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Any closed 3-manifold M may be thought of as a union of 3-cells. Any covering space of M is made up of copies of these 3-cells with boundaries locally identified as in M. Covers of M may be built by piecing together these balls. This paper develops a method to piece together the universal cover of manifolds obtained from the 3-sphere by surgery on a class of pretzel knots in such a way that the cover can be shown to be R^3.

1. It has been shown [4, 6, 14] that any connected orientable 3-manifold may be constructed by surgery along a finite number of knots in S^3, that is, by removing tubular neighborhoods of one or more smooth knots and sewing them back in differently. In this paper we study the 3-manifolds obtained by surgery on certain pretzel knots [2, 8] by showing that the universal cover of such manifolds is R^3. We thus show that these pretzel knots satisfy several hoped for conjectures (property P: nontrivial surgery never yields a simply connected manifold, and property R: surgery never yields a manifold with fundamental group Z).

The manifolds obtained by surgery on pretzel knots have been studied algebraically. In fact, it is known that all pretzel knots (except, of course, the unknot, which has several pretzel knot representations) have property R [5, 7]. The property P question has been answered affirmatively for certain classes of pretzel knots [1, 9, 10, 11], but the question remains open in general. We will take a different, geometric approach to the problem. In §2 we introduce the basic concepts we will use to build the covering spaces of the manifolds, as well as prove a lemma which identifies the covering space in certain situations. In §3 we apply the ideas of §2 to the surgery manifolds of certain pretzel knots and construct their universal covering spaces.

2. In this section we develop concepts which allow us, in certain cases, to build and identify covering spaces inductively. The central idea is "precover" which isolates properties a space needs in order to be part of a covering space (i.e., an incompletely built covering space).

DEFINITION 2.1. Let $p: \tilde{X} \to X$ be a map. $(\tilde{X}, p, X)$ is a *precover* if for each $x \in X$ there is a connected open set U containing x such that

(i) $p^{-1}(U) = \bigcup S_a$ where the S_a's form a collection (nonempty if x is in $p(\tilde{X})$) of mutually disjoint connected open sets in $\tilde{X}$, each containing

one preimage x_a of x:

(ii) for each S_a, $p \mid S_a$: $S_a \rightarrow p(S_a)$ is a homeomorphism, and if x_a is in the interior of $\tilde{X}$, $p(S_a) = U$.

We use interior in the sense that the precover is a subset of a covering space. This need not be the case (though it will be if we are successful in our efforts to build the covering space) so we need to clarify our meaning. Since we will be building covering spaces only for 3-manifolds with no boundary, we will use the following definition.

DEFINITION 2.2. A point in a space which lies in a neighborhood homeomorphic to R^3 is called an *interior point*.

DEFINITION 2.3. A point in a space which is not an interior point is a *boundary point*.

We need criteria to indicate when the covering space has been reached. The following lemma is suitable for the manifolds we will be considering.

LEMMA 2.4. *A precover $(\tilde{X}, p, X)$ such that $p(\tilde{X}) = X$ and $\tilde{X}$ has no boundary points is a covering space for X.*

Observe that for any covering space $(\tilde{X}, p, X)$ and Y a subset of X and for $\tilde{Y}$ any component of $p^{-1}(Y)$, $(\tilde{Y}, p \mid \tilde{Y}, Y)$ is a covering space for Y. In particular, if Y is homeomorphic to a 3-cell, then $p \mid \tilde{Y}$: $\tilde{Y} \rightarrow Y$ is a homeomorphism. Therefore, if X can be divided into 3-cells with boundaries glued together, $\tilde{X}$ can be divided into copies (3-cells that are taken by the covering map onto the 3-cells in X) of these same 3-cells with boundaries identified as they are in X. For our purposes, if this can be done, precovers of X can be built by glueing together copies of these 3-cells. To that end we make the following definitions.

DEFINITION 2.5. A *block subdivision* of a 3-manifold X is a decomposition of X into 3-cells $\{C_i\}$ such that

(i) the collection $\{C_i\}$ is locally finite;

(ii) for $j \neq k$, $C_j \cap C_k$ is empty or a finite number of pairwise disjoint disks (called faces);

(iii) if the intersection of any three distinct 3-cells in $\{C_i\}$ is non-empty, the intersection is a finite number of pairwise disjoint arcs;

(iv) each point $x \in X$ has a closed neighborhood U which is homeomorphic to a 3-cell, such that U intersected with a face of any block is

either empty or a single disk containing x, $U \cap C_i$ is empty or homeomorphic to a 3-cell, and the space remaining after removing any number of $U \cap C_i$'s from U, if nonempty, is homeomorphic to a 3-cell.

DEFINITION 2.6. Let X be a 3-manifold with a block subdivision $\{C_i\}$. A *block* is a pair (B, b) where B is a 3-cell and $b: B \rightarrow X$ is a homeomorphism between B and some C_i. If D is a disk in the boundary of B such that $b(D)$ is a face of $b(B)$, $(D, b|D)$ is a *face* of the block (B, b).

DEFINITION 2.7. A *precover* $(\tilde{X}, p, X)$ *is made of blocks* if, for a block subdivision of X, $\tilde{X}$ has a block subdivision such that for each 3-cell $\tilde{B}$ in the subdivision of $\tilde{X}$, $(\tilde{B}, p | \tilde{B})$ is a block of X.

We now start building covering spaces out of blocks. We need to be careful in the building process so that we do not end up with any branch points.

DEFINITION 2.8. Given blocks (B_1, b_1) and (B_2, b_2) and a face D_1 of B_1, if there exists a face D_2 of B_2 such that $b_1(D_1) = b_2(D_2) = [b_1(B_2) \cap b_2(B_2)]$ then the blocks are *attached* to each other by identifying D_1 and D_2 (d_1 in D_1 and d_2 in D_2 are identified iff $b_1(d_1) = b_2(d_2)$) and combining the maps. Put the quotient topology on $B_1 \cup B_2$.

The result of attaching two blocks is a precover made of blocks (see Lemma 2.10). We will next extend the definition of attach to precovers made of blocks so that the result is also a precover made of blocks. To simplify matters, we will allow only points in the boundaries of the two precovers to be identified so the following procedure yields either a precover or the statement that the two precovers cannot be attached along the given faces.

DEFINITION 2.9. Consider (P_1, p_1, X) and (P_2, p_2, X), two precovers made of blocks compatible with the same block subdivision of X. For $i = 1, 2$, let D_i be a disk on the boundary of P_i and a face of the block $(B_i, p_i | B_i)$ such that the two blocks may be attached along these disks. The two precovers may be *attached along these disks* or found to be *unattachable along these disks* by the following algorithm:

(i) Identify the disks and combine the maps as above to form the space $(P, p, X)_0$.

(ii) If for every pair of distinct faces D_j and D_k in $(P, p, X)_0$ such that $D_j \cap D_k$ is not empty, the image of D_j is not the image of D_k, then the precovers have been attached and $(P, p, X) = (P, p, X)_0$. If there exist such pairs form $(P, p, X)_1$ by identifying the faces of one of the pairs and giving the result the quotient topology.

(iii) Having found $(P, p, X)_{n-1}$, continue inductively as follows: if for every pair of distinct faces D_j and D_k in $(P, p, X)_{n-1}$ such that $D_j \cap D_k$ is not empty, the image of D_j is not the image of D_k, then the precovers have been attached and $(P, p, X) = (P, p, X)_{n-1}$. If the images are the same for at least one pair, pick such a pair; if at least one of the faces is not in the boundary of $(P, p, X)_{n-1}$ the precovers are not attachable along D_1 and D_2; otherwise, form $(P, p, X)_n$ by identifying the two faces and giving the result the quotient topology.

(iv) Either the above process terminates in a finite number of steps or a countable number of identifications are made.

LEMMA 2.10. *If (P, p, X) is the result of attaching two precovers along given disks, then (P, p, X) is also a precover.*

Proof. We first note that the attaching process is well defined. Let (P_1, p_1, X) and (P_2, p_2, X) be the precovers that have been attached along D_1 and D_2. The set $p(P) = p_1(P_1) \cup p_2(P_2)$ is closed in X, therefore each $x \in X - [p(P)]$ is contained in an open neighborhood N_x such that $p^{-1}(N_x)$ is empty.

Let x be in $p(P)$. Relative to the subdivision on X, x is in a closed neighborhood N_x with certain properties (Definition 2.5iv). In the attaching process, if two faces with a common image intersect, the two faces will be identified, which insures that each component of $p^{-1}(N_x)$ will be assembled out of components of $p^{-1}(N_x \cap C_i)$ as N_x is assembled out of the pieces of $N_x \cap C_i$ so each component of $p^{-1}(N_x)$ will be homeomorphic to a 3-cell which is mapped one-to-one into N_x by the map induced by p. Thus, (P, p, X) is a precover.

DEFINITION 2.11. The points which have been identified when two precovers are attached are said to be the *total attachment*.

We will be building covering spaces homeomorphic to R^3 with the intermediate precovers being missing boundary 3-cells.

DEFINITION 2.12. An n-manifold M is a *missing boundary manifold* if M is homeomorphic to $N - L$, where N is a compact n-manifold and L is a closed (possibly empty) set in the boundary of N [12].

LEMMA 2.13. *Suppose* $\{(P_i, p_i, X)\}$ *and* $\{(Q_i, q_i, X)\}$ *are sequences of precovers of blocks such that*
  (i) *P_1 and each Q_i is homeomorphic to a missing boundary 3-cell;*
  (ii) *P_{i+1} is formed by attaching P_i and Q_i;*
  (iii) *the total attachment (D_i' in the boundary of Q_k, D_i'' in the boundary of P_i, and D_i in P_{i+1}) is a closed disk.*
Then for $\tilde{P} = \cup P_i$ and p: $\tilde{P} \to X$ defined by the maps on the sequence of precovers $\{P_i\}$, $\tilde{P}$ is a missing boundary 3-cell and $(\tilde{P}, p, X)$ is a precover made of blocks.

Proof. This proof follows a similar argument given by Waldhausen [**13**, Theorem 8.1]. It will be shown that each (P_i, p_i, X) is a missing boundary 3-cell and then that $\tilde{P}$ has the desired properties.

The proof is given by induction with the case $i = 1$ given in the hypothesis.

Let E be the closed unit ball in R^3 and suppose f_i: $P_i \to E$ is an embedding such that $f_i(P_i)$ contains the interior of E. By hypothesis, there exists an embedding of Q_i in another 3-cell E', g_i: $Q_i \to E'$ such that $g_i(Q_i)$ contains the interior of E'. The set $f_i(D_i'')$ is a disk by hypothesis (iii) and the fact that f_i is an embedding. Also, D_i'' is contained in the boundary of p_i since the attaching process doesn't make any identifications with points in the interior of P_i. Since $f_i(\text{int } P_i) = \text{int } E$, $f_i(D_i'')$ is in the boundary of E. Similarly, $g_i(D_i')$ is a disk in the boundary of E'. Thus E and E' can be sewn together along $f_i(D_i'')$ and $g_i(D_i')$ according to the instructions for attaching P_i and Q_i to form P_{i+1}.

Map $E \cup E'$ onto E by the homeomorphism h_i: $E \cup E' \to E$ with the properties
  (i) for any $x \in E$, the distance from x to the boundary of E is less than or equal to the distance from $h_i(x)$ to the boundary of E;
  (ii) if $x \in E$ is in the cone from the center of E to the closure of the complement of $E \cap E'$, then $h_i(x) = x$;
  (iii) if $x \in E$ is a distance greater than $1/i$ from the boundary of E, then $h_i(x) = x$.
This shows that P_{i+1} is a missing boundary 3-cell.

Because of the properties of the map h_i, a limit map is defined for $\tilde{P}$ into E showing that $\tilde{P}$ is a missing boundary 3-cell.

As shown in the proof of Lemma 2.10, each $x \in X$ has a neighborhood N_x such that for all precovers made of blocks (P_i, p_i, X), the components of $p^{-1}(N_x)$ map homeomorphically into N_x. Since the set of blocks in X is locally finite, each such component C in $(\tilde{P}, p, X)$ is

completely built after i steps, for some i, so $p: C \to N_x$ is identical with each $p_j: C \to N_x$, for $j \geq i$. Thus $(\tilde{P}, p, X)$ is a precover.

Each block is a 3-cell, so it is a missing boundary 3-cell. If precovers can be built from blocks such that the total attachments are always closed disks and that every boundary point is eliminated, a precover homeomorphic to R^3 will result. This will be the universal cover.

3. We now consider the $(4, 3, -5)$ pretzel knot. This knot, as any pretzel knot, may be realized as a curve on the surface of a double torus. S^3 has a genus two Heegaard splitting; hence S^3 may be thought of as two double tori whose boundaries have been identified. Figures 1a and 1b illustrate such a splitting with the markings on the lines indicating the
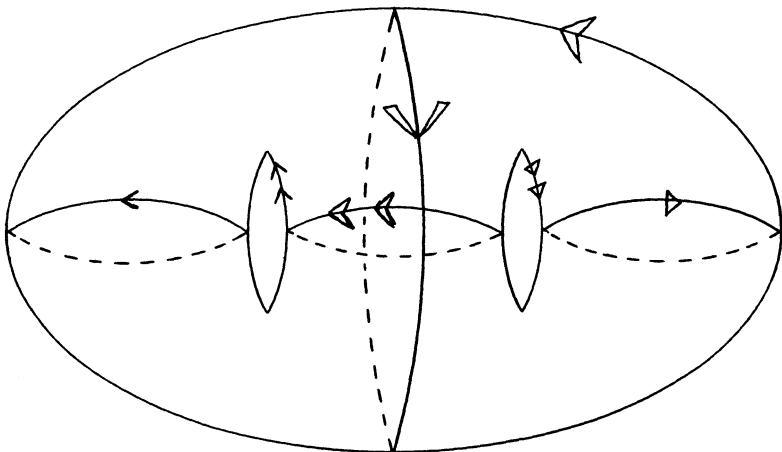
FIGURE 1a
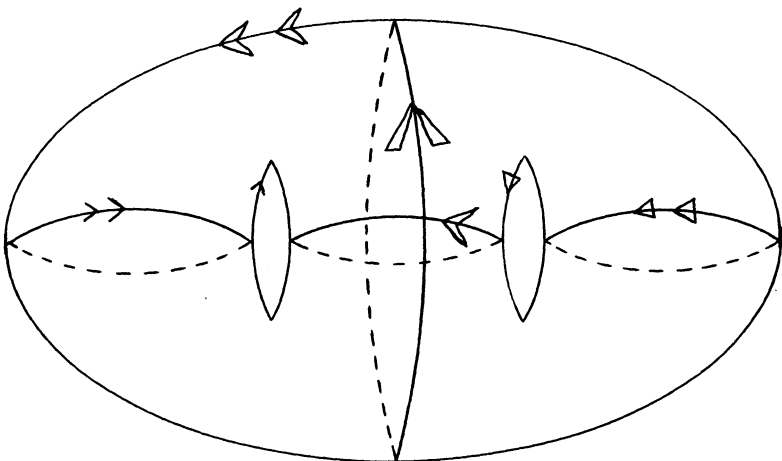
FIGURE 1b

identifications made between the two boundaries. The $(4, 3, -5)$ knot is placed on Figure 1a in the standard manner to produce Figure 2a with Figure 2b indicating how the knot lies on the other handlebody. The knot manifold, that is, the closed complement of a neighborhood of the knot in S^3, is realized as the sum of two double tori with all but an annulus of each boundary identified, as described by Figure 2. Additionally, any surgery manifold for this knot may be described by adding a torus whose boundary is identified with the remaining boundary (the two annuli) in a manner which depends upon the particular surgery. This description of the surgery manifold will be used to build its universal cover. We will refer to the double torus with the pretzel knot placed on its surface in the standard manner as D_a and the other double torus as D_b.
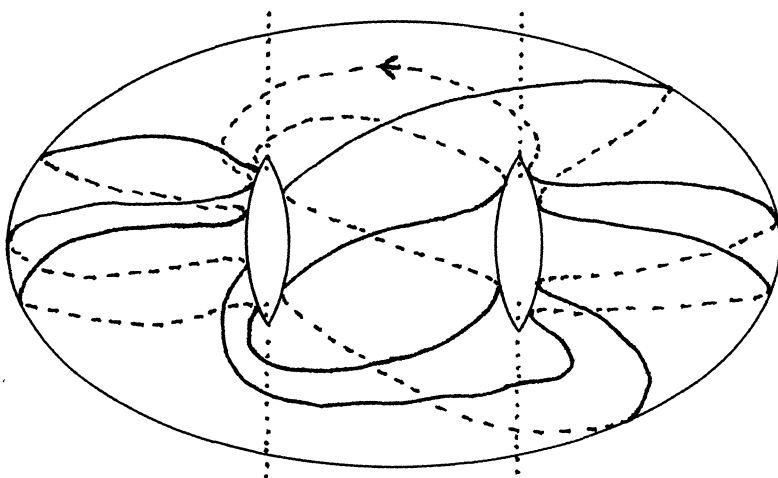
FIGURE 2a
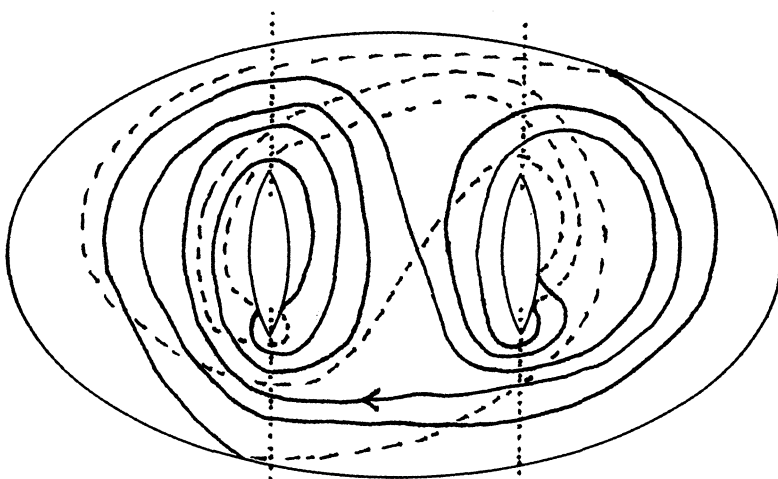
FIGURE 2b

We now form a block subdivision of the surgery manifold by cutting the above handlebodies into blocks. It is convenient to not have the boundary of a block identified to itself, so we cut each double torus into three blocks as indicated by the dotted lines in Figure 2. The torus is cut along two meridian disks so that no point in the manifold lies in more than four blocks. In discussing blocks in M and precovers of M, the following definitions will be used.

DEFINITION 3.1. An *H-disk* is a maximal disk on a block which is mapped properly into one of the three handlebodies (i.e., a disk along which blocks are identified to help form one of the handlebodies).

DEFINITION 3.2. A *D-disk* is a maximal disk on a block which is mapped into the intersection of the two double tori.

DEFINITION 3.3. A *K-disk* is a maximal disk on a block which is mapped to the boundary of the knot manifold, often indicated by a line segment on the various diagrams.

REMARK. The boundary of each block is decomposed into H-disks, D-disks, and K-disks, each of which is a face of the block.

DEFINITION 3.4. If $(\tilde{X}, p, M)$ is a precover, any component of the inverse image of the boundary of the knot manifold is a *knot lifting*.

In any covering space of a surgery manifold, each component of the preimage of the solid torus (closed neighborhood of the knot) will, along with the restricted covering map, be a covering space for the torus, so the space is either a torus or homeomorphic to $D^2 \times R^1$. In building universal covers of the surgery manifold for the $(4, 3, -5)$ pretzel knot, we will assume that the latter is the case and use them as the framework in the construction of the universal cover.

DEFINITION 3.5. A *core* is a universal covering space (precover of the surgery manifold) made of blocks of the solid torus (closed neighborhood of the knot).

DEFINITION 3.6. A *junction block* is a block with four H-disks on its boundary (each double torus is decomposed into one junction block and two others).

DEFINITION 3.7. The intersection of the three handlebodies is a pair of simple closed curves, either of which is called a *copy of the knot*.

The universal covers of the surgery manifolds will be built by starting with a core and covering it with blocks to form a precover M_0. The union of the K-disks on the boundary of M_0 is a collection of pairwise disjoint disks. Along one of these disks a new core is attacked, then covered with blocks. This process is repeated until a certain set of K-disks on the boundary of M_0 have had cores attached and each core has been covered with blocks. The result is the precover M_1. From this, we continue the construction inductively. We will first construct a diagram (Figure 6) to indicate how blocks (from the two double tori) are attached to a core. At any one time, this will give us enough of a view of the space we are constructing to proceed.

The surface of the single torus consists of two annuli, intersecting in two copies of the knot, each being identified with an annulus on the surface of one of the double tori. The surface of a core will consist of preimages of these two annuli, which will either be annuli or strips homeomorphic to $I \times R^1$, depending upon the particular surgery. In either case, the surface of a core can be thought of as having a (two dimensional) block subdivision consisting of two different blocks, $(I \times I, b_1)$ and $(I \times I, b_2)$, where $b_1(I \times I)$ is one of the annuli and $b_2(I \times I)$ is the other. By showing how blocks from D_a attach to copies of $(I \times I, b_1)$, how blocks from D_b attach to copies of $(I \times I, b_2)$, and how blocks from D_a and blocks from D_b which are attached to adjacent strips on the core are attached to each other, we will be able to construct Figure 6.

In order to reduce the amount of detail in the diagrams, we will first consider the $(2, 1, 1)$ pretzel knot. Figure 3 consists of blocks of the $(2, 1, 1)$ pretzel knot on D_b such that the connected union of K-disks represented by the double lined curve is a block of the surface of a core. We
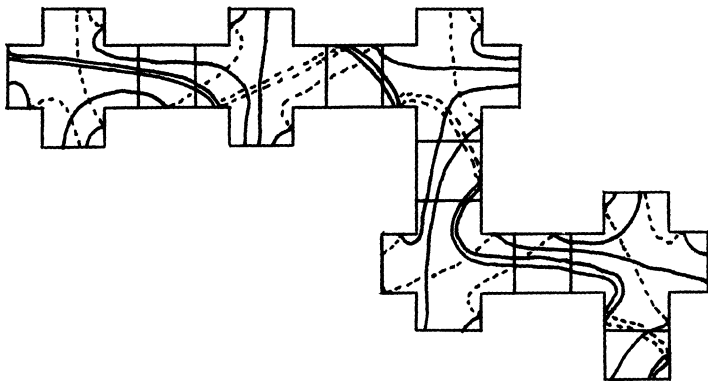

FIGURE 3

next put Figure 3 into a more useable form. First, Figure 3 is straightened out, as in the top diagram of Figure 4. The next diagram is formed by shrinking the secondary tubes back to the main tube. The disks on the top of the surface are *H*-disks which are not attached to other blocks in this diagram. In the third diagram of Figure 4, the tube is twisted so the core block disk is on the bottom, while the other liftings of the knot are on the top. In the next to last diagram, the pairs of *H*-disks have been moved so that one is at the top and the other is at the bottom of the diagram. The final diagram of Figure 4 is formed by marking the position of each pair of *H*-disks with a vertical line. Each vertical line lies on a different junction block. Between any adjacent pair of vertical lines lies one nonjunction block. Line segments representing other knot liftings that terminate in the same vicinity continue across the same *H*-disk, out of the diagram onto the same nonjunction block. When *K*-disks from two blocks
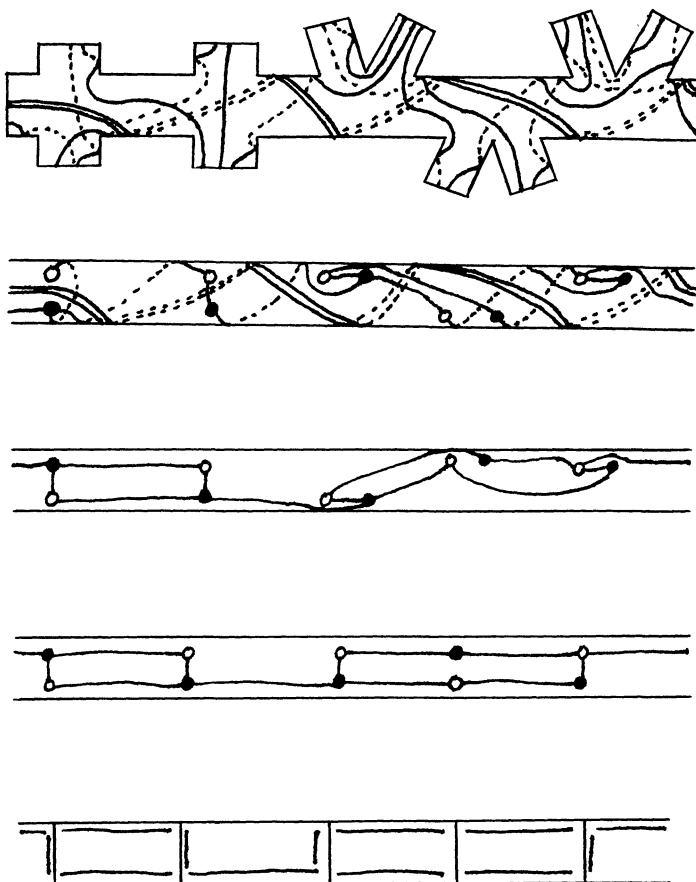
FIGURE 4

intersect (in a line segment) on a core, the attaching process will lead to the identification of *H*-disks, if the blocks map to the same double torus, or *D*-disks if the blocks map to different double tori. We have taken care of the former, since we formed Figure 3 by identifying *H*-disks. To see how *D*-disks are identified, thereby identifying edges of other *K*-disks, when blocks are attached to a core, we check Figure 2, which contains the information on how the blocks are identified. Figures 5a and 5b are the diagrams (from D_a and D_b respectively) for the $(4, 3, -5)$ pretzel knot. The numbering by some of the line segments (representing *K*-disks) indicate which knot liftings on the strips have edges identified when blocks from the two double tori are attached to adjacent strips on the boundary of the core.

DEFINITION 3.8. A knot lifting which lies on blocks attached directly to a given core which map into both D_a and D_b is called a *spanning knot lifting*.

Instead of having a diagram directly representing the blocks attached to a core (which depends upon the particular surgery), we will construct a diagram (Figure 6) representing the blocks which would be attached to the universal cover of the boundary of the knot manifold (a plane). Making the appropriate identifications for a particular surgery manifold produces the pattern of blocks that would be attached to a core in the universal cover of that particular manifold. Figure 6 is made by attaching copies of Figures 5a and 5b to the blocks of the universal cover of the boundary of the knot manifold. As noted above, when these diagrams, which represent
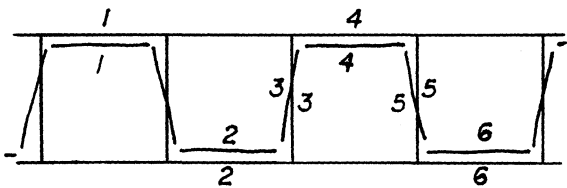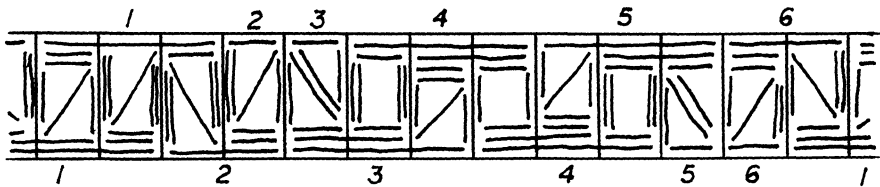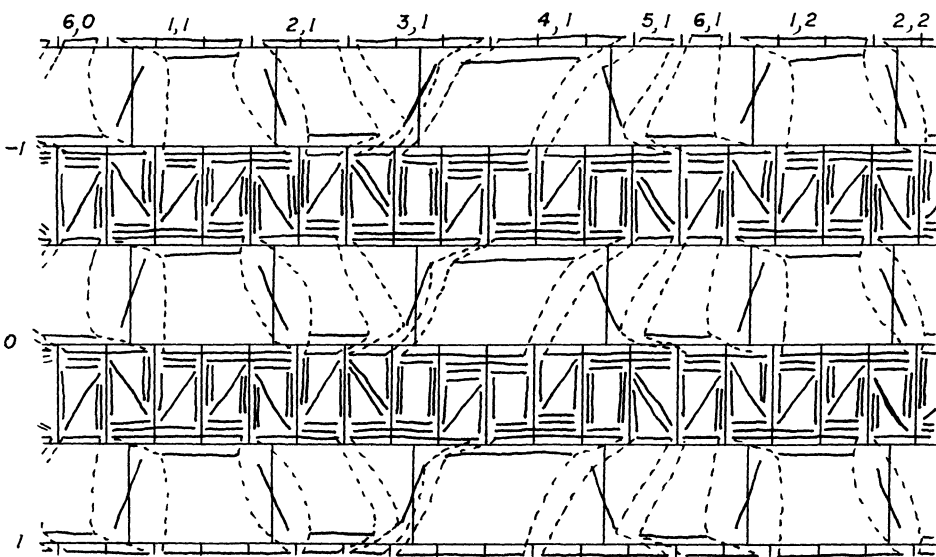

FIGURE 5a

FIGURE 5b

FIGURE 6

strips of blocks of the knot manifold, are used to describe adjacent blocks on a core, some of the knot liftings on the blocks from the knot manifold have common edges and so are part of the same spanning knot lifting. The dotted lines in Figure 6 indicate which of the knot liftings on the various copies of 5a and 5b (represented by the solid line segments) join together to form the knot liftings on Figure 6. The spanning knot liftings are labeled by pairs of numbers across the top and single numbers along the left side of the figure. The second number in each pair together with the vertical coordinates indicate which copy of Figure 5a or 5b the blocks are in. To get the pattern on the cores associates with (p, q) surgery, identify $((a, b), c)$ with $((a, b + p), c + n)$ where $n = q - lp$, where l is the linking number of a copy of the knot and the knot. For this knot $l = 4$ so $n = q - 4p$.

Assume that we have constructed a particular precover M_{i-1} for a manifold obtained by surgery on the $(4, 3, -5)$ pretzel knot. We pick out a subset of the knot liftings on the surface.

DEFINITION 3.9. A knot lifting on the surface of M_{i-1} which is on the surface of at least one block of D_b is called an *i level knot lifting*.

THEOREM 3.10. *The universal cover for any nontrivial surgery manifold M of the $(4, 3, -5)$ pretzel knot is R^3.*

Proof. We choose a nontrivial surgery and make the appropriate identifications on Figure 6. One core, with blocks attached to its boundary will be the precover M_0. Note that M_0 is just Figure 6, with identifications, and nothing else, from which we may see that all blocks are attached along disks so that M_0 is a missing boundary 3-cell. Continue inductively. Assume that M_{i-1} has been formed; M_{i-1} is a precover of the surgery manifold which is a missing boundary 3-cell; and that every core has been covered with blocks.

In order to keep track of the boundary of the various intermediate manifolds, in particular, the size of the knot liftings, we need one more device. For each i level knot lifting K there exists a collection of precovers made of blocks $\{U_j\}$ in M_{i-1} such that

(i) the blocks in a particular U_j map into the same double torus;

(ii) K meets each block of $\cup U_j$;

(iii) each U_j is a maximal (with respect to (i) and (ii)) connected set. Each U_j is represented by a portion of a strip on Figure 6 and made up of copies of Figure 5a or Figuure 5b. We call $U = \cup U_j$ the U set of a knot lifting. Notice that a knot lifting lies entirely on its U set. To pass inductively from M_{i-1} to M_i, assume that each U set in M_{i-1} has the following additional properties which limits its size, and hence the size of K:

(iv) for each U_j, there exists a block B_j in M_{i-1} (not necessarily in U_j) that maps to the same handlebody as U_j such that every $i - 1$ level core which is attached to U_j is also attached to B_j;

(v) if B_k maps to D_b and B_k is one of the blocks making up U_k, then B_k and U are attached to the same $i - 1$ level cores;

(vi) if U_j maps to D_b, B_j is a nonjunction block;

(vii) $\cup B_j$ is connected.

The B_j's are either pictured explicitly in Figure 6 or are nonjunction blocks which would be directly attached to Figure 6 along an H-disk.

Assume that any number of i level cores, along with the blocks that attach directly to them, have been attached and that the result is still a missing boundary 3-cell. We select an i level knot lifting not yet covered and attach a core to it. We need to check not only that the knot lifting is small enough so that we may successfully attach the core, but also that it is narrow enough so that attaching the core will not bring knot liftings to the left and right of the core close enough together so that attaching blocks to the remaining surface of the core will connect formerly distinct knot liftings, that is, conditions (iv)–(vii) hold on the resulting surface. We assume this has been the case for previously attached i level cores. We

now use Figure 6 to represent the point of view of this most recently attached core. The other (non $i - 1$ level) knot liftings on a U set for an i level knot lifting are also knot liftings on the surface of M_{i-1} so they are i level knot liftings provided they lie on at least one block mapping to D_b. Our last assumption above insures that the remaining knot liftings will be on the surface of M_i, and therefore $i + 1$ or higher level knot liftings. These observations are also true when Figure 6 is used to show the point of view of any previously attached core. As we check Figure 6 for the various possible U sets, it is useful to note that the largest U sets consistent with properties (iv)–(vii) have on their surface at most two spanning knot liftings with at least one block in common attached to $i - 1$ level cores. We first check to see that for any permissible U set where any or all of the other i level cores have been attached (along with the blocks which attach to the core) the resulting knot lifting is such that the core in question, and all blocks which will be attached to it, may be attached by making identification along disks. We next check that attaching blocks to the core, thereby filling the gap between the left and right side of the core, does not connect any distinct knot liftings from the previous surface. For most surgeries, this does not present a problem. For (p, q) surgeries with $|p| > 1$ the gap is clearly wider than any knot liftings lying on the blocks we add, so no knot liftings are joined together. For most values of q the two sides of the gap are not sufficiently aligned so knot liftings will not be joined together. For this particular knot, the only surgeries that lead to narrow, aligned gaps are $(1, 3)$, $(1, 4)$, and $(1, 5)$ surgery. If we are doing one of these surgeries, we carefully check Figure 6 and find that again, no knot liftings are joined together. Therefore, we may form M_i by attaching cores to $i - 1$ level knot liftings and the associated blocks (one at a time) such that each identification is along a disk and the U sets associated with the $i + 1$ level knot liftings meet conditions (iv)–(vii), hence M_i meets all the conditions of M_{i-1}.

M_0 meets all the conditions set forth, so the induction process is started. By Lemma 2.13 $M = \lim_{k \to \infty} M_k$ is a missing boundary 3-cell which is a precover of M. The boundary of M_{i-1} is in the interior of M_{i+1}, so $\tilde{M}$ has no boundary which implies that $\tilde{M}$ is homeomorphic to R^3, and that $\tilde{M}$ is the universal cover of the knot manifold M.

The above procedure works for any nontrivial surgery, so we conclude that the covering space of every nontrivial surgery manifold of this knot is homeomorphic to R^3.

THEOREM 3.11. *The universal cover of any nontrivial surgery on any pretzel knot of the type* $(4 + 2p, 3 + 2q, -5 - 2r)$ *for* p, q, r *in* Z^+ *is* R^3.
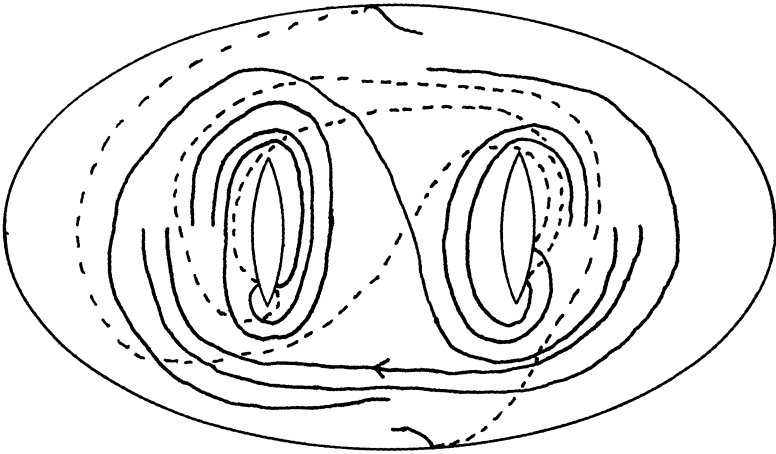
FIGURE 7

Proof. Changing the knots on D_a by full twists doesn't change the blocks at all since they may be untwisted. The effect on D_b is indicated in Figure 7, with the gaps being filled in by p loops around the left hole, q loops around both holes, and r loops around the right hole. A diagram analogous to Figure 5b may be produced from Figure 7 for any of these knots, and from that a diagram analogous to Figure 6, which we shall call the core diagram. As can be seen from Figure 7, the spanning knot liftings and a subset of the nonspanning knot liftings (including those adjacent to the spanning knot liftings) will produce the same pattern of knot liftings on the core diagram of any of these knots as that on Figure 6, if the number of blocks of D_b on which the knot liftings lie is disregarded. Each remaining knot lifting lies on a subset of the blocks of the core diagram on which one of the included nonspanning knot liftings lies, so these remaining knot liftings will not be relevant in determining the size of U sets, the level of knot liftings, etc. Therefore, if the proof of Theorem 3.10 is successful for a particular surgery on the $(4, 3, -5)$ pretzel knot, then the methods of Theorem 3.10 will be successful for surgery on any $(4 + 2p, 3 + 2q, -5 - 2r)$ pretzel knot which has the same identifications on its core diagram as would be made on Figure 6 (surgeries on the various knots do not correspond directly to each other since the linking number of copies of the knot and the knot varies).

4. Other pretzel knots (and knots in general) may be handled similarly. When each of the three braids of a pretzel knot has at least four half twists, the interplay between the various cores is similar to our example in §3 and the covering spaces may be similarly built. For pretzel

knots with fewer twists, this is not always the case. Individual cores have a stronger effect on each other which may cause the need for a more elaborate scheme for piecing together the covering space or the universal cover may not be R^3. The latter occurs for the $(p, -2, 3)$ pretzel knots where p is a positive odd integer greater than 3 $((1, -2, 3)$ and $(3, -2, 3)$ are torus knots), the case $p = 7$ being an example of Fintushel-Stern of a knot in S^3 for which two surgeries give lens spaces. When one follows the procedures of Theorem 3.10 for these knots for $(1, 2(p + 3) - 1)$ or $(1, 2(p + 3) - 2)$ surgery adding cores at certain points in the construction produce narrow, aligned gaps which, when filled in by adding blocks to the remaining surface of one of these cores connects formerly disjoint knot liftings making them large enough so that at the next level when cores are attached to these knot liftings the precover folds over on itself, forming a compact universal cover (of order $2(p + 3) - 1$ and $2(p + 3) - 2$ respectively). Not having a result similar to Lemma 2.13 for this situation, it is difficult to identify these universal covers by the methods of this paper.

Finally we note that we could easily modify Theorem 3.10 to get a version of the Cyclic Surgery Theorem of Culer-Shalen/Gordon-Luecke [3], that is, that only a very few surgeries on any given pretzel knot in S^3 have any possibility of producing a surgery manifold with finite fundamental group since the details of a Figure 6 corresponding to a particular class of pretzel knots comes into play for only a few of the surgeries.

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