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DOUGLAS MURRAY PICKRELL

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Let G denote the infinite dimensional group consisting of all unitary operators which are compact perturbations of the identity (on a fixed separable Hilbert space). Kirillov showed that G has a discrete spectrum (as a compact group does). The point of this paper is to show that there are analogues of the Peter-Weyl theorem and Frobenius reciprocity for G. For the left regular representation, the only reasonable candidate for Haar measure is a Gaussian measure. The corresponding L^2 decomposition is analogous to that for a compact group. If X is a flag homogeneous space for G, then there is a unique invariant probability measure on (a completion of) X. Frobenius reciprocity holds, for our surrogate Haar measure fibers over X precisely as in finite dimensions (this is the key observation of the paper). When X is a symmetric space, each irreducible summand contains a unique invariant direction, and this direction is the L^2 limit of the corresponding (L^2 normalized) finite dimensional spherical functions.

1. Introduction. Let H be a separable complex Hilbert space, $U(H)_\infty = \{g \in U(H): g = 1 + \text{compact operator}\}$. This group is a basic example of an infinite dimensional Banach Lie group. Kirillov proved that this group is type 1 and has a discrete spectrum ([**4**], [**6**]).

Fix an orthonormal basis $e_1, e_2, \ldots$ for H. Then $U(H)_\infty$ is the closure in the operator norm topology of $U(\infty) = \bigcup_n U(n)$, where $U(n) \cong \{g \in U(H): ge_j = e_j,\ j > n\}$.

Relative to this basis, view $U(H) \to M$, where M is the space of matrices $(E_{ij})_{1 \le i, j < \infty}$, and which we identity with the space of linear operators mapping H^{alg}, the algebraic span of the $\{e_j\}$, to $\mathbf{C}^\infty$, the space of all formal linear combinations of the $\{e_j\}$. The left action of $U(\infty)$ on $U(H)_\infty$ extends to an action of $U(\infty)$ on M.

Let ν_G denote the Gaussian measure for the linear space $\mathscr{L}_2(H)$. We recall the following facts established in [**8**]: (a) every ergodic invariant probability measure for the left action of $U(\infty)$ on M is a linear equivariant image of ν_G (and itself Gaussian), (b) ν_G is the weak limit of the uniform distributions on the spaces $\sqrt{n}\, U(n)$, and (c) up to scaling ν_G is the only $U(\infty)$ ergodic biinvariant measure on M. For these reasons it is natural to view ν_G as a kind of Haar measure for $U(H)_\infty$, relative to its left regular action. In this paper we will exploit the existence of this Haar type measure to decompose various regular representations of $U(H)_\infty$.

Of course the first step is to decompose the representation

$$U(H)_\infty \to U\big(L^2(M, d\nu_G)\big).$$

This is done in §2, and the decomposition is analogous to the Peter-Weyl decomposition for a compact group.

In §3 we use the Peter-Weyl decomposition to decompose the regular representations for $U(H)_\infty$ on homogeneous spaces (flag manifolds) ((3.2)). The key idea can be described in terms of the simplest example. Via the basis above view $H \cong l^2 \to \mathbf{C}^\infty$. The natural projection $\pi\colon \mathbf{C}^\infty \setminus \{0\} \to \mathbf{P}(\mathbf{C}^\infty)$ is $U(\infty)$ equivariant, and it pushes the Gaussian measure for H to the unique $U(\infty)$ invariant probability measure on $\mathbf{P}(\mathbf{C}^\infty)$. Now it is frequently said that Gaussian measure behaves as a uniform distribution on a sphere of infinite radius. In particular we should expect

$$L^2(\mathbf{P}(\mathbf{C}^\infty)) \cong L^2(\mathbf{C}^\infty)^{U(1)}, \tag{1.1}$$

where the right hand side denotes those functions invariant under the scalar action of $U(1)$. This is correct. The key ((3.8)) is to fiber the Gaussian over the invariant measure on projective space; the fiber is the Haar measure for the unitary stabilizer (in general), in this case $U(1)$. The right hand side of (1.1) is easy to understand because of the Peter-Weyl decomposition, and this leads to Frobenius reciprocity.

In §4 we consider the special case of a symmetric space, i.e. a Grassmann manifold $\mathrm{Gr}(n, H)$. In this case the decomposition is multiplicity free. Each irreducible component contains a unique invariant direction for the isotropy group, and this direction is the L^2 limit of the corresponding (L^2 normalized) finite dimensional spherical functions.

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Notation. $dm(\cdot)$ denotes Lebesgue measure, $\mathscr{P}(\cdot)$ the polynomial algebra. If π_i is a representation for G_i, then $\pi_1 \times \pi_2$ is the (outer) tensor product representation for $G_1 \times G_2$. If $G_1 = G_2$, $\pi_1 \otimes \pi_2$ is the usual tensor product representation for $G_1 = G_2$.

2. Peter-Weyl theorem. In this section it will be convenient to view ν_G as a cylinder measure (i.e. weak distribution) on $\mathscr{L}_2(H)$ (see **[5]** or **[9]**). A function on $\mathscr{L}_2(H)$ of the form $\phi(E) = \Phi(P(E))$, where P is an orthogonal projection of rank $n < \infty$ and Φ is a bounded Borel function, will be called tame; we let $\mathscr{V}$ denote the algebra of all tame functions. If we set

$$E(\phi) = \int \phi\, d\nu_G = \int_{\mathscr{R}(P)} \Phi(x)\pi^{-n}e^{-|x|^2}\, dm(x)$$

where ϕ is as above, then $(\mathscr{V}, E)$ is an integration algebra. There is a natural representation of $O(\mathscr{L}_2(H))$ as automorphisms of $(\mathscr{V}, E)$, hence a unitary representation on $L^2(\nu_G)$, the completion of $\mathscr{V}$ in the norm $E(\bar{\phi}\phi)$.

We view $U(H) \times U(H) \subset O(\mathscr{L}_2(H))$ by $g \times h \cdot E = g \circ E \circ h^{-1}$. Our goal in this section is to decompose the action of $U(H) \times U(H)$ (and $U(H)_\infty \times U(H)_\infty$) on $L^2(\nu_G)$. Of course there is a natural $U(\infty) \times U(\infty)$ equivariant isomorphism of $L^2(\nu_G)$, as constructed above, and $L^2(M, \nu_G)$, when we view ν_G as a probability measure on M.

Let $\mathscr{T}$ denote the transform defined by

$$(\mathscr{T}\phi)(w) = \int \phi(L)e^{iRe(w,E)}\, d\nu_G(E)$$

for $\phi \in \mathscr{V}$ and $w \in \mathscr{L}_2(H)^*$. By the corollary of Theorem 6.4 of [**2**], $\mathscr{T}$ extends to a $U(H) \times U(H)$ equivariant isomorphism

$$L^2(\nu_G) \cong \mathbf{C}e^{-1/4|w|^2} \otimes \sum_{j=0}^{\infty} \hat{\mathscr{P}}^j \otimes \sum_{k=0}^{\infty} \bar{\hat{\mathscr{P}}}^k \tag{2.1}$$

where $\hat{\mathscr{P}}^j$ is $(j!)^{1/2}$ times the completion of $\mathscr{P}^j(\mathscr{L}_2(H))$ in the norm it inherits from the tensor algebra.

Suppose λ is a partition, i.e. a decreasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $\lambda_j = 0$ for all sufficiently large j. If $\lambda_{n+1} = 0$, then we denote by $\rho_{\lambda,n}$ the representation of $U(\mathbf{C}^n)$ with signature $(\lambda_1 \geq \cdots \geq \lambda_n)$, by ρ_λ the direct limit of the $\rho_{\lambda,n}$, which extends canonically from $U(\infty)$ to a representation of $U(H)$.

In the proof of (3.1) of [**8**] it is shown that as a representation of $U(H)_\infty \times U(H)_\infty$ (or $U(H) \times U(H)$)

$$\hat{\mathscr{P}}^j(\mathscr{L}_2(H)) = \sum \rho_\lambda^* \times \rho_\lambda \tag{2.2}$$

where the sum is over all partitions λ with $\sum\lambda_i = j$. This proves the following

(2.3) Proposition. *As a representation of $U(H) \times U(H)$ or $U(H)_\infty \times U(H)_\infty$,*

$$L^2(\nu_G) = \sum \rho_\lambda^* \otimes \rho_\mu \times \rho_\lambda \otimes \rho_\mu^*$$

where the sum is over all partitions λ, μ.

By Kirillov's classification ([4]) of the irreducible representations of $U(H)_\infty$, this decomposition is analogous to the Peter-Weyl theorem for compact groups.

The physical space for $\rho_\lambda^* \times \rho_\lambda$, as a subrepresentation of (2.2), consists of matrix coefficients for ρ_λ (ρ_λ is a subrepresentation of the action of $U(H)$ on the tensor algebra of H, so this action extends naturally to an action of $GL(H)$; the matrix coefficients restrict to polynomials on $\mathscr{L}_2(H)$). Hence the physical space for $\rho_\lambda^* \otimes \rho_\mu \times \rho_\lambda \otimes \rho_\mu^*$, as a subrepresentation of the right hand side of (2.1), consists of matrix coefficients for the action of $\mathrm{GL}(H)$ on $\mathscr{L}_2(H_\lambda, H_\mu)$ given by $g\colon T \to \rho_\mu(g) \circ T \circ \rho_\lambda(g)^*$ (where ρ_λ is realized on H_λ).

To describe the corresponding subspace in $L^2(\nu_G)$, one must invert the transform $\mathscr{T}$. Whether this can be done in a reasonably explicit manner in general, I do not know. In the case of spherical functions, there does exist a relatively simple inversion formula (see §4, especially (4.6)).

3. Frobenius reciprocity. In this section we fix a finite set of integers $0 < n_1 < n_2 < \cdots < n_l < \infty$. Let $\mathrm{Flag}(H) \subset \mathrm{Gr}(n_1, H) \times \cdots \times \mathrm{Gr}(n_l, H)$ denote the set of points (flags) $\{W_i\}$ such that $W_1 \subset W_2 \subset \cdots \subset W_l$, where $\mathrm{Gr}(n_i, H)$ denotes the set of all n_i dimensional subspaces of H. $\mathrm{Flag}(H)$ is a homogeneous space for $U(H)$ and $U(H)_\infty$. We let $\mathrm{Flag}(\mathbf{C}^N)$ and $\mathrm{Flag}(\mathbf{C}^\infty)$ denote the analogous objects for $\mathbf{C}^N \cong \mathrm{span}\{e_j\colon j \leq N\}$ $(N > n_l)$ and $\mathbf{C}^\infty \cong \{$formal linear combinations of $e_j\}$. The action of $U(\infty)$ extends from $\mathrm{Flag}(H)$ to $\mathrm{Flag}(\mathbf{C}^\infty)$, and there are natural embeddings $\mathrm{GL}(N) \to \mathrm{GL}(H)$, $\mathrm{Flag}(\mathbf{C}^N) \to \mathrm{Flag}(\mathbf{C}^\infty)$.

Our first task is to recall why there is a unique $U(\infty)$ invariant probability measure on $\mathrm{Flag}(\mathbf{C}^\infty)$. In the process we will develop notation which we will employ in the remainder of the paper.

A generic flag (i.e. a point in the largest cell) of $\mathrm{Flag}(\mathbf{C}^\infty)$ can be characterized in two ways: (a) it is of the form $\{W_j\} = \{L\mathbf{C}^{n_j}\}$, where L is a lower triangular block matrix with identity matrices on the diagonal, the block sizes being $n_1, n_2 - n_1, \ldots, n_l - n_{l-1}$ along the top, $n_1, \ldots, n_l - n_{l-1}, \infty$ along the side; (b) each W_j is of the form $\mathrm{graph}(z_j)$, where $z_j \in \mathscr{L}(\mathbf{C}^{n_j}, \mathbf{C}^\infty \ominus \mathbf{C}^{n_j})$. The operator L and the set $\{z_j\}$ determine one another via the relations

$$\gamma_j \alpha_j^{-1} = z_j \quad \text{where } L = \begin{pmatrix} \alpha_j & * \\ \gamma_j & * \end{pmatrix}$$

with respect to the splittings of the domain $= \mathbf{C}^{n_j} \oplus (\mathbf{C}^{n_l} \ominus \mathbf{C}^{n_j})$ and range $= \mathbf{C}^{n_j} \oplus (\mathbf{C}^\infty \ominus \mathbf{C}^{n_j})$.

Let $L^{(N)}$ denote the projection of L to $\mathscr{L}(\mathbf{C}^{n_l}, \mathbf{C}^N)$ ($L^{(N)} = Q \circ L$, where Q: $\mathbf{C}^\infty \to \mathbf{C}^N$ is the obvious projection) (similarly for z). The diagram

$$\begin{array}{ccc} L & \rightarrow & L^{(N)} \\ \downarrow & & \downarrow \\ \{z_j\} & \rightarrow & \left\{z_j^{(N)}\right\} \end{array} \tag{3.1}$$

is commutative (the cutoff is on the left, whereas the α's act from the right). In §4 of **[8]** it is shown how $U(N)$ equivariance of the map $Z_j \to Z_j^{(N)}$ implies uniqueness for the $U(\infty)$ invariant probability measure on $\mathrm{Gr}(n_j, \mathbf{C}^\infty)$. The above diagram shows the same argument applies to flags.

Conversely, the projection

$$\pi\colon \mathscr{L}(\mathbf{C}^{n_l}, \mathbf{C}^\infty)' \to \mathrm{Flag}(\mathbf{C}^\infty)\colon E \to \{E(\mathbf{C}^{n_j})\},$$

where the prime indicates we exclude those E which are singular, is $U(\infty)$ equivariant. Thus the Gaussian measure associated to the linear space $\mathscr{L}(\mathbf{C}^{n_l}, H)$ will be mapped by π to a $U(\infty)$ invariant probability measure on $\mathrm{Flag}(\mathbf{C}^\infty)$. This proves existence.

Let μ_0 denote the unique invariant measure on $\mathrm{Flag}(\mathbf{C}^\infty)$. Our task is to decompose $L^2(\mathrm{Flag}(C^\infty))$.

Let $K_l = \times_1^l U(\mathbf{C}^{n_j} \ominus \mathbf{C}^{n_{j-1}})$ and $K = K_l \times U(H \ominus \mathbf{C}^{n_l})$. Let P: $M \to \mathscr{L}(\mathbf{C}^{n_l}, \mathbf{C}^\infty)$ denote the obvious projection, and $\nu = P_* \nu_G$. In this section we will ultimately prove the following

(3.2) Proposition. *The pullbacks*

$$L^2(\mathrm{Flag}(\mathbf{C}^\infty)) \xrightarrow{\pi^*} L^2(\mathscr{L}(\mathbf{C}^{n_l}, \mathbf{C}^\infty), \nu)^{K_l} \xrightarrow{P_*} L^2(M, \nu_G)^K$$

are isomorphisms, where the superscripts indicate the sets of vectors invariant under the right action of K_l and K, respectively. As a representation of $U(H)_\infty$

$$L^2(\mathrm{Flag}(\mathbf{C}^\infty)) = \sum m(\lambda, \mu) \rho_\lambda \otimes \rho_\mu^*$$

where the sum is over those partitions with $\lambda_{n_l+1} = \mu_{n_l+1} = 0$ and

$$m(\lambda, \mu) = \mathrm{dimn}\left(\left(\rho_\lambda^* \otimes \rho_\mu\right)^K\right) = \mathrm{dimn}\left(\left(\rho_{\lambda, n_l}^* \otimes \rho_{\mu, n_l}\right)^{K_l}\right).$$

The proof of the analogue of this proposition for a compact group is trivial, because of the existence of Haar measure. Our proof will be trivial as well, once we understand how ν is fibered over μ_0 (see (3.8) below). We will prove (3.2) at the end of this section.

We will need the following computational lemma.

For $E \in \mathscr{L}(\mathbf{C}^r, \mathbf{C}^\infty)$, recall that $E^{(N)}$ is the projection of E to $\mathscr{L}(\mathbf{C}^r, \mathbf{C}^N)$.

(3.3) LEMMA. *The scalar* $(\det N^{-1}E^{(N)*}E^{(N)})^{-1}$ *converges to* 1 *and the* $r \times r$ *matrices* $N^{-1}E^{(N)*}E^{(N)}$ *and their inverses converge to the identity in* $L^p(\mathscr{L}(\mathbf{C}^r, \mathbf{C}^\infty), d\nu)$ *as* $N \to \infty$, *for all* $1 \leq p < \infty$.

Proof. First consider $N^{-1}E^{(N)*}E^{(N)}$. For the diagonal entries

$$\int \left|N^{-1}(E^{(N)*}E^{(N)})_{jj} - 1\right|^p d\nu(E)$$

$$= \int_{\mathbf{C}^N} \left|N^{-1}|x|^2 - 1\right|^p \pi^{-N} e^{-|x|^2}\, dm(x)$$

$$= \sum_0^p \binom{p}{k} (-1)^{p-k} N^{-k} \int_0^\infty s^{N+k+1} e^{-s}\, ds \Big/ \int_0^\infty s^{N-1} e^{-s}\, ds$$

$$= \sum_0^p \binom{p}{k} (-1)^{p-k} N^{-k} (N)_k$$

and this tends to 0.

For off diagonal entries

$$\int \left|N^{-1}(E^{(N)*}E^{(N)})_{ij}\right|^{2p} d\nu(E)$$

$$= \int_{\mathbf{C}^N + \mathbf{C}^N} \left|N^{-1} \sum_1^N x_j \bar{y}_j\right|^{2p} \pi^{2N} e^{-|x|^2 - |y|^2}\, dm(x)\, dm(y)$$

$$= N^{-2p} \int_{\substack{1 \leq i_k, j_k \leq N \\ 1 \leq k \leq p}} \sum_{l=1}^p x_{i_l} \bar{x}_{j_l} y_{i_l} \bar{y}_{j_l}$$

$$= N^{-2p} \int_{\substack{1 \leq i_k \leq N \\ 1 \leq k \leq p}} \left(\int_{\mathbf{C}} |z|^2 \pi^{-1} e^{-|z|^2}\, dm(z) \right)^2 = N^{-2p} N^p$$

which tends to zero.

We now consider

$$\int \left|(\det N^{-1}E^{(N)*}E^{(N)})^{-1} - 1\right|^{2p} d\nu(E). \tag{3.4}$$

We use the integral formula

$$\int \phi(E^{(N)})\,d\nu(E)$$

$$= c\int_{(\mathbf{R}^+)^r}\left\{\int \phi(k_1\lambda k_2)\,dk_1\,dk_2\right\}\prod_{i<j}\left|\lambda_i^2-\lambda_j^2\right|^2\prod_1^r 2\lambda_j\lambda_j^{2(N-r)}e^{-\lambda_j^2}\,d\lambda_j$$

where $\lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_r)$, dk_1 and dk_2 denote the unitarily invariant probability measures on $\mathrm{Isom}(\mathbf{C}^r,\mathbf{C}^N)$ and $U(\mathbf{C}^r)$, respectively, and c is a normalization constant (see Chapter I of [1]). We now see (3.4) equals

$$c\int\left|\prod_1^r Nu_j^{-1}-1\right|^{2p}\prod_{i<j}(u_i-u_j)^2\prod_1^r u_j^{N-r}e^{-u_j}\,du_j$$

$$=\sum_0^{2p}\binom{2p}{k}(-1)^{2p-k}cN^{rk}\int\prod_{i<j}(u_i-u_j)^2\prod_1^r u_j^{N-r-k}e^{-u_j}\,du_j.$$

Let $s = N-r-k$. The kth integral equals

$$cN^{rk}\int \det{}^2\left(\mathscr{L}_i^{(s)}(u_j)\right)\prod_1^r u_j^s e^{-u_j}\,du_j$$

$$= cN^{rk}\det\left(\int \mathscr{L}_i^{(s)}(u)\mathscr{L}_j^{(s)}(u)u^s e^{-u}\,du\right)$$

$$= N^{rk}\prod_1^r\left(\int \mathscr{L}_i^{(s)2}u^s e^{-u}\,du\Big/\int \mathscr{L}_i^{(N-r)2}u^{N-r}e^{-u}\,du\right)$$

where the $\mathscr{L}_i^{(s)}$ are the Laguerre polynomials. The lemma now follows from

$$\int\left|\mathscr{L}_i^{(s)}\right|^2 u^s e^{-u}\,du = \Gamma(s+1)\binom{s+i}{i}$$

(see Chapter 5 of **[10]**). □

(3.5) LEMMA. *For a generic flag* $\{W_j\} = \{L\mathbf{C}^{n_j}\}$ *in* Flag($\mathbf{C}^\infty$), *let* $g^{(N)}(L)$ *be the isometry from* $\mathbf{C}^{n_l}$ *to* $\mathbf{C}^N$ *obtained by applying the (block) Gram-Schmidt orthonormalization process to* $L^{(N)}$. *Then entry by entry* $N^{1/2}g_N(L)$ *has a limit in probability* $g(L)(\in\mathscr{L}(\mathbf{C}^{n_l},\mathbf{C}^\infty))$. *In the case* $l=1$, *we actually have* $L^p(\mu_0)$ *convergence, for each* $1\le p<\infty$.

REMARK. It is almost certainly the case that the limit above is $L^p(\mu_0)$ in general. However, for $l>1$ this seems to complicate the proof immensely. The reason is essentially that the function $\{W_j\}\to L$, which is well-defined a.e. $[\mu_0]$, does not have integrable entries. It would be desirable to establish L^p convergence, because this would yield a second proof of (3.8) below (see the remark following the proof of (3.8)). The meaning of the convergence when $l = 1$ is explored in the next section.

Proof of (3.5). Let $E \in \mathscr{L}(\mathbf{C}^{n_l}, \mathbf{C}^{\infty})$, $E \cdot \mathbf{C}^{n_j} = W_j$, so that $E = LU$ where U is (block) upper triangular. Note $E^{(N)} = L^{(N)}U$. Write $E = [E_1, \dots, E_l]$, where the E_j are the columns (similarly for L, etc.).

Let $\alpha = \alpha_1(E)$. Then, as a function of E,

$$g_1^{(N)} = L_1^{(N)}\left|L_1^{(N)}\right|^{-1} = E_1^{(N)}\alpha^{-1}\left(\alpha\left(E_1^{(N)*}E_1^{(N)}\right)^{-1}\alpha^*\right)^{1/2}.$$

The entries of E_1 are in all $L^p(\nu)$, and

$$\operatorname{tr}\left|\alpha^{-1}\left(\alpha\left(E_1^{(N)*}E_1^{(N)}\right)^{-1}\alpha^*\right)^{1/2}\right|^2 = \operatorname{tr}\left(E_1^{(N)*}E_1^{(N)}\right)^{-1}.$$

By (3.3) we have L^p convergence

$$N^{1/2}g_1^{(N)} \to L_1\left(\alpha_1(E)\alpha_1(E)^*\right)^{1/2}$$

entry by entry as $N \to \infty$ (note the existence of the limit shows the RHS is equal to a function of L, a.e. $[\nu]$).

Now suppose we have established that $g_i^{(N)}$ has a limit g_i in probability for $1 \le i \le j$. We have

$$g_j^{(N)} = \left(1 - \sum_{i<j} g_i^{(N)}g_i^{(N)*}\right)L_j^{(N)}\left|\left(1 - \sum g_i^{(N)}g_i^{(N)*}\right)L_j^{(N)}\right|^{-1} \tag{3.6}$$

(here 1 is the $N \times N$ identity matrix).

Consider the $N \times (n_j - n_{j-1})$ matrix $g_i^{(N)}g_i^{(N)*}L_j^{(N)}$. The Hilbert-Schmidt norm is dominated by

$$\left(\operatorname{tr}\left(g_i^{(N)*}g_i^{(N)}\right)^2\right)^{1/2}\left(\operatorname{tr} L_j^{(N)*}L_j^{(N)}\right)^{1/2}. \tag{3.7}$$

By induction the first factor is $O(N^{-1})$ in probability. On the other hand $L_j^{(N)*}L_j^{(N)}$ is the (j, j) (block) entry of $U^{-1*}E^{(N)*}E^{(N)}U^{-1}$, which is $O(N)$ by (3.3). Therefore (3.7) is $O(N^{-1/2})$ in probability. So we certainly have $(1 - \Sigma_{i<j} g_i^{(N)}g_i^{(N)*})L_j^{(N)} \to L_j$ in probability, entry by entry.

Now let $\phi_N = N^{-1/2}L_j^{(N)}$, $\psi_N = -N^{-1/2}\Sigma_{i<j} g_i^{(N)}g_i^{(N)*}L_j^{(N)}$. We know that $\phi_N^*\phi_N \to ((UU^*)^{-1})_{jj}$ and $\psi_N^*\psi_N \to 0$ in probability. The generalized Holder inequality

$$\operatorname{tr}\left|\psi_N^*\phi_N\right| \le \left(\operatorname{tr}\left|\psi_N^*\right|^2\right)^{1/2}\left(\operatorname{tr}\left|\phi_N\right|^2\right)^{1/2}$$

shows that $\psi_N^*\phi_N$ and $\phi_N^*\psi_N$ tend to zero in probability as well. This implies that $|\phi_N + \psi_N|^2 \to ((UU^*)^{-1})_{jj}$, which is strictly positive. This implies

$$N^{1/2}\left|\left(1 - \sum g_i^{(N)}g_i^{(N)*}\right)L_j^{(N)}\right|^{-1} = \left|\phi_N + \psi_N\right|^{-1}$$

has a limit in probability. Hence (3.6), scaled by $N^{1/2}$, has a limit in probability, entry by entry. This completes the induction. $\square$

(3.8) PROPOSITION. *The decomposition of the Gaussian measure ν with respect to the projection π: $\mathscr{L}(\mathbf{C}^{n_l}, \mathbf{C}^\infty)' \to \mathrm{Flag}(\mathbf{C}^\infty)$ is given by*

$$\int \phi\, d\nu = \int_{\mathrm{Flag}(\mathbf{C}^\infty)} \int_{K_l} \phi(g(w)k)\, dk\, d\mu_0(w).$$

Proof. Assume ϕ is a bounded continuous function based on $\mathscr{L}(\mathbf{C}^{n_l}, \mathbf{C}^m)$. For $N > m$, we have

$$\begin{aligned}\int_{\mathrm{Isom}(\mathbf{C}^{n_l}, \mathbf{C}^N)} & \phi(N^{1/2}E)\, d\omega_N(E) \\ &= \int_{\mathrm{Flag}(\mathbf{C}^N)} \int_{K_l} \phi\left(N^{1/2} g_N(W)k\right) dk\, d\mu_{0,N}(W) \\ &= \int_{\mathrm{Flag}(\mathbf{C}^\infty)} \int_{K_l} \phi\left(N^{1/2} g_N(W)k\right) dk\, d\mu_0(W)\end{aligned}$$

where $\omega_N(\mu_{0,N})$ denotes the unique invariant probability measure for $U(N)$. Take the limit as $N \to \infty$. By (2.1) of [**8**] the LHS converges to the LHS of (3.8). By (3.5) the RHS converges to the RHS of (3.8). This proves (3.8). $\square$

(3.9) REMARK. It is possible to give a more direct, but formal, argument for (3.8) as follows.

First, via direct calculation, we fiber the Gaussian on $\mathscr{L}(\mathbf{C}^{n_l}, \mathbf{C}^N)$ over $\mu_{0,N}$ on $\mathrm{Flag}(\mathbf{C}^N)$ (we let $L = L^{(N)}$).

$$\text{(3.10)} \qquad \int_{\mathscr{L}(\mathbf{C}^{n_l}, \mathbf{C}^N)} \phi\, d\nu(E) = c \int \phi(LU) e^{-\mathrm{tr}|LU|^2}\, dm(LU).$$

Now $dm(LU) = \prod_1^l (\det|U_{jj}|^2)^{N-n_j}\, dm(L)\, dm(U)$. To separate the L and U variables in the exponential in (3.10), we (block) orthonormalize L, which amounts to multiplying L on the right by a (block) upper triangular matrix, and then we change the U variable:

$$U = L^{-1} g_N(L) V, \quad U_{jj} = \left(L^{-1} g_N(L)\right)_{jj} V_{jj},$$

$$dm(U) = \prod_1^l \left(\det\left|\left(L^{-1} g_N(L)\right)_{jj}\right|^2\right)^{n_l - n_{j-1}} dm(V) \qquad (n_0 = 0).$$

This implies (3.10) equals

$$\int_{\mathrm{Flag}(\mathbf{C}^N)} \int \phi(g_N(W)V)\, dv_{0,N}(V)\, d\mu_{0,N}(W)$$

where

$$dv_{0,N}(V) = c\prod_1^l \left(\det|V_{jj}|^2\right)^{N-n_j} e^{-\operatorname{tr} V^*V} dm(V)$$

is a probability measure on the (block) upper triangular matrices. The formula (3.8) then formally follows from the fact that

(i) $ce^{-\operatorname{tr} N|V_{ij}|^2} dm(N^{1/2}V_{ij}) \to \delta_0$ as $N \to \infty$ for $1 \le i < j \le l$, and

(ii) $c(\det N|V_{jj}|^2)^{N-n_j} e^{-\operatorname{tr} N|V_{jj}|^2} dm(N^{1/2}V_{jj}) \to dk_j$, the Haar invariant probability measure on $U(\mathbf{C}^{n_j} \ominus \mathbf{C}^{n_{j-1}})$, which can be verified using the integral formulae in the proof of (3.3).

Proof of (3.2). We first consider P^*. We have $L^2(\nu_G) = L^2(\nu) yx L^2(\nu^\perp)$, where $\nu^\perp = (1-P)_*\nu$. Thus

$$L^2(\nu_G)^{U(H\ominus\mathbf{C}^{n_l})} = L^2(\nu) \otimes L^2(\nu^\perp)^{U(H\ominus\mathbf{C}^{n_l})} = L^2(\nu).$$

This shows P^* induces an isomorphism

The fact π^* induces an isomorphism follows immediately from (3.8).

(2.2) implies the claims about the multiplicity. □

4. Symmetric space. In this section we consider the special case of a Grassmannian, $\mathrm{Gr}(n, \mathbf{C}^\infty)$. Recall that if z is the graph coordinate, the map

$$\mathrm{Gr}(n, \mathbf{C}^\infty) \to \mathrm{Gr}(n, \mathbf{C}^N)\colon z \to z^{(N)}, \tag{4.1}$$

which is defined almost everywhere, is $U(\mathbf{C}^N)$ equivariant. This is equivalent to saying that the pullback defines a $U(\mathbf{C}^N)$ equivariant isometric map

$$L^2(\mathrm{Gr}(n, \mathbf{C}^N)) \to L^2(\mathrm{Gr}(n, \mathbf{C}^\infty)).$$

We want to study how the decomposition for $\mathrm{Gr}(n, \mathbf{C}^N)$ converges to that for $\mathrm{Gr}(n, \mathbf{C}^\infty)$.

Because the irreducible summands of $L^2(\mathrm{Gr}(n, \mathbf{C}^N))$ consist of algebraic functions and the projection

$$\mathrm{Gr}(n, \mathbf{C}^{N+k}) \to \mathrm{Gr}(n, \mathbf{C}^N)$$

defined by (4.1) is not globally continuous, it is not the case that the irreducible summands coherently embed as $N \to \infty$. Thus the convergnce is somewhat subtle. It is most easily understood in terms of spherical functions.

Now $\mathrm{Gr}(n, H) = U/K$, where $U = U(H)$, $K = U(\mathbf{C}^n) \times U(\mathbf{C}^{n^\perp})$. It is a symmetric space of rank n. The Cartan involution is given by $\theta(x) = \begin{pmatrix} \alpha & \\ & \delta \end{pmatrix} - \begin{pmatrix} & \beta \\ \gamma & \end{pmatrix}$, where $x = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ relative to $H = \mathbf{C}^n \oplus (\mathbf{C}^n)^\perp$, $x \in \mathrm{gl}(H)$. Let $\mathfrak{a}$ denote the set of all operators of the form $T = \sum_1^n t_j(e_j \otimes e_{n+j}^* + e_{n+j} \otimes e_j^*)$ with $t_j \in \mathbf{R}$. This set is a maximal abelian subalgebra of $P = \{\begin{pmatrix} & x \\ x^* & \end{pmatrix} \colon x \in \mathscr{L}(\mathbf{C}^{n^\perp}, \mathbf{C}^n)\}$, the real "noncompact" part of the $\theta = -1$ eigenspace.

Suppose $T \in \mathfrak{a}$. Then

$$\exp(it) = \sum_1^n \Big(\cos t_j\big(e_j \otimes e_j^* + e_{n+j} \otimes e_{n+j}^*\big) + i \sin t_j\big(e_j \otimes e_{n+j}^* + e_{n+j} \otimes e_j^*\big)\Big)$$

plus the identity on $\{e_j \colon 1 \leq j \leq 2n\}^\perp$. Hence generically we have $\exp(it) = \mathrm{graph}(z)$, where $z = \sum_1^n i \tan t_j e_j \otimes e_j^*$. Note the spectrum of $(1 + z^*z)^{-1}$ is $\{u_j\}$, where $u_j = \cos^2 t_j$.

We now recall the formulae of Berezin-Karpelevic for the spherical functions (these are proven by Hoogenboom in [3]). Let $N \geq 2n$.

(4.2) LEMMA. *The spherical functions of* $\mathrm{Gr}(n, \mathbf{C}^N)$ *are parameterized by partitions* μ *with* $\mu_{n+1} = 0$. *The funcion corrsponding to* μ *is a multiple of the function*

$$\psi(z) = \frac{\det\big\{L_{i-1+\bar{\mu}_i}^{(k)}(u_j)\big\}}{\det\big\{u_j^{i-1}\big\}}$$

where $u_1, \ldots, u_n$ *is the spectrum of* $(1 + z^*z)^{-1}$, *the* $L_i^{(k)}$ *are the (Legendre) orthogonal polynomials for the probability measure* $(k+1)(1-x)^k\,dx$ *on* $[0,1]$, $k = N - 2n$, *and* $\bar{\mu}_j = \mu_{n+1-j}$.

Using integration in polar coordinates (see Chapter I of [1]), it is easily checked that the L^2 normalized spherical function corresponding to the partition μ is given by

$$\psi_{\mu,N}(z) = \frac{\det\big\{\tilde{L}_{i-1+\bar{\mu}_i}^{(k)}(u_j)\big\}}{\det\big\{\tilde{L}_{i-1}^{(k)}(u_j)\big\}},$$

where $\tilde{L}_i$ denotes the L^2 normalization of L_i.

(4.3) PROPOSITION. *For each partition* μ *with* $\mu_{n+1} = 0$, *the functions* $\psi_{\mu,N}$ *have a limit* ψ_μ *in all* $L^p(\mu_0)$, $1 \leq p < \infty$. *As a function of* $E \in \mathscr{L}(\mathbf{C}^n, \mathbf{C}^\infty)$,

$$\psi_\mu(E) = \frac{\det\big\{\tilde{\mathscr{L}}_{i-1+\bar{\mu}_i}(v_j)\big\}}{\det\big\{\tilde{\mathscr{L}}_{i-1}(v_j)\big\}}$$

where $\{v_i\}$ is the spectrum of $\alpha(E)^*\alpha(E)$ and the $\tilde{\mathscr{L}}_i$ are the L^2 normalized Laguerre polynomials.

Proof. $\{\tilde{L}_i^{(k)}(k^{-1}y)\}$ is the system of L^2 normalized orthogonal polynomials for the probability measure $(k+1)/k(1-y/k)^k\,dy$ on $0 \le y \le k$, which tends to $e^{-y}\,dy$ as k (or N) $\to \infty$. From this it follows easily (or one can check the well-known formulae directly) that the coefficients of $\tilde{L}_i^{(k)}(k^{-1}y)$ converge to those of $\tilde{\mathscr{L}}_i(y)$ as $k \to \infty$.

We also know that $k(1+Z^{(N)*}Z^{(N)})^{-1}$ ($\cong N|L_1^{(N)}|^{-2}$ in the notation of (3.5)) converges to the $n \times n$ matrix $(\alpha\alpha^*)(E)$ in all L^p. Hence we have L^p convergence

$$\sigma\left(ku_1^{(N)}, \dots, ku_n^{(N)}\right) \to \sigma(v_1, \dots, v_n)$$

for any symmetric polynomial. This implies

$$\psi_{\mu,N} = \frac{\det\left\{\tilde{L}_{i-1+\bar{\mu}_i}^{(k)}\left(ku_i^{(N)}/k\right)\right\}}{\det\left\{\tilde{L}_{i-1}^{(k)}\left(ku_i^{(N)}/k\right)\right\}} \to \psi_\mu$$

in all L^p, $1 \le p < \infty$. □

By (3.2) we know that

$$L^2(\mathrm{Gr}(n, \mathbf{C}^\infty)) \cong \sum \rho_\lambda^* \otimes \rho_\mu \times \left(\rho_\lambda \otimes \rho_\mu^*\right)^K$$
$$= \sum \rho_\lambda^* \otimes \rho_\mu \times \left(\rho_{\lambda,n} \otimes \rho_{\mu,n}^*\right)^{U(n)} \cong \sum \rho_\mu^* \otimes \rho_\mu,$$

where the second and third sums are over those partitions with $(n+1)$th term $= 0$.

(4.4) Proposition. *For each partition μ with $\mu_{n+1} = 0$, ψ_μ is, up to a multiple, the unique K invariant vector in $\rho_\mu^* \otimes \rho_\mu$.*

Proof. We will use the transform $\mathscr{T}$ of §2, which induces an equivariant isometry

$$L^2(\mathscr{L}(\mathbf{C}^n, \mathbf{C}^\infty)) \cong \mathbf{C}e^{-1/4|w|^2} \otimes \sum_0^\infty \hat{\mathscr{P}}^j \otimes \sum_0^\infty \overline{\hat{\mathscr{P}}}^k$$

where $\hat{\mathscr{P}}^j = (j!)^{1/2}\mathscr{P}^j(\mathscr{L}(\mathbf{C}^n, H))$. We must show that $\mathscr{T}\psi_\mu$ is in the one dimensional space

$$(4.5) \quad \mathbf{C}e^{-1/4|w|^2} \otimes \left(\rho_\mu^* \otimes \rho_\mu \times \rho_{\mu,n} \otimes \rho_{\mu,n}^*\right)^{K \times U(n)}$$
$$= \mathbf{C}e^{-1/4|w|^2} \otimes \left(\rho_{\mu,n}^* \times \rho_{\mu,n} \otimes \rho_{\mu,n} \times \rho_{\mu,n}^*\right)^{U(n) \times U(n)}$$

Let $\rho = \rho_{\lambda,n}$. One vector in the space $\rho^* \times \rho$ is the character $\chi = \operatorname{trace}\rho$ (where we have holomorphically extended ρ to a representation of $\mathrm{GL}(n)$). Thus $|\chi|^2$ is a vector in $\rho^* \times \rho \otimes \rho \times \rho^*$. Let $\{\varepsilon_i\}$ be an orthonormal basis for a realization V of ρ. Any hermitian form on V is a multiple of a fixed $U(n)$ invariant positive form $\langle\ ,\ \rangle$. A standard argument shows that this implies

$$\int_{U(n)} \left\langle v_1, \rho(k)^*\varepsilon_i\right\rangle\left\langle v_2, \rho(k)^*\varepsilon_j\right\rangle dk = \delta(i-j)\operatorname{dimn}(V)^{-1}\langle v_1, v_2\rangle$$

for $v_1, v_2 \in V$. Thus

$$\int_{U(n)} |\chi|^2(kw)\,dk = \sum_{i,j}\int \left\langle \rho(w)\varepsilon_i, \rho(k)^*\varepsilon_j\right\rangle\left\langle \rho(w)\varepsilon_i, \rho(k)^*\varepsilon_j\right\rangle dk$$

$$= \sum_j \operatorname{dimn}(V)^{-1}\left\langle \rho(w^*w)\varepsilon_j, \varepsilon_j\right\rangle.$$

Thus, by the Weyl character formula, a nonzero vector spanning the space in (4.5) is

$$\phi_\lambda(w) = \exp\left(-\frac{1}{4}|w|^2\right)\operatorname{tr}\rho(w^*w)$$

$$= \exp\left(-\frac{1}{4}|w|^2\right)\frac{\det\left(v_j^{i-1+\mu_i}\right)}{\det\left(v_j^{i-1}\right)},$$

where $\{v_j\}$ is the spectrum of w^*w.

Any K invariant vector in the range of $\mathscr{T}$ must be a linear combination of the ϕ_λ, in particular, $\psi_\mu = \sum c_\lambda \psi_\lambda$ (where a priori we only know $\lambda_{n+1} = 0$). Now asymptotically,

$$\operatorname{tr}\rho_{\lambda,n}\left(\operatorname{diag}(t_j)\right) \sim \left\langle \rho_{\lambda,n}\left(\operatorname{diag}(t_j)\right)v_0, v_0\right\rangle = \prod_1^n t_j^{\lambda_j}$$

where v_0 is a highest weight vector, we set $t = t_1 = \cdots = t_k$, $t_{k+j} = 1$, and we let $t \to \infty$. This is also the asymptotic behavior of ψ_μ (with $\mu = \lambda$ above). The theory of homogeneous chaos (Section 6.3 of **[2]**) shows that if f is a polynomial of degree (p, q) (in E, $\overline{E}$), then $\exp(\frac{1}{4}|w|^2)\mathscr{T}f$ is of the same degree. Since ψ_μ is a symmetric function in the eigenvalues of $\alpha(E)^*\alpha(E)$, it follows that $\mathscr{T}\psi_\mu$ has the same asymptotic behavior above as ϕ_μ. Thus we must have

$$\mathscr{T}\psi_\mu = c\phi_\mu \tag{4.6}$$

which proves (4.4). □

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