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We prove that a space X is an absolute extensor for the class of all zero-dimensional spaces if and only if X is an upper semi-continuous compact-valued retract of a power of the real line.

1. Introduction. Dugundji spaces were introduced by Pelczynski [**5**]. Later Haydon [**4**] proved that the class of Dugundji spaces coincides with the class of all compact absolute extensors for zero-dimensional compact spaces (briefly, AE(0)). After Haydon's paper, compact AE(0)-spaces have been extensively studied (see Ščepin's review [**9**]); let us note the following result of Dranishnikov [**3**]: a compact X is an AE(0)-space if and only if for every embedding of X in a Tychonoff cube I^τ there exists an upper semi-continuous compact-valued (br. usco) mapping r from I^τ to X such that $r(x) = \{x\}$, for each $x \in X$ (such a usco mapping will be called a usco retraction).

Chigogidze [**2**] extended the notion of AE(0) from the class of compact spaces to that of completely regular spaces and gave a characterization of such AE(0)-spaces.

The aim of the present paper is to give another characterization of completely regular AE(0)-spaces which is similar to the above mentioned result of Dranishnikov. We prove that $X \in$ AE(0) iff X is a usco retract of R^τ for some τ, where R is the real line with the usual topology. Our technique is different from Dranishnikov's.

The author is indebted to S. Nedev and M. Ganster for useful discussions.

2. Notations and terminology. All spaces considered are completely regular and all single-valued mappings are continuous. A set-valued mapping r from X to Y is called upper semi-continuous (br. u.s.c.) if the set $r^{\#}(U) = \{x \in X\colon r(x) \subset U\}$ is open in X whenever U is open in Y. We say that a usco mapping r is minimal if every usco selection for r coincides with r. It follows from the Kuratowski-Zorn lemma that every usco mapping has a minimal usco selection.

A mapping f from Y to X, where $Y \subset Z$, is called Z-normal if, for every continuous function g on X, the function $g \circ f$ is continuously extendable to Z. A space X is called an absolute extensor for zero-dimensional spaces [**2**], if every Z-normal mapping f from Y to X, where $Y \subset Z$

and $\dim Z = 0$, is continuously extendable to Z; if f is continuously extendable only to a neighbourhood of Y in Z, the space X is called an absolute neighbourhood extensor for 0-dimensional space, briefly ANE(0). Here, dim stands for the dimension defined by finite functionally open covers.

A mapping f from X to Y will be called 0-soft [**2**], if for every 0-dimensional space Z and every two Z-normal mappings g: $Z_0 \to X$, h: $Z_1 \to Y$ with $Z_0 \subset Z_1 \subset Z$ and $f \circ g = h|Z_0$, there exists a Z-normal mapping k: $Z_1 \to X$ such that $g = k|Z_0$ and $f \circ k = h$. In the case Z is paracompact and Z_0 and Z_1 are closed subsets of Z, one gets Ščepin's notion [**8**] of a 0-soft mapping, defined earlier.

A space X is said to be a multivalued absolute (resp. neighbourhood) extensor (br. $X \in$ MA(N)E) if every Z-normal mapping f: $Z_0 \to X$ with $Z_0 \subset Z$, can be extended to a usco mapping from Z (resp. from a neighbourhood of Z_0 in Z) to X.

A mapping f: $X \to Y$ is said to be functionally open if $f(U)$ is functionally open in Y for every functionally open subset U of X.

Let A be a subset of X. We dentoe by $G_\delta(A)$ the G_δ-closure of A in X; i.e. the set $\{x \in X$: every G_δ-subset of X containing x intersects $A\}$. Finally, let $X = \prod\{X_s: s \in S\}$ and $B \subset S$. Then p_B stands for the natural projection from X onto $X_B = \prod\{X_s: s \in B\}$. If U is a subset of X, then $k(U)$ denotes the family $\{B: p_B^{-1}(p_B(U)) = U\}$.

3. AE(0)-spaces.

LEMMA 1. *Let $X = \prod\{X_s: s \in S\}$ be a product of separable metric spaces and let U be a G_δ-set in X. Then there exists a countable set $B \subset S$ such that $p_B(U)$ is a G_δ-set in X_B and $G_\delta(U) = X_{S\setminus B} \times p_B(U)$. If U is open in X then $G_\delta(U)$ is functionally open in X.*

Proof. Put $M = X \setminus G_\delta(U)$. By a result of R. Pol and E. Pol [**6**] there exists a countable set $B \subset S$ such that $p_B(U)$ is a G_δ-set in X_B and $p_B(U) \cap p_B(M) = \varnothing$. Hence $p_B^{-1}(p_B(U)) \cap M = \varnothing$. Since $p_B(G_\delta(U)) = p_B(U)$, we have $B \in k(G_\delta(U))$, so $G_\delta(U) = p_B(U) \times X_{S\setminus B}$. If U is open in X then $p_B(U)$ is functionally open in X_B. Thus, $G_\delta(U)$ is functionally open too.

The proof of the follwing (actually known) lemma is an easy exercise on the definition of a minimal usco mapping.

LEMMA 2. *Let r be a minimal usco mapping from X to Y and let U be an open set in Y. Then the following holds*:

(i) $r(x) \subset \mathrm{cl}(U)$ *for every* $x \in \mathrm{Int}(\mathrm{cl}(r^{\#}(U)))$;

(ii) $\mathrm{cl}(r^{-1}(U)) = \mathrm{cl}(r^{\#}(U))$, *where* $r^{-1}(U) = \{x \in X\colon\ r(x) \cap U \neq \varnothing\}$.

Let $Y = \prod\{Y_s\colon s \in S\}$ be a product of separable metric spaces and let $X \subset Y$. Let r be a u.s.c. mapping from Y to X. A subset B of S is called r-admissible if $B \in k(\mathrm{cl}(r^{\#}(U \cap X)))$ for every standard open subset U of Y with $B \in k(U)$. The above definition is a simple modification of the definition of e-admissible set, given by Shirokov [**11**]. The following lemma was actually proved by Shirokov [**11**].

LEMMA 3. *Let* $Y = \prod\{Y_s\colon s \in S\}$ *be a product of separable metric spaces,* $X \subset Y$ *and let* r *be a u.s.c. mapping from* Y *to* X. *Then we have*:

(i) *for every set* $B \subset S$ *there is a r-admissible set* A *containing* B *and* $\mathrm{card}\, A = \mathrm{card}\, B$;

(ii) *a union of r-admissible subsets of* S *is r-admissible too.*

LEMMA 4. *Let* $Y = \prod\{Y_s\colon s \in S\}$ *be a product of separable metric spaces,* $X \subset Y$ *and let* r *be a minimal usco mapping from* Y *to* X. *Suppose* B *is a r-admissible subset of* S. *Then the following conditions are fulfilled*:

(i) $B \in k(\mathrm{cl}(r^{\#}(\bigcup_{i=1}^{n} U_i \cap X)))$ *for every finite family* $\{U_i\colon i = 1, \ldots, n\}$ *of standard open subsets of* Y *with* $B \in \bigcap_{i=1}^{n} k(U_i)$;

(ii) $p_B(r(x)) = p_B(r(y))$ *whenever* $p_B(x) = p_B(y)$.

Proof. (i) Let $U = \bigcup_{i=1}^{n} U_i$. By Lemma 2(ii) we have

$$\mathrm{cl}\big(r^{\#}(U \cap X)\big) = \mathrm{cl}\big(r^{-1}(U \cap X)\big) = \mathrm{cl}\left(\bigcup_{i=1}^{n} r^{-1}(U_i \cap X)\right)$$

$$= \bigcup_{i=1}^{n} \mathrm{cl}\big(r^{-1}(U_i \cap X)\big) = \bigcup_{i=1}^{n} \mathrm{cl}\big(r^{\#}(U_i \cap X)\big).$$

Since B is r-admissible, $B \in k(\mathrm{cl}(r^{\#}(U_i \cap X)))$ for each i. Thus, $B \in k(\mathrm{cl}(r^{\#}(U \cap X)))$.

(ii) Let $p_B(x) = p_B(y)$ and $p_B(r(y)) \subset p_B(V)$, where V is open in Y. Since $r(y)$ is compact, V can be considered as a finite union $\bigcup_{i=1}^{n} V_i$ of standard open subsets of Y with $B \in \bigcap_{i=1}^{n} k(V_i)$. Then, by (i), we have $B \in k(\mathrm{cl}(r^{\#}(V \cap X)))$. Consequently, $B \in k(\mathrm{Int}(\mathrm{cl}(r^{\#}(V \cap X))))$. Thus, $x \in \mathrm{Int}(\mathrm{cl}(r^{\#}(V \cap X)))$ because $y \in r^{\#}(V \cap X)$. Hence, by Lemma 2(i), $r(x) \subset \mathrm{cl}(V \cap X)$ i.e. $p_B(r(x)) \subset \mathrm{cl}(p_B(V))$. The last inclusion shows that $p_B(r(x)) \subset p_B(r(y))$. Analogously, $p_B(r(y)) \subset p_B(r(x))$. Therefore $p_B(r(x)) = p_B(r(y))$.

A mapping $f\colon X \to Y$ is said to have a polish kernel [2], if there exists a polish (i.e. complete separable metric) space P such that X is C-embedded in $Y \times P$ and f coincides with the restriction $p_Y|X$, where $p_Y\colon Y \times P \to Y$ is the natural projection. The following lemma is proved by Chigogidze [2].

LEMMA 5. *Let the mapping f from X to Y have a polish kernel, where X and Y are* AE(0)*-spaces. Then f is* 0*-soft if and only if f is functionally open.*

LEMMA 6. *Let $Y = \prod\{Y_s\colon s \in S\}$ be a product of separable metric spaces and let r be a minimal usco retraction from Y to X. Then for every r-admissible set $B \subset S$ the following conditions are fulfilled:*
(i) *the restriction $p_B|X$ is functionally open;*
(ii) *$p_B(X)$ is a usco retract of Y_B.*

Proof. (i) First we prove that for every $C \subset S$ the projection p_C is functionally open. Let U be a functionally open subset of Y. Then, by Lemma 1, there exists a countable set $D \subset S$ such that $U = p_D^{-1}(p_D(U))$. This permits us to present U as a countable union $\bigcup_{i=1}^{\infty} U_i$ of standard open subsets of Y with $D \in k(U_i)$, for each i. Hence, $p_C(U) = \bigcup_{i=1}^{\infty} p_C(U_i)$. Since every $p_C(U_i)$ is a standard open subset of Y_C, the set $p_C(U)$ is a countable union of functionally open subsets of Y_C. Therefore $p_C(U)$ is functionally open.

Now, suppose B is r-admissible and U is functionally open in X. Since $G_\delta(r^{\#}(U))$ is functionally open in Y (by Lemam 1), in order to prove that $p_B|X$ is functionally open it suffices to show that $p_B(U) = p_B(G_\delta(r^{\#}(U))) \cap p_B(X)$. Let $x \in X$ and let $p_B(x) = p_B(y)$ for some $y \in G_\delta(r^{\#}(U))$. If we assume $r(y) \subset X \setminus U$ then $y \in r^{\#}(X \setminus U)$. However $r^{\#}(X \setminus U)$ is a G_δ-set in Y because $X \setminus U$ is a zero-set in X. Hence, $r^{\#}(X \setminus U) \cap r^{\#}(U) \neq \varnothing$, which is impossible. Thus, $r(y) \cap U \neq \varnothing$. By Lemma 4(ii), we have $p_B(x) = p_B(r(x)) = p_B(r(y))$, so $p_B(x) \in p_B(U)$. Therefore $p_B(G_\delta(r^{\#}(U))) \cap p_B(X) \subset p_B(U)$. The inverse inclusion is obvious.

(ii) Let B be a r-admissible set. Define a compact-valued mapping $r_1\colon Y_B \to p_B(X)$ by letting $r_1(p_B(x)) = p_B(r(x))$. Lemma 4(ii) implies that this definition is correct and that $r_1(p_B(x)) = p_B(x)$ for every $x \in X$. It remains to prove that r_1 is u.s.c. Let $r_1(p_B(x_0)) \subset U$ for some $x_0 \in Y$, where U is open in Y_B. Then, by Lemma 4(i), we have $B \in k(\mathrm{cl}(r^{\#}(p_B^{-1}(U) \cap X)))$. Consequently, $B \in k(V)$, where $V = \mathrm{Int}(\mathrm{cl}(r^{\#}(p_B^{-1}(U) \cap X)))$. The set $p_B(V)$ is a neighbourhood of $p_B(x_0)$

because $x_0 \in r^{\#}(p_B^{-1}(U) \cap X)$. Let $p_B(x) \in p_B(V)$. Then $x \in V$ and, by Lemma 2(i), $r(x) \subset \mathrm{cl}(p_B^{-1}(U) \cap X)$; so $r_1(p_B(x)) \subset \mathrm{cl}(U)$. Therefore, r_1 is u.s.c.

LEMMA 7. *Let $Y = \prod\{Y_s\colon s \in S\}$ be a product of separable metric spaces and let X be a usco retract of Y. Then the following conditions are fulfilled*:

(i) *X is C-embedded in Y*;

(ii) *there exists a set $B \subset S$ of cardinality $w(X)$ such that $p_B|X$ is a homeomorphism and $p_B(X)$ is a usco retract of Y_B.*

Proof. (i) Suppose f is a continuous function on X. Consider the family $\mathscr{L}$ of all open intervals in R with rational endpoints. Using Lemma 1, for every $U \in \mathscr{L}$ choose a countable set $B(U) \subset S$ such that $B(U) \in k(G_\delta(r^{\#}(f^{-1}(U))))$, where r is a minimal usco retraction from Y to X. It follows from Lemma 3(i) that there exists a countable r-admissible set C containing $\bigcup\{B(U)\colon U \in \mathscr{L}\}$. One can easily see that $p_C(x) = p_C(y)$ implies $f(x) = f(y)$ for every $x, y \in X$. Since $p_C|X$ is open, there exists a continuous function g on $p_C(X)$ such that $f(x) = g(p_C(x))$, for each $x \in X$. Since $p_C(X)$ is a usco retract of Y_C, it is closed in Y_C. Hence, g is continuously extendable on Y_C; so f is continuously extendable on Y.

(ii) Suppose r is a minimal usco retraction from Y to X. Let $\mathscr{Q}$ be a family of standard open subsets of Y such that $\mathrm{card}\,\mathscr{Q} = w(X)$ and $\{U \cap X\colon U \in \mathscr{Q}\}$ is a base for X. Put $B_1 = \bigcup\{m(U)\colon U \in \mathscr{Q}\}$, where $m(U) = \{s \in S\colon p_s(U) \neq Y_s\}$. Clearly, $\mathrm{card}\,B_1 = w(X)$. By Lemma 3(i), pick a r-admissible set B containing B_1 and such that $\mathrm{card}\,B = w(X)$. Observe that $p_B|X$ is one-to-one. Since $p_B|X$ is open (by Lemma 6(i), we conclude that $p_B|X$ is a homeomorphism. Next, by Lemma 6(ii), $p_B(X)$ is a usco retract of Y_B.

THEOREM 1. *For a space X, the following conditions are equivalent*:

(i) $X \in \mathrm{AE}(0)$;

(ii) $X \in \mathrm{MAE}$;

(iii) *X is a usco retract of R^A, for some A.*

Proof. (i) $\to$ (ii) Let $f\colon H \to X$ be a Z-normal mapping, where $H \subset Z$. Consider the absolute aZ of Z and the natural projection $g\colon aZ \to Z$. Put $Y = g^{-1}(H)$. Observe that $f \circ g$ is aZ-normal. Since $\dim aZ = 0$ and $X \in \mathrm{AE}(0)$, there exists an extension $h\colon aZ \to X$ of $f \circ g$. Then the usco mapping $r\colon Z \to X$, defined by $r(z) = h(g^{-1}(z))$, is an extension of f. Thus, $X \in \mathrm{MAE}$.

(ii) $\rightarrow$ (iii) Denote by $C(X)$ the family of all continuous functions on X. Consider X as a C-embedded subset of $R^{C(X)}$. Hence, there exists a usco retraction from $R^{C(X)}$ to X.

(iii) $\rightarrow$ (i) Let $\mathscr{K}$ be the class of all spaces Y with the following property: Y is a usco retract of R^A, for some A. We will prove (by transfinite induction) that every element of $\mathscr{K}$ is an AE(0)-space. Let $X \in \mathscr{K}$ and $w(X) = \aleph_0$. In this case, by Lemma 7(ii), X is a usco retract of $R^{\aleph_0}$. Hence, X is a polish space and, by a result of Chigogidze [2], $X \in$ AE(0). Assume that $\tau > \aleph_0$ and that for every $X \in \mathscr{K}$ with $w(X) < \tau$ we have $X \in$ AE(0). Consider a space $X \in \mathscr{K}$ with $w(X) = \tau$. By Lemma 7(ii), X is a usco retract of $R^\tau = \prod\{R_\alpha\colon \alpha < \omega(\tau)\}$, where $\omega(\tau)$ is the initial ordinal of cardinality τ. Let r be a minimal usco retraction from R^τ to X. By Lemma 3(i), for every $\alpha < \omega(\tau)$ there exists a countable r-admissible set B_α containing α. Next, denote $A(\alpha) = \bigcup\{B_\beta\colon \beta < \alpha\}$, $q_\alpha = p_{A(\alpha)}|X$ and $X_\alpha = q_\alpha(X)$ for each $\alpha < \omega(\tau)$. If $\alpha > \beta$ we put $p^\alpha_\beta = q_\beta \circ q_\alpha^{-1}$. Thus, we actually construct a continuous inverse system $S = \{X_\alpha, q^\alpha_\beta, \beta < \alpha < \Omega(\tau)\}$, in the sense of Ščepin [8], such that $X = \varprojlim S$. According to Lemmas 3(ii) and 6, we have that, for every $\alpha < \omega(\tau)$, $X_\alpha \in \mathscr{K}$ and q_α is functionally open. Hence, $q^{\alpha+1}_\alpha$ is functionally open. But $w(X_\alpha) < \tau$, so $X_\alpha \in$ AE(0) for each $\alpha < \omega(\tau)$. Finally, Lemma 7(i) implies that $q^{\alpha+1}_\alpha$ has a polish kernel. Therefore, it follows from Lemma 5 that $q^{\alpha+1}_\alpha$ is 0-soft for every $\alpha < \omega(\tau)$. So, all spaces X_α and all mappings $q^{\alpha+1}_\alpha$ are AE(0) and 0-soft, respectively. Therefore, $X \in$ AE(0).

LEMMA 8. *Let r be a usco mapping from M to a compact space X and let M be a dense subset of Y. Then r can be extended to a usco mapping from Y to X.*

Proof. For every $y \in Y$ denote by $U(y)$ the local base at y in Y. Then the usco mapping r_1, defined by $r_1(y) = \bigcap\{\mathrm{cl}(r(U \cap M))\colon U \in U(y)\}$, is the required extension.

LEMMA 9. *Suppose $Z = \prod\{Z_s\colon s \in S\}$ is a product of separable metric spaces and Y is closed in Z. Let r be a minimal usco mapping from Z to Y and let X be a subset of Y such that $r(x) = \{x\}$ for every $x \in X$. Then the following holds*:

(i) $r(x) = \{x\}$ *for every* $x \in G_\delta(X)$;

(ii) $r(G_\delta(M)) \subset G_\delta(H)$ *for every* $H \subset Y$ *and every* $M \subset r^{\#}(H)$.

Proof. (i) Suppose $r(x_0) \neq x_0$ for some $x_0 \in G_\delta(x)$. Take a point $y \in r(x_0) \setminus \{x_0\}$ and a countable r-admissible set $B \subset S$ such that $p_B(y) \neq p_B(x_0)$. Since $p_B^{-1}(p_B(x_0)) \cap X \neq \varnothing$, choose $x \in p_B^{-1}(p_B(x_0)) \cap X$. Lemma 4(ii) implies $p_B(x) = p_B(r(x_0))$. This is impossible because x_0, $y \in r(x_0)$ and $p_B(x_0) \neq p_B(y)$. Hence, $r(x) = \{x\}$ for every $x \in G_\delta(X)$.

(ii) Assume $H \subset Y$ and $M \subset r^{\#}(H)$. Let $r(x_0) \setminus G_\delta(H) \neq \varnothing$ for some $x_0 \in G_\delta(M)$. Take a point $y \in r(x_0) \setminus G_\delta(H)$ and a countable r-admissible set $B \subset S$ such that $p_B(y) \notin p_B(H)$. Next choose a point $x \in p_B^{-1}(p_B(x_0)) \cap M$. Then, by Lemma 4(ii), we have $p_B(r(x)) = p_B(r(x_0))$. But $r(x) \subset H$; so $p_B(r(x_0)) \subset p_B(H)$. This contradicts $p_B(y) \notin p_B(H)$. Therefore, $r(G_\delta(M)) \subset G_\delta(H)$.

THEOREM 2. *For a space X, the following conditions are equivalent*:
(i) $X \in$ ANE(0);
(ii) $X \in$ MANE;
(iii) *X is open in its Hewitt-realcompactification νX and $\nu X \in$* AE(0).

Proof. (i) → (ii) This implication can be proved as the implication (i) → (ii) of Theorem 1.

(ii) → (iii) Consider X as a C-embedded subset of R^A, where A is the family of all continuous functions on X. Clearly, $\nu X = \mathrm{cl}(X)$. Since $X \in$ MANE there exists a usco retraction r_1 from an open subset U of R^A to X. It is easily seen that $U \cap \nu X = X$ i.e. X is open in νX. Identifying R with $(0, 1)$, we consider R^A as a dense subset of I^A, where $I = [0, 1]$. Put $Y = \mathrm{cl}_{I^A}(X)$. By Lemma 8, there exists a usco extension r_2: $\mathrm{Int}_{I^A}(\mathrm{cl}_{I^A}(U)) \to Y$ of r_1. Let r_3 be a usco mapping from I^A to Y defined by letting $r_3(y) = r_2(y)$, for $y \in \mathrm{Int}_{I^A}(\mathrm{cl}_{I^A}(U))$, and $r_3(y) = Y$, otherwise. Denote by r a minimal usco selection for r_3. Since each point $z \in I^A \setminus R^A$ is contained in a G_δ-subset $H(z)$ of I^A with $H(z) \cap R^A = \varnothing$, the G_δ-closure $G_\delta(X)$ of X in I^A coincides with νX. So, by Lemma 9, r is a usco retraction from $G_\delta(U)$ to νX. Here, $G_\delta(U)$ is the G_δ-closure of U in R^A. It follows from Lemma 1 that there exists a countable set $B \subset A$ such that $G_\delta(U) = p_B(U) \times R^{A \setminus B}$. The space $p_B(U)$, being a polish space, is an AE(0). Hence, $G_\delta(U) \in$ AE(0) as a product of AE(0)-spaces. Thus, νX is a usco retract of an AE(0)-space. Therefore, by Theorem 1, $\nu X \in$ AE(0).

(iii) → (i) This implication is obvious.

COROLLARY 1. *Let* $X \in$ A(N)E(0) *and let* F *be a* G_δ*-subset of* X. *Then the* G_δ*-closure of* F *in* X *is also an* A(N)E(0)*-space.*

Proof. Let $X \in$ ANE(0). Since $\nu X \in$ AE(0) there is a minimal usco retraction r from R^A to νX for some A. The set F is G_δ in νX because X is open in νX. Hence, $r^{\#}(F)$ is a G_δ-subset of R^A. By Lemma 1, $G_\delta(r^{\#}(F))$ is a product of polish spaces, so $G_\delta(r^{\#}(F)) \in$ AE(0). Next, Lemma 9 implies that the G_δ-closure $G_\delta(F)$ of F in νX is a usco retract of $G_\delta(r^{\#}(F))$. Thus, $G_\delta(F)$ is also an AE(0)-space. But $G_\delta(F) \cap X$ is open and dense in $G_\delta(F)$. Consequently $G_\delta(F) \cap X \in$ ANE(0). However, $G_\delta(F) \cap X$ is the G_δ-closure of F in X.

By the same arguments one can prove that the G_δ-closure of F in X is an AE(0)-space if $X \in$ AE(0).

THEOREM 3. *Let* X *be a pinnate in the sense of Arhangel'skii* [**1**] ANE(0)*-space. Then* νX *is Lindelöf and Čech-complete.*

Proof. First we will prove that X is Čech-complete. Consider the Stone-Čech compactification βX of X. Denote by Z the space obtained from βX by means of making the points of $\beta X \setminus X$ isolated. We observe that X is a closed C-embedded subset of Z. Since $X \in$ ANE(0), there is a usco retraction from U to X, where U is an open set in Z containing X. Now, to prove that X is Čech-complete one can use the arguments of Przymusinski [7, the proof of Lemma 2].

Next, let r_1 be a usco mapping from R^A to νX for some A. Consider R^A as a dense subset of I^A by identifying R with $(0,1)$, and put $Y = \mathrm{cl}_{I^A}(\nu X)$. By Lemma 8, r_1 is extendable to a usco mapping r from I^A to Y. Wlog, we assume that r is minimal. Put $H = r^{\#}(X)$. H is a G_δ-subset of I^A because X is Čech-complete. Since $G_\delta(X) = \nu X$, it follows from Lemma 9 that r is a usco retraction from $G_\delta(H)$ to νX. So, νX is closed in $G_\delta(H)$. But, by Lemma 1, $G_\delta(H)$ is a Lindelöf G_δ-subset of I^A. Therefore, νX is Lindelöf and Čech-complete.

COROLLARY 2. *Every pinnate* AE(0)*-space is Lindelöf and Čech-complete.*

An embedding j of X in Y is said to be d-regular [**11**] (br. a d-embedding) if for every open subset U of $j(X)$ there exists an open subset $e(U)$ of Y such that the following conditions are fulfilled:

(1) $e(\varnothing) = \varnothing$;

(2) $e(U) \cap j(X) = U$;

(3) $e(U) \cap e(V) = e(U \cap V)$;

Shirokov [**11**] proved that X is a Dugundji space if and only if every embedding of X in a Tychonoff cube is a d-embedding. We give a similar characterization of Čech-complete AE(0)-spaces.

THEOREM 4. *For a Čech-complete space X the following conditions are equivalent*:

(i) *νX is a Čech-complete Lindelöf* AE(0)-*space*;

(ii) *every C-embedding of X in any space is a d-embedding*;

(iii) *X is a d-embedded subset of R^A, for some A.*

Proof. (i) → (ii) Suppose X is a C-embedded subsert of a space Y. Then there exists a mapping $h\colon Y \to R^{C(X)}$ such that $h|X$ is a homeomorphism and $\mathrm{cl}_{R^{C(X)}}(h(X)) = \nu X$. Let r be a usco retraction from $R^{C(X)}$ to νX. For every open set U in X, we let $e(U) = h^{-1}(r^{\#}(V(U)))$, where $V(U) = \bigcup\{W\colon W$ is open in νX and $W \cap h(X) = h(U)\}$. It is easily seen that this operator satisfies the above three conditions. Thus, X is d-embedded in Y.

(ii) → (iii) This implication is obvious.

(iii) → (i) Let X be a d-embedded subset of R^A for some A. So, there exists a d-regular operator e from the topology of X to the topology of R^A. Consider R^A as a dense subset of I^A and put $Y = \mathrm{cl}_{I^A}(X)$. Define a usco mapping r_1 from R^A to Y by letting $r_1(x) = \bigcap\{\mathrm{cl}_Y(U)\colon x \in e(U)\}$, for $x \in \bigcup\{e(U)\colon U$ is open in $X\}$, and $r_1(x) = Y$, otherwise. Clearly, $r_1(x) = \{x\}$ for every $x \in X$. Next, by Lemma 8, r_1 is extendable to a usco mapping r from I^A to Y. We assume that r is minimal. Since X is Čech-complete, the set $H = r^{\#}(X)$ is G_δ in I^A. Lemma 9 implies that r is a usco retraction from $G_\delta(H)$ to $G_\delta(X)$. By Lemma 1, $G_\delta(H)$ is a Lindelöf Čech-complete AE(0)-space. Therefore, $G_\delta(X)$ being a usco retract of $G_\delta(H)$, is a Lindelöf Čech-complete AE(0)-space too. It remains to prove that $G_\delta(X)$ is the Hewitt-realcompactification of X. It is known [**2**] that every AE(0)-space is perfectly k-normal in the space of Ščepin [**10**] and that every G_δ-dense subset of a perfectly k-normal space Z is C-embedded in Z [**12**]. Hence, X is C-embedded in $G_\delta(X)$. Therefore, $G_\delta(X)$ is the Hewitt-realcompactification of X.

COROLLARY 3. *For a Čech-complete realcompact space X the following conditions are equivalent*:

(i) *X is a Lindelöf* AE(0)-*space*;

(ii) *every C-embedding of X in any space is a d-embedding*;

(iii) *X is a d-embedded subset of R^A, for some A.*

Let us note that the completeness in Theorem and Corollary 3 is essential. Indeed, every non-complete subspace of $R^{\aleph_0}$ is d-embedded in $R^{\aleph_0}$ but is not an AE(0)-space.

We have been unable to decide the following problems: Is every Lindelöf AE(0)-space Čech-complete? Is every normal AE(0)-space Lindelöf?

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