any  $T > 0$ , and for any Caccioppoli set  $V \subset \Omega \times \mathbf{R}$  with

support of  $\phi_U - \phi_V$  contained in  $\Omega \times [-T, T]$  we have

$$(1.6) \quad F_T(U; \gamma) \leq F_T(V; \gamma)$$

where

$$\begin{aligned} F_T(U; \gamma) \equiv & \int_{\Omega \times [-T, T]} |D\phi_U| \\ & + \int_{\Omega \times [-T, T]} H_\gamma \phi_U dx dt - \cos \gamma \int_{\partial\Omega \times [-T, T]} \phi_U dH_n. \end{aligned}$$

**DEFINITION 1.2.** A function  $u: \Omega \rightarrow [-\infty, \infty]$  is said to be a generalized solution of  $\Phi(u; \gamma)$  if its subgraph  $U = \{(x, t) \in \Omega \times \mathbf{R} \mid t < u(x)\}$  is a solution of  $F(U; \gamma)$ .

**DEFINITION 1.3.**  $A \subset \Omega$  is a solution of  $G(A; \gamma)$  if  $G(A; \gamma) \leq G(E; \gamma)$  for any Caccioppoli set  $E \subset \Omega$ .

The relation between  $\Phi(u; \gamma)$  and  $F(U; \gamma)$  is the following theorem by M. Miranda [16], see also [11] and [17].

**THEOREM 1.1.** *Let  $\Omega$  be a Lipschitz domain and  $u \in BV(\Omega)$ , then  $u$  is a solution of  $\Phi(u; \gamma)$  if and only if  $u$  is a generalized solution of  $\phi(u; \gamma)$ .*

**REMARK 1.1.** In the definitions of  $\Phi(u; \gamma)$  and  $F(u; \gamma)$ , Definition 1.1–1.3 and Theorem 1.1 there is no need to restrict  $\gamma$  so that  $0 \leq \gamma < \pi/2$ .

By Theorem 1.1, in order to find a solution of (1.2) we may first find a generalized solution and then prove that it actually belongs to  $BV(\Omega)$ .

We need several lemmas. First of all, we say that an open portion  $\Gamma_1$  of  $\partial\Omega$  has Lipschitz constant  $L$  if for any point  $x \in \Gamma_1$ , there exists a ball  $B_\rho(x)$  such that  $B_\rho(x) \cap \partial\Omega = \text{graph } g$  for some Lipschitz function  $g: A \subset \mathbf{R}^{n-1} \rightarrow \mathbf{R}$  with  $A$  open and the Lipschitz constant of  $g$  is  $L$ .

**LEMMA 1.1.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbf{R}^n$ ,  $\Gamma_1$  is an open portion of  $\partial\Omega$  with Lipschitz constant  $L$ . Suppose  $\Gamma$  is a closed subset of  $\Gamma_1$ , then there is a constant  $\delta_0 > 0$  such that for all  $0 < \delta < \delta_0$  we can find a constant  $C_1$  depending only on  $\delta$ ,  $\Omega$ ,  $\Gamma_1$ , and  $\Gamma$  such that for any  $T > 0$*

$$\begin{aligned} (1.7) \quad & \int_{\Gamma \times (-T, T)} |f| dH_n \leq \sqrt{1 + L^2} \int_{\Omega_\delta \times (-T, T)} |Df| \\ & + C_1 \int_{\Omega \times (-T, T)} |f| dx dt \end{aligned}$$

for all  $f \in BV(\Omega \times (-T, T))$ , where  $\Omega_\delta = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq \delta\}$ . Note that  $\delta_0$  and  $C_1$  do not depend on  $T$ .

The proof of the lemma is the same as the proof of Lemma 1.1 in [3], so we omit it.

**LEMMA 1.2.** *With the same assumptions as in Lemma 1.1 and let  $\{f_j\}$  be a sequence of functions in  $BV(\Omega \times (-T, T))$ ,  $T > 0$ , such that the functions  $f_j$  converge in  $L^1(\Omega \times (-T, T))$  to some function  $f \in BV(\Omega \times (-T, T))$ . If  $\cos \gamma \sqrt{1 + L^2} \leq 1$  then*

$$(1.8) \quad \int_{\Omega \times (-T, T)} |Df| - \cos \gamma \int_{\Gamma \times (-T, T)} f dH_n \\ \leq \liminf_{j \rightarrow \infty} \left( \int_{\Omega \times (-T, T)} |Df_j| - \cos \gamma \int_{\Gamma \times (-T, T)} f_j dH_n \right).$$

*Proof.* (See [11].) Define

$$J(u) = \int_{\Omega \times (-T, T)} |Du| - \cos \gamma \int_{\Gamma \times (-T, T)} u dH_{n-1}.$$

By Lemma 1.1, there are constants  $\delta_0$  and  $C_1$  not depending on  $j$ , such that for any  $0 < \delta < \delta_0$  if  $\Sigma_\delta = \Omega - \Omega_\delta$ ,

$$\begin{aligned} J(f) - J(f_j) &\leq \int_{\Omega \times (-T, T)} |Df| - \int_{\Omega \times (-T, T)} |Df_j| \\ &\quad + \cos \gamma \int_{\Gamma \times (-T, T)} |f - f_j| dH_n \\ &\leq \int_{\Omega \times (-T, T)} |Df| - \int_{\Omega \times (-T, T)} |Df_j| \\ &\quad + \cos \gamma \sqrt{1 + L^2} \int_{\Omega_\delta \times (-T, T)} |Df - Df_j| \\ &\quad + C_1 \cos \gamma \int_{\Omega \times (-T, T)} |f - f_j| dx \\ &\leq \int_{\Sigma_\delta \times (-T, T)} |Df| - \int_{\Sigma_\delta \times (-T, T)} |Df_j| \\ &\quad + 2 \int_{\Omega_\delta \times (-T, T)} |Df| + C_1 \int_{\Omega \times (-T, T)} |f - f_j| dx. \end{aligned}$$

where we have used the fact  $\cos \gamma \sqrt{1 + L^2} \leq 1$ . Since  $f_j$  converge to  $f$  in  $L^1(\Omega \times (-T, T))$ , by semicontinuity Theorem 1.9 of [12] and let  $j \rightarrow \infty$  we get

$$J(f) - \liminf_{j \rightarrow \infty} J(f_j) \leq 2 \int_{\Omega_\delta \times (-T, T)} |Df|.$$

Since  $f \in BV(\Omega \times (-T, T))$ , let  $\delta \rightarrow 0$  the lemma follows.  $\square$

In application, we need a refined version of Lemma 1.2. Let us assume  $\partial\Omega$  satisfies

$$(1.9) \quad \begin{aligned} &\text{there is an } \varepsilon_0 > 0 \text{ such that for all } 0 < \varepsilon < \varepsilon_0 \text{ we can} \\ &\text{find } L_\varepsilon \text{ such that } \partial\Omega \text{ has Lipschitz constant } L_\varepsilon \text{ and} \\ &\sqrt{1 + L_\varepsilon^2}(\cos \gamma - \varepsilon) \leq 1. \end{aligned}$$

**REMARK 1.2.** If  $\partial\Omega$  has Lipschitz constant  $L$  with  $\cos \gamma \sqrt{1 + L^2} \leq 1$ , then  $\partial\Omega$  obviously satisfies (1.9). If  $\partial\Omega$  is smooth and  $\gamma = 0$  then there will be no  $L$  such that  $\partial\Omega$  has Lipschitz constant  $L$  and  $\cos \gamma \sqrt{1 + L^2} \leq 1$ . However  $\partial\Omega$  satisfies (1.9). Also if  $n = 2$ ,  $\partial\Omega$  is piecewise smooth and the smallest interior angle  $2\alpha$  satisfies  $\alpha + \gamma = \pi/2$  (we assume every interior angle is less than  $\pi$ ) then  $\partial\Omega$  again satisfies condition (1.9).

**LEMMA 1.3.** *Same assumptions as in Lemma 1.2 except that now we only assume  $\Gamma_1 = \partial\Omega$  satisfying (1.9). If  $\int_{\Gamma \times (-T, T)} f_j dH_n$  are uniformly bounded, then the conclusion of Lemma 1.2 is still true.*

*Proof.* By assumption, there is a constant  $M > 0$  such that

$$(1.10) \quad \left| \int_{\Gamma \times (-T, T)} f_j dH_n \right| + \left| \int_{\Gamma \times (-T, T)} f dH_n \right| \leq M \quad \text{for all } j.$$

Let  $J(u)$  be the functional defined in Lemma 1.2. For any  $0 < \varepsilon < \varepsilon_0$ ,

$$\begin{aligned} J(f) - J(f_j) &\leq \int_{\Omega \times (-T, T)} |Df| - (\cos \gamma - \varepsilon) \int_{\Gamma \times (-T, T)} f dH_n \\ &\quad - \int_{\Omega \times (-T, T)} |Df_j| + (\cos \gamma - \varepsilon) \int_{\Gamma \times (-T, T)} f_j dH_n \\ &\quad + \varepsilon \left( \left| \int_{\Gamma \times (-T, T)} f dH_n \right| + \left| \int_{\Gamma \times (-T, T)} f_j dH_n \right| \right). \end{aligned}$$

By Lemma 1.2, (1.9) and (1.10), let  $j \rightarrow \infty$

$$J(f) - \liminf_{j \rightarrow \infty} J(f_j) \leq \varepsilon M.$$

Since  $\varepsilon$  can be arbitrary small, the lemma follows.

## 2. Existence of generalized solution.

LEMMA 2.1. Suppose  $\Omega$  is a bounded Lipschitz domain in  $\mathbf{R}^n$  satisfying (1.3), then  $\Phi(u; \beta)$  has a bounded solution  $u_\beta$  for all  $\pi/2 > \beta > \gamma$ . Furthermore  $u_\beta$  is analytic and satisfies  $\operatorname{div} Tu_\beta = H_\beta$  in  $\Omega$ .

*Proof.* Since  $G(A; \gamma) > 0$  for all  $A \neq \emptyset$  or  $\Omega$ , for such an  $A$

$$\int_{\Omega} |D\phi_A| + \int_{\Omega} H_\beta \cdot \frac{\cos \gamma}{\cos \beta} \cdot \phi_A dx - \cos \beta \int_{\partial\Omega} \frac{\cos \gamma}{\cos \beta} \cdot \phi_A dH_{n-1} > 0.$$

Hence,

$$(1 - \varepsilon) \int_{\Omega} |D\phi_A| + \int_{\Omega} H_\beta \phi_A dx - \cos \beta \int_{\partial\Omega} \phi_A dH_{n-1} > 0$$

$$\text{for } \varepsilon = 1 - \frac{\cos \beta}{\cos \gamma} > 0.$$

From [9], we conclude that  $\Phi(u; \gamma)$  has a bounded solution  $u_\beta$  which is in  $C^2(\Omega)$  and satisfies  $\operatorname{div} Tu_\beta = H_\beta$  in  $\Omega$ . Since  $H_\beta$  is a constant, therefore  $u_\beta$  is analytic.  $\square$

THEOREM 2.1. Let  $\Omega$  be a bounded Lipschitz domain satisfying (1.3) and (1.9). For any sequence  $\pi/2 > \gamma_j > \gamma$  and  $\gamma_j \searrow \gamma$ , if  $u_j$  is a solution of  $\Phi(u; \gamma_j)$  for each  $j$ , then we can find a subsequence of  $u_j$  which converges pointwise almost everywhere to a generalized solution of  $\Phi(u; \gamma)$ .

*Proof.* (See [11].) Since  $\Omega$  satisfies (1.3), the solutions  $u_j$  of  $\Phi(u; \gamma_j)$  exist by Lemma 2.1. Let  $U_j$  be the subgraph of  $u_j$ ,  $U_j$  is a solution of  $F(U; \gamma_j)$  by Theorem 1.1. For any  $T > 0$ , compare  $U_j$  with  $U_j - \Omega \times (-T, T)$ :

$$\int_{\Omega \times [-T, T]} |D\phi_{U_j}| + \int_{\Omega \times [-T, T]} H_{\gamma_j} \phi_{U_j} dx dt$$

$$- \cos \gamma_j \int_{\partial\Omega \times [-T, T]} \phi_{U_j} dH_n \leq 2|\Omega|.$$

$$\int_{\Omega \times [-T, T]} |D\phi_{U_j}| \leq 2|\Omega| + \cos \gamma_j \int_{\partial\Omega \times [-T, T]} \phi_{U_j} dH_n \leq 2|\Omega| + 2T|\partial\Omega|.$$

Also  $\int_{\Omega \times [-T, T]} \phi_{U_j} dx dt \leq 2T|\Omega|$ . Hence

$$(2.1) \quad \int_{\Omega \times [-T, T]} |D\phi_{U_j}| + \int_{\Omega \times [-T, T]} \phi_{U_j} dx dt$$

$$\leq (2 + 2T)|\Omega| + 2T|\partial\Omega| \quad \text{for all } j.$$

By compactness Theorem 1.19 of [12], we can select a subsequence of  $\phi_{U_j}$  which converges almost everywhere in  $\Omega \times (-T, T)$  to the characteristic function of some set. Take a sequence  $T_k \rightarrow \infty$ , and use diagonal process, we can find a subsequence of  $\phi_{U_j}$ , which we also call it  $\phi_{U_j}$ , converging almost everywhere to the characteristic function of some set  $w$ , which may take the values  $-\infty$  or  $+\infty$ , and  $\lim_{j \rightarrow \infty} u_j = w$  almost everywhere in  $\Omega$ . By (2.1) and semicontinuity, we know that  $W$  is a Caccioppoli set.

It remains to prove that  $w$  is a generalized solution of  $\Phi(u; \gamma)$ . Note that for almost all  $T > 0$ ,

(i) the traces of  $\phi_{U_j}$  and  $\phi_W$  on  $\Omega \times \{-T, T\}$  are  $\phi_{U_j}$  and  $\phi_W$   $H_n$ -almost everywhere;

(ii)  $\int_{\Omega \times \{-T, T\}} |D\phi_{U_j}| = \int_{\Omega \times \{-T, T\}} |D\phi_W| = 0$ ; and

(iii)  $\lim_{j \rightarrow \infty} \int_{\Omega \times \{-T, T\}} |\phi_{U_j} - \phi_W| dH_n = 0$ .

(i) and (ii) follow from the fact that  $U_j$  and  $W$  are Caccioppoli sets and (iii) follows from the fact that  $\lim_{j \rightarrow \infty} \int_{\Omega \times [-T, T]} |\phi_{U_j} - \phi_W| dH_n = 0$  for all  $T > 0$ .

For any  $T > 0$  satisfying (i), (ii) and (iii), and for any Caccioppoli set  $V \subset \Omega \times \mathbf{R}$ ,  $V = W$  outside  $\Omega \times [-T, T]$ , define

$$V_j = \begin{cases} U_j & \text{outside } \Omega \times [-T, T], \\ V & \text{in } \Omega \times [-T, T] \end{cases}.$$

Then

$$\begin{aligned} & \int_{\Omega \times [-T, T]} |D\phi_{U_j}| + \int_{\Omega \times [-T, T]} H_{\gamma_j} \phi_{U_j} dx dt - \cos \gamma_j \int_{\partial\Omega \times [-T, T]} \phi_{U_j} dH_n \\ & \leq \int_{\Omega \times [-T, T]} |D\phi_{V_j}| + \int_{\Omega \times [-T, T]} H_{\gamma_j} \phi_{V_j} dx dt \\ & \quad - \cos \gamma_j \int_{\partial\Omega \times [-T, T]} \phi_{V_j} dH_n. \end{aligned}$$

Therefore

$$\begin{aligned} (2.2) \quad & \int_{\Omega \times [-T, T]} |D\phi_{U_j}| + \int_{\Omega \times [-T, T]} H_{\gamma_j} \phi_{U_j} dx dt \\ & - \cos \gamma \int_{\partial\Omega \times [-T, T]} \phi_{U_j} dH_n + (\cos \gamma - \cos \gamma_j) \int_{\partial\Omega \times [-T, T]} \phi_{U_j} dH_n \\ & \leq \int_{\Omega \times [-T, T]} |D\phi_V| + \int_{\Omega \times [-T, T]} H_{\gamma_j} \phi_V dx dt \\ & - \cos \gamma_j \int_{\partial\Omega \times [-T, T]} \phi_V dH_n + \int_{\Omega \times \{-T, T\}} |\phi_W - \phi_{U_j}| dH_n. \end{aligned}$$



Since  $T$  satisfies (ii) and (iii), by Lemma 1.3 with  $\Gamma = \Gamma_1 = \partial\Omega$  and the fact that  $\lim_{j \rightarrow \infty} \gamma_j = \gamma$ , let  $j \rightarrow \infty$  in (2.2):

$$\begin{aligned}
 (2.3) \quad & \int_{\Omega \times [-T, T]} |D\phi_w| + \int_{\Omega \times [-T, T]} H_\gamma \phi_w dx dt \\
 & - \cos \gamma \int_{\partial\Omega \times [-T, T]} \phi_w dH_n \\
 & \leq \int_{\Omega \times [-T, T]} |D\phi_v| + \int_{\Omega \times [-T, T]} H_\gamma \phi_v dx dt \\
 & - \cos \gamma \int_{\partial\Omega \times [-T, T]} \phi_v dH_n.
 \end{aligned}$$

(2.3) is true for almost all  $T > 0$ , so it is easy to see that (2.3) holds for all  $T > 0$ . Therefore  $w$  is a generalized solution of  $\Phi(u; \gamma)$ .  $\square$

Take any sequence  $\gamma_j \searrow \gamma$ , since the solution  $u_j$  of  $\Phi(u; \gamma_j)$  is unique up to an additive constant, we normalize  $u_j$  in Theorem 2.1 by

$$\begin{aligned}
 (2.4) \quad & \left| \{x \in \Omega \mid u_j(x) \geq 0\} \right| \geq |\Omega|/4 \quad \text{and} \\
 & \left| \{x \in \Omega \mid u_j(x) \leq 0\} \right| \geq |\Omega|/4.
 \end{aligned}$$

By passing to a subsequence,  $\lim_{j \rightarrow \infty} u_j = w$  almost everywhere in  $\Omega$  where  $w$  is a generalized solution of  $\Phi(u; \gamma)$ . *In the remaining part of this paper we always assume  $\Omega$  satisfies (1.3) and (1.9), and we fix the sequence  $u_j$  and the function  $w$  described above unless otherwise specified.*

Let  $P = \{x \in \Omega \mid w(x) = +\infty\}$ , and  $N = \{x \in \Omega \mid w(x) = -\infty\}$ . Then we have:

**LEMMA 2.2.**  *$P$  is a solution of  $G(A; \gamma)$  and  $N$  is a solution of  $G(A; \pi - \gamma)$ .*

*Proof.* For any positive integer  $j$ ,  $w - j$  is also a generalized solution of  $\Phi(u; \gamma)$ . As in the proof of Theorem 2.1, we can find a subsequence of  $w - j$  which converges to a generalized solution  $w'$  of  $\Phi(u; \gamma)$ . But the subgraph of  $w'$  is  $P \times \mathbf{R}$ . Therefore  $P \times \mathbf{R}$  is a solution of  $F(U; \gamma)$ . Since  $P \times \mathbf{R}$  is a cylinder, so  $P$  is a solution of  $G(A; \gamma)$ . By considering  $-w$  which is a generalized solution of  $\Phi(u; \pi - \gamma)$ , one can similarly prove that  $N$  is a solution of  $G(A; \pi - \gamma)$ .  $\square$

**COROLLARY 2.1.**  *$w$  is finite almost everywhere.*

*Proof.* By Lemma 2.2,  $G(P; \gamma) \leq G(\Omega; \gamma) = 0$ . By (1.3) we conclude  $P = \emptyset$  or  $\Omega$ . But each  $u_j$  satisfies (2.4) so  $P = \emptyset$ . Similarly we can prove  $N = \emptyset$ .  $\square$

### 3. Boundedness of $w$ : case when $\cos \gamma \sqrt{1 + L^2} < 1$ .

LEMMA 3.1. *Let  $U$  be a solution of  $F(U; \beta)$  and  $\Gamma_1$  be an open portion of  $\partial\Omega$ . Assume (i)  $\Gamma_1 = \emptyset$  or  $\Gamma_1$  has a Lipschitz constant  $L$  with  $|\cos \beta \sqrt{1 + L^2}| \leq a < 1$ ; and (ii)  $|H_\beta| \leq b$ . For any subdomain  $\Omega'$  of  $\Omega$  with  $\text{dist}(\Omega', \partial\Omega - \Gamma_1) > \sigma > 0$ , we can find constants  $C_3 > 0$  and  $r_0 > 0$ , which depend only on  $n, b, \sigma$ , and  $\Omega$  if  $\Gamma_1 = \emptyset$  and depend also on  $\Gamma_1, L$  and  $a$  if  $\Gamma_1 \neq \emptyset$ , such that for all  $(x_0, t_0) \in \bar{\Omega}' \times \mathbf{R}$ , the following are true:*

(3.1) *if  $|U_r(x_0, t_0)| = |C_r(x_0, t_0) \cap U| > 0$  for all  $r > 0$ , then*

$$|U_r(x_0, t_0)| \geq C_3 r^{n+1} \quad \text{for all } r \leq r_0;$$

(3.2) *if  $|U'_r(x_0, t_0)| = |C_r(x_0, t_0) - U| > 0$  for all  $r > 0$ , then*

$$|U'_r(x_0, t_0)| \geq C_3 r^{n+1} \quad \text{for all } r \leq r_0,$$

where  $C_r(x_0, t_0) = \{(x, t) \in \mathbf{R}^{n+1} \mid |x - x_0| < r \text{ and } |t - t_0| < r\}$ .

*Note.* We define the distance between a set and the empty set to be  $+\infty$ .

*Proof.* The proof is essentially the same as the proof of Theorem 3.2 in [11]. Let us first assume  $\Gamma_1 \neq \emptyset$ . For simplicity we write  $_r = U_r(x_0, t_0)$ ,  $C_r = C_r(x_0, t_0)$  and  $H = H_\beta$ . Now compare  $U$  with  $U - C_r$ ,

$$\begin{aligned} (3.3) \quad & \int_{(\Omega \times \mathbf{R}) \cap C_r} |D\phi_U| + \int_{(\Omega \times \mathbf{R}) \cap C_r} H\phi_U dx dt - \cos \beta \int_{(\partial\Omega \times \mathbf{R}) \cap C_r} \phi_U dH_n \\ & \leq \int_{\partial C_r} \phi_U dH_n. \end{aligned}$$

So for almost all  $r > 0$ ,

$$\begin{aligned} (3.4) \quad & \int_{\Omega \times \mathbf{R}} |D\phi_U| + \int_{\Omega \times \mathbf{R}} H\phi_U dx dt - \cos \beta \int_{\partial\Omega \times \mathbf{R}} \phi_U dH_n \\ & \leq 2 \int_{\partial C_r} \phi_U dH_n. \end{aligned}$$

Let  $\Gamma = \{x \in \Gamma_1 \mid \text{dist}(x, \partial\Omega - \Gamma_1) \geq \sigma/2\}$ . Suppose  $0 < r < \sigma/2$ , then by Lemma 1.1,

$$(3.5) \quad \int_{\partial\Omega \times \mathbf{R}} \phi_{U_r} dH_n = \int_{\Gamma \times \mathbf{R}} \phi_{U_r} dH_n \leq \sqrt{1 + L^2} \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}| + C_1 |U_r| \\ \leq \sqrt{1 + L^2} \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}| + C_1 k(n+1) |C_r|^{1/(n+1)} \int |D\phi_{U_r}|,$$

where  $C_1$  is the constant in Lemma 1.1 which depends only on  $\sigma$ ,  $\Omega$  and  $\Gamma_1$ , and  $k(n+1)$  is the isoperimetric constant in  $\mathbf{R}^{n+1}$ . In the last inequality we have used the isoperimetric inequality Corollary 1.29 in [12]. Now choose  $r_1 > 0$  small enough such that

$$(3.6) \quad r_1 \leq \sigma/2 \quad \text{and} \quad C_1 \cdot k(n+1) |C_r|^{1/(n+1)} < 1/2.$$

Since  $\int |D\phi_{U_r}| = \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}| + \int_{\partial\Omega \times \mathbf{R}} \phi_{U_r} dH_n$ , so from (3.5) and (3.6) we have

$$(3.7) \quad \int_{\partial\Omega \times \mathbf{R}} \phi_{U_r} dH_n \leq \frac{\sqrt{1 + L^2} + C_1 k(n+1) |C_r|^{1/(n+1)}}{1 - C_1 k(n+1) |C_r|^{1/(n+1)}} \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}|,$$

and

$$(3.8) \quad \int |D\phi_{U_r}| = \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}| + \int_{\partial\Omega \times \mathbf{R}} \phi_{U_r} dH_n \\ \leq \frac{1 + \sqrt{1 + L^2}}{1 - C_1 k(n+1) |C_r|^{1/(n+1)}} \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}| \\ \leq 2(1 + \sqrt{1 + L^2}) \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}|,$$

for all  $0 < r \leq r_1$ .

By Lemma 2.1 of [11]:

$$(3.9) \quad \int_{\Omega \times \mathbf{R}} H\phi_{U_r} dx dt = \int_{U_r} H dx dt \geq -k(n) \cdot b \cdot |C_r|^{1/n} \int |D\phi_{U_r}|.$$

Combining (3.7)–(3.9):

$$(3.10) \quad \int_{\Omega \times \mathbf{R}} H\phi_{U_r} dx dt - \cos\beta \cdot \int_{\partial\Omega \times \mathbf{R}} \phi_{U_r} dH_n \\ \geq - \left\{ 2b(1 + \sqrt{1 + L^2}) \cdot k(n) |C_r|^{1/n} \right. \\ \left. + \cos\beta \left( \frac{\sqrt{1 + L^2} + C_1 k(n+1) |C_r|^{1/(n+1)}}{1 - C_1 k(n+1) |C_r|^{1/(n+1)}} \right) \right\} \cdot \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}|,$$

for almost all  $0 < r \leq r_1$ .

The expression in the whole bracket above tends to  $\cos \beta \sqrt{1 + L^2}$  as  $r \rightarrow 0$ . Since  $\cos \beta \sqrt{1 + L^2} \leq a < 1$ , we may choose  $0 < r_0 \leq r_1$  such that the expression in the bracket is less than  $(1 + a)/2$  for all  $0 < r \leq r_0$ . We see that  $r_0$  depends only on  $n, a, b, \sigma, L, \Omega$ , and  $\Gamma_1$ . By (3.4), (3.6), (3.8) and (3.10) we get

$$\begin{aligned} 2 \frac{d}{dr} |U_r| &= 2 \int_{\partial C_r} \phi_U dH_n \geq \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}| - \frac{1+a}{2} \int_{\Omega \times \mathbf{R}} |D\phi_{U_r}| \\ &\geq \frac{1-a}{4(1 + \sqrt{1 + L^2})} \int |D\phi_{U_r}| \\ &\geq \frac{1-a}{4 \cdot k(n+1)(1 + \sqrt{1 + L^2})} |U_r|^{n/(n+1)}, \end{aligned}$$

for almost all  $0 < r \leq r_0$ . If  $|U_r| > 0$  for all  $r > 0$ , then we have  $|U_r| \geq C_3 r^{n+1}$  for all  $r \leq r_0$  where  $C_3$  depends only  $n, a$ , and  $L$ , which can be computed explicitly.

If  $\Gamma_1 = \emptyset$ , then there is no need to estimate boundary term  $\int_{\partial\Omega \times \mathbf{R}} \phi_{U_r} dH_n$ . So in this case we do not need the inequality  $\cos \beta \sqrt{1 + L^2} \leq a < 1$ , the rest of the proof is similar. Hence (3.1) is true. Similarly by considering  $U' = \Omega \times \mathbf{R} - U$ , we can also prove (3.2).  $\square$

**THEOREM 3.1.** *Suppose  $\Omega$  is a bounded Lipschitz domain satisfying (1.3) and (1.9). Let  $\gamma_j, u_j$  and  $w$  be as before, and let  $\Gamma_1$  be an open portion of  $\partial\Omega$  such that either  $\Gamma_1 = \emptyset$  or  $\Gamma_1$  has a Lipschitz constant  $L$  with  $\cos \gamma \sqrt{1 + L^2} < 1$ . Then for any subdomain  $\Omega' \subset \Omega$  with  $\text{dist}(\Omega', \partial\Omega - \Gamma_1) > 0$ , there exists a constant  $C_4$  independent of  $j$  such that*

$$(3.11) \quad \sup_{\Omega'} |u_j| \leq C_4 \quad \text{and}$$

$$(3.12) \quad \sup_{\Omega'} |w| \leq C_4.$$

*Proof.* We assume  $\Gamma_1 \neq \emptyset$ , the case when  $\Gamma_1 = \emptyset$  can be proved similarly. Suppose the functions  $u_j$  are not uniformly bounded above in  $\Omega'$ . Since each  $u_j$  is bounded in  $\Omega$ , by passing to a subsequence if necessary, we can find  $x_j \in \Omega'$ ,  $\lim_{j \rightarrow \infty} x_j = x_0 \in \overline{\Omega'}$  and  $\lim_{j \rightarrow \infty} u_j(x_j) = +\infty$ . Let  $C_3$  and  $r_0$  be the constants in Lemma 3.1 corresponding to  $a = \cos \gamma \sqrt{1 + L^2}$  and  $b = H_\gamma$ . For any  $t > 0$ , we have  $u_j(x_j) > t$  and  $|x_j - x_0| < r_0/2$  if  $j$  is large enough. Let  $U_j$  be the subgraph of  $u_j$ . Then

$U_j$  is a solution of  $F(U; \gamma_j)$ , and if  $j$  is large enough  $|U_{j,r}(x_j, t)| = |U_j \cap C_r(x_j, t)| > 0$  for all  $r > 0$ , because  $u_j$  is regular in  $\Omega$ . Since  $\cos \gamma_j \sqrt{1 + L^2} < a$  and  $0 < H_{\gamma_j} < b$ , by Lemma 3.1 if  $j$  is large enough then

$$|U_{j,r_0}(x_0, t)| \geq |U_{j,r_0/2}(x_j, t)| \geq C_3 \left( \frac{r_0}{2} \right)^{n+1}.$$

Let  $j \rightarrow \infty$ , we conclude that

$$|W_{r_0}(x_0, t)| \geq C_3 \left( \frac{r_0}{2} \right)^{n+1},$$

where  $W$  is the subgraph of  $w$ . Since  $t$  is arbitrary, this contradicts Corollary 2.1 that  $w$  is finite almost everywhere. Therefore  $u_j$  are uniformly bounded above in  $\Omega'$ . Similarly we can prove that  $u_j$  are uniformly bounded below in  $\Omega'$ , and (3.11) is proved. (3.12) is an immediate consequence of (3.11) because  $u_j$  converge to  $w$  almost everywhere in  $\Omega$ .  $\square$

**THEOREM 3.2.** *Let  $\Omega$  be a bounded Lipschitz domain having Lipschitz constant  $L$  with  $\cos \gamma \sqrt{1 + L^2} < 1$ . If  $\Omega$  satisfied (1.3), then  $\Phi(u; \gamma)$  has a bounded solution which is analytic in  $\Omega$ .*

*Proof.* By Theorem 3.1 with  $\Gamma_1 = \partial\Omega$ , we conclude that  $w$  is bounded and hence is a bounded solution of  $\Phi(u; \gamma)$ . As in the proof of Lemma 2.1, we see that  $w$  is analytic in  $\Omega$ .  $\square$

If  $\Omega$  only satisfies (1.9) (and (1.3)) then  $w$  may be unbounded. For example, if  $\Omega$  is smooth and  $\gamma = 0$ , then  $w$  may be unbounded, see [10]. However we have the following:

**THEOREM 3.3.** *Let  $\Omega$  be a bounded Lipschitz domain satisfying (1.3) and (1.9), then  $\Phi(u; \gamma)$  has a locally bounded generalized solution which is analytic and satisfies  $\operatorname{div} Tw = H_\gamma$  in  $\Omega$ .*

*Proof.* By Theorem 3.1, for any subdomains  $\Omega' \subset \subset \Omega'' \subset \subset \Omega$ , there exists a constant  $C_4$  not depending of  $j$  such that  $\sup_{\Omega''} |u_j| \leq C_4$  for all  $j$ .

Since each  $u_j$  is analytic and satisfies  $\operatorname{div} Tu = H_{\gamma_j}$  in  $\Omega$ . By Corollary 15.7 of [8], we can find a constant  $C_5$  not depending on  $j$  such that

$$\sup_{\bar{\Omega}'} |Du_j| + \sup_{\bar{\Omega}'} |D^2 u_j| + \sup_{\bar{\Omega}'} |D^3 u_j| \leq C_5,$$

for all  $j$ . By passing to a subsequence,  $u_j$ ,  $Du_j$  and  $D^2 u_j$  converge uniformly in  $\Omega'$ . Hence  $w \in C^2(\Omega')$  and satisfies  $\operatorname{div} Tw = H_\gamma$ , so  $w$  is analytic in  $\Omega'$ . Since  $\Omega'$  can be any subdomain of  $\Omega$  such that  $\bar{\Omega}' \subset \Omega$ , the theorem follows.  $\square$

Now we are going to investigate the boundary behavior of  $w$ . Following [7] and [13] we have the following definitions:

**DEFINITION 3.1.** We say a family of domains  $\Omega_k$  exhausting  $\Omega$  if  $\partial\Omega_k \in C^1$ ,  $\bar{\Omega}_k \subset \Omega_{k+1}$  and  $\bigcup_k \Omega_k = \Omega$ .

If  $\Omega$  is Lipschitz we can always find such an exhausting sequence by [14].

**DEFINITION 3.2.** Let  $\Omega$  be a Lipschitz domain,  $u \in C^2(\Omega)$  is said to satisfy  $Tu \cdot \nu = \cos \gamma$  weakly on  $\partial\Omega$  if for any exhausting sequence  $\Omega_k$  of  $\Omega$ ,

$$(3.13) \quad \lim_{k \rightarrow \infty} \int_{\partial\Omega_k} fTu \cdot \nu_k dH_{n-1} = \int_{\partial\Omega} f \cos \gamma dH_{n-1}$$

for any function  $f$  belonging to the Sobolev space  $W^{1,1}(\Omega)$ , where  $\nu_k$  is the outward normal of  $\partial\Omega_k$ .

**THEOREM 3.4.** *If  $\Omega$  satisfies (1.3) and (1.9), and if  $u_j$  and  $w$  are as before, then  $w$  satisfies  $Tw \cdot \nu = \cos \gamma$  weakly on  $\partial\Omega$ .*

*Proof.* For each  $j$ ,  $u_j$  is a variational solution of  $\Phi(u; \gamma_j)$ . By the proof of Lemma 2 in [7], for all  $f \in W^{1,1}(\Omega)$ , we have

$$(3.14) \quad \int_{\Omega} Df \cdot \frac{Du_j}{\sqrt{1 + |Du_j|^2}} dx + \int_{\Omega} fH_{\gamma_j} dx - \int_{\partial\Omega} f \cos \gamma_j dH_{n-1} = 0.$$

From the proof of Theorem 3.3, by passing to a subsequence  $\lim_{j \rightarrow \infty} Du_j = Dw$  in  $\Omega$ . As  $|Du_j|/\sqrt{1 + |Du_j|^2} < 1$  in  $\Omega$  for all  $j$ , noting that  $\lim_{j \rightarrow \infty} \cos \gamma_j = \cos \gamma$  and  $\lim_{j \rightarrow \infty} H_{\gamma_j} = H_{\gamma}$ , let  $j \rightarrow \infty$  in (3.14), by Lebesgue dominated convergence theorem, we get

$$(3.15) \quad \int_{\Omega} Df \cdot \frac{Dw}{\sqrt{1 + |Dw|^2}} dx + \int_{\Omega} fH_{\gamma} dx - \int_{\partial\Omega} f \cos \gamma dH_{n-1} = 0.$$

Let  $\Omega_k$  be an exhausting sequence of  $\Omega$ . Multiply  $\operatorname{div} Tw = H_{\gamma}$  by  $f$  and integrating by parts over  $\Omega_k$ :

$$(3.16) \quad \int_{\Omega_k} Df \cdot \frac{Dw}{\sqrt{1 + |Dw|^2}} dx + \int_{\Omega_k} fH_{\gamma} dx - \int_{\partial\Omega_k} fTw \cdot \nu_k dH_{n-1} = 0.$$

By Lebesgue dominated convergence theorem again,

$$\lim_{k \rightarrow \infty} \int_{\Omega_k} Df \cdot \frac{Dw}{\sqrt{1 + |Dw|^2}} dx = \int_{\Omega} Df \cdot \frac{Dw}{\sqrt{1 + |Dw|^2}} dx \quad \text{and}$$

$$\lim_{k \rightarrow \infty} \int_{\Omega_k} fH_{\gamma} dx = \int_{\Omega} fH_{\gamma} dx.$$

Therefore combine (3.15) and (3.16), we have

$$\lim_{k \rightarrow \infty} \int_{\partial\Omega_k} fTw \cdot \nu_k dH_{n-1} = \int_{\partial\Omega} f \cos \gamma dH_{n-1}. \quad \square$$

**4. Boundedness of  $w$ : general case.** As we noted before, if there is no  $L$  with  $\cos \gamma \sqrt{1 + L^2} < 1$  such that  $\Omega$  has Lipschitz constant  $L$ , then the generalized solution  $w$  may fail to be bounded. However, if we impose some reasonable assumptions on  $\partial\Omega$ , we can still prove that  $w$  is bounded and hence  $w$  is a solution of  $\Phi(u; \gamma)$ .

As before we assume  $\Omega$  satisfies (1.3) and (1.9). In addition we assume  $\partial\Omega = \Gamma' \cup \Gamma''$  such that

(4.1)  $\Gamma'$  is an open portion of  $\partial\Omega$  with Lipschitz constant  $L$ ,  $\cos \gamma \sqrt{1 + L^2} < 1$ ;

(4.2)  $\Gamma'' = \Gamma^1 \cup \Gamma^2 \cup \dots \cup \Gamma^N$ , each  $\Gamma^k$  is closed;

(4.3) for each  $k = 1, \dots, N$ , there is an open set  $O_k \subset \mathbf{R}^n$  containing  $\Gamma^k$  such that  $\text{dist}(O_k, \bigcup_{m \neq k} \Gamma^m) > 0$ ,  $O_k \cap \Omega$  is Lipschitz and connected; also there are functions  $v_k^{(1)}$  and  $v_k^{(2)}$  belonging to  $C^2(\mathbf{R}^n)$  and satisfying:

(i)  $\text{div} Tv_k^{(1)} = \text{div} Tv_k^{(2)} = H_{\gamma}$  in  $O_k \cap \Omega$ ; and

(ii)  $Tv_k^{(1)} \cdot \nu \geq \cos \gamma \geq Tv_k^{(2)} \cdot \nu$   $H_{n-1}$ -almost everywhere on  $O_k \cap \partial\Omega$ .

$v_k^{(1)}$  and  $v_k^{(2)}$  will serve as upper and lower comparison surfaces for  $w$  respectively. In some cases, for example,  $n = 2$  and  $\Omega$  is piecewise smooth, lower comparison surfaces always exist, see [6]. Our assumptions on  $\partial\Omega$  is a generalization of the hypothesis  $\alpha(\gamma)$  introduced in §1 to higher dimensions.

**THEOREM 4.1.** *Suppose  $\Omega$  satisfies (1.3), (1.9) and (4.1)–(4.3). Let  $u_j$  and  $w$  be as before, then  $w$  is bounded and is a variational solution of  $\Phi(u; \gamma)$ .*

*Proof.* By Theorem 3.1 and (4.1), we know that  $w$  is bounded in every subdomain  $\Omega' \subset \Omega$  with  $\text{dist}(\Omega', \partial\Omega - \Gamma') > 0$ . It remains to prove that  $w$  is bounded in  $O_k \cap \Omega$  for  $k = 1, \dots, N$ . Firstly, we want to prove that  $w$

is bounded above. For simplicity, we write  $O = O_k$ ,  $\Gamma = \Gamma^k$  and  $v = v_k^{(1)}$ . Let  $O' \subset \subset O$  be an open set containing  $\Gamma$ , then  $\text{dist}(O - O', \Gamma'') > 0$  by assumption (4.3), and so  $\text{dist}((O - O') \cap \Omega, \Gamma'') > 0$ . By Theorem 3.1, the functions  $u_j$  are uniformly bounded in  $(O - O') \cap \Omega$ . By adding a constant to  $v$ , we may assume the  $u_j < v$  on  $(O - O') \cap \Omega$ . We assert that  $w \leq v$  on  $O \cap \Omega$ .

From the proof of Theorem 3.1, we know that by passing to a subsequence  $u_j$  converge to  $w$  and  $Du_j$  converge to  $Dw$ , both uniformly on compact subsets of  $\Omega$ . If  $v(x_0) < w(x_0)$  for some  $x_0 \in O \cap \Omega$ , then there exist positive numbers  $\rho$ ,  $M$ , and a positive integer  $j_0$  such that  $\overline{B_\rho(x_0)} \subset \Omega$ , and we have  $0 < u_j - v < M$  in  $B_\rho(x_0)$ , for  $j \geq j_0$ . Define

$$w_j = \begin{cases} 0 & u_j - v \leq 0 \\ u_j - v & 0 < u_j - v < M \\ M & M \leq u_j - v \end{cases} \quad \text{in } O \cap \Omega.$$

Extend  $w_j$  to be zero in  $\Omega - O$ . Since  $w_j = 0$  in  $(O - O') \cap \Omega$  and  $\text{dist}(\Omega - O, O') > 0$ , so  $w_j \in W^{1,1}(\Omega)$ . From the proof of Lemma 2 of [7]:

$$(4.4) \quad \int_{O \cap \Omega} Dw_j \cdot \frac{Du_j}{\sqrt{1 + |Du_j|^2}} dx \\ = - \int_{O \cap \Omega} H_{\gamma_j} w_j dx + \int_{O \cap \partial \Omega} w_j \cos \gamma_j dH_{n-1},$$

bearing in mind that  $w_j = 0$  in  $\Omega - O$ . On the other hand by assumption (4.3)(i) and (ii),

$$(4.5) \quad \int_{O \cap \Omega} Dw_j \cdot \frac{Dv}{\sqrt{1 + |Dv|^2}} dx \\ \geq - \int_{O \cap \Omega} H_{\gamma} w_j dx + \int_{O \cap \partial \Omega} w_j \cos \gamma dH_{n-1}.$$

Subtracting (4.5) from (4.4):

$$(4.6) \quad \int_{O \cap \Omega} Dw_j \cdot \left( \frac{Du_j}{\sqrt{1 + |Du_j|^2}} - \frac{Dv}{\sqrt{1 + |Dv|^2}} \right) \\ \leq \int_{O \cap \Omega} (H_{\gamma} - H_{\gamma_j}) w_j dx + \int_{O \cap \partial \Omega} w_j (\cos \gamma_j - \cos \gamma) dH_{n-1}.$$



By the definition of  $w_j$  and the particular structure of the operator  $Tu$ , it is easy to see

$$Dw_j \cdot \left( \frac{Du_j}{\sqrt{1 + |Du_j|^2}} - \frac{Dv}{\sqrt{1 + |Dv|^2}} \right) \geq 0.$$

By (4.6), for  $j \geq j_0$ :

$$\begin{aligned} (4.7) \quad & \int_{B_\rho(x_0)} D(u_j - v) \cdot \left( \frac{Du_j}{\sqrt{1 + |Du_j|^2}} - \frac{Dv}{\sqrt{1 + |Dv|^2}} \right) dx \\ & \leq \int_{O \cap \Omega} Dw_j \cdot \left( \frac{Du_j}{\sqrt{1 + |Du_j|^2}} - \frac{Dv}{\sqrt{1 + |Dv|^2}} \right) dx \\ & \leq \int_{O \cap \Omega} (H_\gamma - H_{\gamma_j}) w_j dx + \int_{O \cap \partial \Omega} w_j (\cos \gamma_j - \cos \gamma) dH_{n-1}. \end{aligned}$$

Let  $j \rightarrow \infty$ , noting that  $0 \leq w_j \leq M$  for all  $j$ ,

$$(4.8) \quad \int_{B_\rho(x_0)} D(w - v) \cdot \left( \frac{Dw}{\sqrt{1 + |Dw|^2}} - \frac{Dv}{\sqrt{1 + |Dv|^2}} \right) \leq 0.$$

But

$$\begin{aligned} & D(w - v) \cdot \left( \frac{Dw}{\sqrt{1 + |Dw|^2}} - \frac{Dv}{\sqrt{1 + |Dv|^2}} \right) \\ & \geq |Dw - Dv|^2 \cdot \int_0^1 \frac{dt}{(1 + |Dv_t|^2)^{3/2}}, \end{aligned}$$

where  $v_t = tw + (1 - t)v$ . Therefore  $Dw - Dv = 0$  in  $B_\rho(x_0)$  and  $w = v + K$  there, where  $K = w(x_0) - v(x_0) > 0$ .

Let  $A = \{x \in O \cap \Omega \mid w(x) = v(x) + K\}$ .  $A$  is obviously closed in  $O \cap \Omega$  since  $w$  and  $v$  are continuous. Also from the proof above, we see that  $A$  is open. Since  $A \neq \emptyset$ , and by assumption  $O \cap \Omega$  is connected we conclude that  $A = O \cap \Omega$ . This is impossible because  $u_j < v$  in  $(O - O') \cap \Omega$  which is not empty. Hence  $w \leq v$  in  $O \cap \Omega$ , and  $w$  is bounded above in  $\Omega$ . Similarly, we can prove that  $w$  is bounded below. By Theorem 1.1,  $w$  is a solution of  $\Phi(u; \gamma)$ .  $\square$

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Dale Edward Alspach, On $\mathcal{L}_{p,\lambda}$ spaces for small $\lambda$ .....	257
Jong Sook Bae and Sangsuk Yie, Range of Gateaux differentiable operators and local expansions .....	289
Hubert Berens and Lothar Hetzelt, On accretive operators on $l_n^\infty$ .....	301
Marie-Françoise Bidaut-Véron, Global existence and uniqueness results for singular solutions of the capillarity equation .....	317
Donald M. Davis and Mark Mahowald, Classification of the stable homotopy types of stunted real projective spaces .....	335
Aad Dijkema, Heinz K. Langer and Hendrik S. V. de Snoo, Unitary colligations in $\Pi_\kappa$ -spaces, characteristic functions and Štraus extensions .....	347
Michel Enock and Jean-Marie Schwartz, Algèbres de Kac moyennables .....	363
Seppo Granlund, Peter Lindqvist and Olli Martio, Note on the PWB-method in the nonlinear case .....	381
Palle E. T. Jorgensen, Analytic continuation of local representations of Lie groups .....	397
Robert P. Kaufman, Plane curves and removable sets .....	409
José M. Montesinos and Wilbur Carrington Whitten, Constructions of two-fold branched covering spaces .....	415
Benedict Seifert, Highly transitive group actions on trees and normalizing Tits systems .....	447
Charles Stuart Stanton, Counting functions and majorization for Jensen measures .....	459
Luen-Fai Tam, On existence criteria for capillary free surfaces without gravity .....	469
Zhuocheng Yang, Exposed points of left invariant means .....	487