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Rigid elements in a field, should they exist, have strong influence on structure of the Witt ring of the field. We generalize rigid elements to the context of n-fold Pfister forms in two ways and study the relations between n-rigid and super n-rigid elements in various classes of fields including global fields, and in abstract Witt rings. In special cases the existence of higher rigidities turns out to be equivalent to important properties of fields known in literature. Among these are the property A_n, torsion freeness of I^nF and finite stability index of WF.

Introduction. Rigid elements have proved to be of importance in studying the structure of the Witt ring of a field (see, for instance, [**1**], [**2**], [**3**], [**4**], [**18**]). However, the notion of rigidity has strictly binary character and almost all existing approaches to rigid elements revolve around the defining condition $D_F\langle 1, x\rangle = \dot{F}^2 \cup x\dot{F}^2$. It is easy to find equivalent conditions which suggest the general notion of rigidity defined in terms of n-fold Pfister forms and powers of the fundamental ideal of Witt ring. Thus an element $x \in \dot{F}$ is said to be n-rigid if every n-fold Pfister form ϕ annihilated by $\langle 1, x\rangle$ in WF has to have the factor $\langle 1, -x\rangle$ making $\phi \cdot \langle 1, x\rangle = 0$ a triviality. Super n-rigidity is obtained by replacing Pfister forms with elements of I^nF. It turns out that what is easily seen to be equivalent for $n = 1$ is presumably a hard problem when $n > 2$. In this paper we make an attempt to understand what lies behind these general conditions.

In the first section we work in arbitrary fields (of characteristic not two). We find a characterization of n-rigid elements by value sets of quadratic forms and show that the sets of n-rigid elements form an ascending chain for $n = 1, 2, \ldots$. The same is proved for super n-rigid elements for special classes of fields. This leads to a new characterization of linked fields (Proposition 1.23). In special cases we arrive at interesting properties of fields studied in the literature. Thus n-rigidity of $x = 1$ means the field satisfies Elman and Lam's property A_n (cf. [**8**]) and super n-rigidity of $x = 1$ means I^nF is torsion free. And for $x = -1$ the two rigidity properties coincide with WF being $(n-1)$-stable (cf. [**6**]). As a by-product we obtain a characterization of stability in terms of value groups of Pfister forms (see (1.11) below).

Section 2 proves that n-rigid and super n-rigid elements coincide for every n, if the field in question is amenable (cf. [**9**]) or linked (cf. [**11**]). We also compute explicitly the sets of n-rigid elements for finite, $\mathfrak{p}$-adic and global fields.

In §3 we study higher rigidities in abstract Witt rings in the sense of Marshall [**15**]. The main result asserts that n-rigid and super n-rigid elements coincide in every Witt ring of elementary type. This result includes, in particular, all Pythagorean fields with finite group of square classes and all fields with group of square classes of order ≤ 32. We conclude this paper with relating our generalization of rigid elements to Arason, Elman and Jacob's generalization of birigid elements ([**1**]).

We use standard notation and terminology (except for higher rigidities). The fields considered are always assumed to have characteristic different from two and $\dot{F} = F \setminus \{0\}$. P_nF denotes the set of all n-fold Pfister forms over F and we often think of P_nF as a subset of the Witt ring WF. For $\phi \in P_nF$, we write ϕ' for the pure subform of ϕ, thus $\phi \cong \langle 1 \rangle \perp \phi'$. The nth radical R_nF is the intersection of value groups of all n-fold Pfister forms over F. Annihilator ideals $\operatorname{Ann}_{WF}\langle 1, x \rangle$ are wirtten simply $\operatorname{Ann}\langle 1, x \rangle$. Unexplained notation and terminology follows [**13**], [**14**] and [**15**].

1. Rigidity and super rigidity. We begin with recalling conditions characterizing rigid elements in fields.

PROPOSITION 1.1. *For an arbitrary $x \in \dot{F}$, the following statements are equivalent.*

(i) $D_F\langle 1, x \rangle = \dot{F}^2 \cup x\dot{F}^2$, *i.e., x is rigid in F.*
(ii) $0 \neq \phi \in P_1F$ *and* $-x \in D_F\phi \Rightarrow -x \in D_F\phi'$.
(iii) $P_1F \cap \operatorname{Ann}\langle 1, x \rangle = \langle 1, -x \rangle P_0F$.
(iv) $IF \cap \operatorname{Ann}\langle 1, x \rangle = \langle 1, -x \rangle I^0F$.

Proof. Here $P_0F = \{0, \langle 1 \rangle\}$ and $I^0F = WF$. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are all easily verified. For $\langle 1, x \rangle$ anisotropic, (i) $\Rightarrow$ (iv) is an immediate consequence of Witt Annihilator Theorem [**14**, p. 71]. For $x \in -\dot{F}^2$, (i) implies F has at most two square classes and then $IF = 2 \cdot WF$ which is (iv).

Conditions (ii), (iii) and (iv) are easily generalized to the context of n-fold Pfister forms. We will also generalize (i) but that requires introducing an appropriate analogue of the nth radical of a field.

DEFINITION 1.2. For $x \in \dot{F}$ and any non-negative integer n, $R_0(F,x) = \dot{F}^2 \cup x\dot{F}^2$, and

$$R_n(F,x) = \bigcap\{D_F\phi \cdot D_F(\langle x\rangle \perp \phi') : \phi \in P_nF\}, \qquad \text{for } n \geq 1.$$

$R_n(F,x)$ is said to be the nth radical of F centered at x.

Observe that for $x = 1$, $R_n(F,1) = R_nF$, is the nth radical of F. R. Bos [3, p. 65] proved that for every $\phi \in P_nF$ the set $D_F\phi \cdot D_F(\langle x\rangle \perp \phi')$ is a subgroup of $\dot{F}$ containing $D_F\phi$. Hence $R_n(F,x)$ is a subgroup of $\dot{F}$ containing R_nF. Clearly $x \in R_n(F,x)$ hence also

$$R_nF \cup xR_nF \subseteq R_n(F,x).$$

Now we consider the four conditions generalizing (i) through (iv) in Proposition 1.1. For $x \in \dot{F}$ and $n \geq 1$ these are

(A) $D_F\langle 1,x\rangle \subseteq R_{n-1}(F,x)$.

(B) $0 \neq \phi \in P_nF$ and $-x \in D_F\phi \Rightarrow -x \in D_F\phi'$.

(C) $P_nF \cap \operatorname{Ann}\langle 1,x\rangle \subseteq \langle 1,-x\rangle P_{n-1}F$.

(D) $I^nF \cap \operatorname{Ann}\langle 1,x\rangle \subseteq \langle 1,-x\rangle I^{n-1}F$.

Observe that for $n > 1$ the assumption $\phi \neq 0$ in (B) is superfluous (but it is necessary when $n = 1$). Also in (C) and (D) the right-hand sides are obviously contained in left-hand sides so that in (C) and (D) we can put the equality sign as well.

THEOREM 1.3. (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C) $\Leftarrow$ (D).

Proof. We may assume $n > 1$. For $\phi \in P_nF$ we have $\phi \cdot \langle 1,x\rangle = 0 \Leftrightarrow -x \in D_F\phi$ and $-x \in D_F\phi' \Leftrightarrow \phi \in \langle 1,-x\rangle P_{n-1}F$ (Pure Subform Theorem [14, p. 64]). This verifies (B) $\Leftrightarrow$ (C). To prove (D) $\Rightarrow$ (C), let $\phi \in P_nF \cap \operatorname{Ann}\langle 1,x\rangle$. Then by (D), $\phi \in \langle 1,-x\rangle I^{n-1}F$, hence by [6, Theorem 2.1], $\phi \in \langle 1,-x\rangle P_{n-1}F$ as needed. Two lemmas will be needed to finish the proof of 1.3.

LEMMA 1.4. *Let $\phi \in P_nF$ and $n \geq 1$. Then for x, $y \in \dot{F}$,*

$$-x \in D_F(\phi' \perp -y\phi) \Leftrightarrow y \in D_F\phi \cdot D_F(\langle x\rangle \perp \phi').$$

Proof. If $-x \in D_F(\phi' \perp -y\phi)$, then $-x \in D_F\langle a,-yb\rangle$ for some $a \in D_F\phi'$ and $b \in D_F\phi$. Hence $yb \in D_F\langle x,a\rangle$ and $y \in bD_F\langle x,a\rangle \subseteq D_F\phi \cdot D_F(\langle x\rangle \perp \phi')$. Conversely, if $y \in D_F\phi \cdot D_F(\langle x\rangle \perp \phi')$, then $by \in D_F(\langle x\rangle \perp \phi')$ for some $b \in D_F\phi$ and so $-x \in D_F(\phi' \perp \langle -by\rangle) \subseteq D_F(\phi' \perp -y\phi)$.

LEMMA 1.5. *Let $\phi \in P_nF$, $n \geq 1$ and $\langle 1,x\rangle\phi = 0$. Then $\phi = \langle 1,-y\rangle\theta$ for some $y \in D_F\langle 1,x\rangle$ and $\theta \in P_{n-1}F$.*

Proof. This is well known. If $x \notin -\dot{F}^2$, we have $-x = t^2 - y$ with $-y \in D_F\phi'$ and so $\phi \in \langle 1, -y\rangle P_{n-1}F$ by Pure Subform Theorem. Also $y = t^2 + x \in D_F\langle 1, x\rangle$. If $x \in -\dot{F}^2$, the result is trivially true.

Now we prove (C) $\Rightarrow$ (A). Let $y \in D_F\langle 1, x\rangle$. Then for every $\phi \in P_{n-1}F$, $\phi\langle 1, -y\rangle\langle 1, x\rangle = 0$, i.e., $\phi\langle 1, -y\rangle \in P_nF \cap \operatorname{Ann}\langle 1, x\rangle$. Using (C), there exists $\theta \in P_{n-1}F$ such that $\phi\langle 1, -y\rangle = \langle 1, -x\rangle\theta$. It follows

$$\phi' \perp -y\phi = \theta' \perp -x\theta,$$

hence $-x \in D_F(\phi' \perp -y\phi)$. By Lemma 1.4, $y \in D_F\phi \cdot D_F(\langle x\rangle \perp \phi')$. Since this holds for every $\phi \in P_{n-1}F$, we get $y \in R_{n-1}(F, x)$.

(A) $\Rightarrow$ (C). Let $\phi \in P_nF \cap \operatorname{Ann}\langle 1, x\rangle$. By Lemma 1.5, $\phi = \langle 1, -y\rangle\theta$ with $y \in D_F\langle 1, x\rangle$ and $\theta \in P_{n-1}F$. Using now (A), we get $y \in D_F\theta \cdot D_F(\langle x\rangle \perp \theta')$ and so, by Lemma 1.4, $-x \in D_F(\theta' \perp -y\theta) = D_F\phi'$. Hence $\phi \in \langle 1, -x\rangle P_{n-1}F$. This finishes proof of Theorem 1.3.

DEFINITION 1.6. We say $x \in \dot{F}$ is n-rigid if it satisfies one (hence all) of the conditions (A), (B), (C). The set of all n-rigid elements in F will be written R^nF. We say $x \in \dot{F}$ is super n-rigid if it satisfies (D). The set of all super n-rigid elements is denoted $\operatorname{Sup} R^nF$.

REMARKS (1.7). $R^1F = \operatorname{Sup} R^1F$ is the set of usual rigid elements in F satisfying conditions of Proposition 1.1.

(1.8) For every $n \geq 1$, $\operatorname{Sup} R^nF \subseteq R^nF$, according to (D) $\Rightarrow$ (C). A large portion of this paper is devoted to studying the cases where the converse inclusion holds.

(1.9) n-rigidity of $x = 1$ is equivalent to

$$P_nF \cap \operatorname{Ann}\langle 1, 1\rangle = 0.$$

This is precisely the definition of property A_n studied by Elman and Lam in [**8**]. The equivalence of (A) and (C) includes as a special case the following result (cf. [**8**, Theorem 4.3]): F satisfies A_n iff $D_F\langle 1, 1\rangle \subseteq R_{n-1}F$.

(1.10) Super n-rigidity of $x = 1$ means $I^nF \cap \operatorname{Ann}\langle 1, 1\rangle = 0$. This is easily seen to be equivalent with $I^nF \cap W_tF = 0$. Thus the question of whether $1 \in R^nF \Rightarrow 1 \in \operatorname{Sup} R^nF$ turns out to be equivalent to the important open problem: does A_n imply that I^nF is torsion free? Elman and Lam [**8**, p. 37] say that this "seems to be a rather difficult and elusive question". Certainly, the same applies to the general question raised by Theorem 1.3: does (C) imply (D)? or equivalently, for any field F, is $R^nF \subseteq \operatorname{Sup} R^nF$?

(1.11) For $x = -1$ we also arrive at an interesting and important point in quadratic form theory. Indeed, $-1 \in R^nF \Leftrightarrow P_nF = 2P_{n-1}F \Leftrightarrow I^nF = 2I^{n-1}F \Leftrightarrow -1 \in \operatorname{Sup} R^nF$. Thus, for $x = -1$, (C) and (D) are equivalent and reduce to the condition: "WF is $(n-1)$-stable" (cf. [**6**, Def,. 3.8]). In terms of value groups of Pfister forms we get the following result (using (A) with $x = -1$): WF is $(n-1)$-stable iff for every $(n-1)$-fold Pfister form ϕ,

$$D_F\phi D_F(\langle -1\rangle \perp \phi') = \dot{F}.$$

In case $n = 2$, the condition says

$$D_F\langle 1, a\rangle \cdot D_F\langle 1, -a\rangle = \dot{F}, \quad \text{for every } a \in \dot{F}.$$

This complements the list of equivalent statements for 1-stability of WF given in [**6**, Prop. 3.9].

(1.12) If F satisfies A_n (i.e., $1 \in R^nF$), then $\Sigma\dot{F}^2 \subseteq R^nF$. Indeed, if $x \in \Sigma\dot{F}^2$ and F satisfies A_n, then by [**8**, Theorem 4.3], F satisfies $A_n(x)$, i.e., $\operatorname{Ann}\langle 1, x\rangle \cap P_nF = 0$. Hence x satisfies (C) and so $x \in R^nF$.

(1.13) If F is non-real and satisfies A_n, then $R^nF = \dot{F}$. This follows immediately from (1.12).

(1.14) R^nF and $\operatorname{Sup} R^nF$ are unions of cosets of $R_{n-1}F$ in $\dot{F}$. For, if $q \in I^{n-1}F$ and $y \in R_{n-1}F$, then $yq = q$ and so for $q \in I^nF$, $q \in \operatorname{Ann}\langle 1, x\rangle \Leftrightarrow q \in \operatorname{Ann}\langle 1, xy\rangle$ and $q \in \langle 1, -x\rangle I^{n-1}F \Leftrightarrow q \in \langle 1, -xy\rangle I^{n-1}F$. Hence if x satisfies (D), so does xy. The same applies to R^nF.

(1.15) If $1 \in R^nF$ ($1 \in \operatorname{Sup} R^nF$), then $R_{n-1}F \subseteq R^nF$ ($R_{n-1}F \subseteq \operatorname{Sup} R^nF$). This follows from (1.14).

PROPOSITION 1.16. For every $n \geq 1$, $R^nF \subseteq R^{n+1}F$. In particular, $R^1F \subseteq R^nF$, for every $n \geq 1$.

Proof. Assume $x \in R^nF$ and $\phi \in P_{n+1}F \cap \operatorname{Ann}\langle 1, x\rangle$. Lemma 1.5 yields $\phi = \langle\langle -y, y_2, \ldots, y_{n+1}\rangle\rangle$ with $y \in D_F\langle 1, x\rangle$ and some $y_i \in \dot{F}$. Then $\theta := \langle\langle -y, y_2, \ldots, y_n\rangle\rangle \in P_nF \cap \operatorname{Ann}\langle 1, x\rangle$ and since $x \in R^nF$ it follows that $\theta \in \langle 1, -x\rangle P_{n-1}F$. Thus $\phi = \theta\langle\langle y_{n+1}\rangle\rangle \in \langle 1, -x\rangle P_nF$, i.e., $x \in R^{n+1}F$.

REMARK (1.17). A special case of 1.16 says $1 \in R^nF$ implies $1 \in R^{n+1}F$, i.e., if F satisfies A_n, it also satisfies A_m for all $m \geq n$ (cf. [**5**, Lemma 2.10]). Combining (1.14) and Proposition 1.16 we get for every $n \geq 2$,

$$R_{n-1}F \cdot R^{n-1}F \subseteq R^nF.$$

It is to be remarked that the analogue of Proposition 1.16 for super rigid elements is not known to hold. What can be proved for an arbitrary field F is summarized below.

PROPOSITION 1.18. *For any field F,*
(i) $\operatorname{Sup} R^1F = R^1F$.
(ii) $\operatorname{Sup} R^2F = R^2F$.
(iii) $\operatorname{Sup} R^1F \subseteq \operatorname{Sup} R^2F \subseteq \operatorname{Sup} R^3F$.

Proof. (i) has already been noticed in (1.7). (ii) We want $R^2F \subseteq \operatorname{Sup} R^2F$. So let $x \in R^2F$ and assume first that $x \notin -\dot{F}^2$. Then $\operatorname{Ann}\langle 1, x\rangle$ is a 1-Pfister ideal (cf. [**11**, 2.4(a)]) and so by [**11**, Corollary 2.15], we get

$$I^2F \cap \operatorname{Ann}\langle 1, x\rangle = \operatorname{Ann}\langle 1, x\rangle \cdot IF.$$

That this implies $x \in \operatorname{Sup} R^2F$ is shown in the proof of Theorem 2.2 in the next section. Now, if $x \in -\dot{F}^2$, then for every $n \geq 1$, $x \in R^nF \Rightarrow x \in \operatorname{Sup} R^nF$ according to (1.11). This proves (ii). (iii) With (i), (ii) and Proposition 1.16, we only need to show that $\operatorname{Sup} R^2F \subseteq \operatorname{Sup} R^3F$. This follows from Theorem 1.22 and Example 1.21 below.

For the balance of this section, we concentrate on conditions under which $\operatorname{Sup} R^nF \subseteq \operatorname{Sup} R^{n+1}F$ holds.

DEFINITION 1.19. An ideal A in WF is said to be n-Pfister neighbor ideal, if for every anisotropic $q \in A$ with $\dim q > 2^{n-1}$, there is an isometry

$$q \cong a\phi \perp \theta \tag{1.19.1}$$

where $a \in \dot{F}$, ϕ is a Pfister neighbor associated to an n-fold Pfister form and θ is an arbitrary form.

EXAMPLES. (1.20). Every strong n-Pfister ideal is (trivially) n-Pfister neighbor ideal. More interestingly, every strong n-Pfister ideal A is $(n+1)$-Pfister neighbor ideal. Indeed, let $q \in A$, $\dim q > 2^n$ and $q = a_1\phi_1 \perp \cdots \perp a_r\phi_r$, where $a_i \in \dot{F}$, $\phi_i \in P_nF$ and $r > 1$. Consider $a_1\phi_1 \perp a_2\phi_2 = a_1(\phi_1 \perp a_1a_2\phi_2)$ and put $\phi = \phi_1 \perp \langle a_1a_2\rangle$. Then ϕ is a Pfister neighbor associated to $\phi_1 \cdot \langle 1, a_1 a_2 \rangle$ and (1.19.1) holds with obvious choices for a and θ.

(1.21). IF is 2-Pfister neighbor ideal, since IF is strong 1-Pfister. For the cases where I^2F is 3-Pfister neighbor ideal, see Proposition 1.23 below.

THEOREM 1.22. *If $I^{n-1}F$ is n-Pfister neighbor ideal, then*

$$\operatorname{Sup} R^n F \subseteq \operatorname{Sup} R^{n+1} F.$$

Proof. Let $x \in \operatorname{Sup} R^n F$ and $q \in I^{n+1}F \cap \operatorname{Ann}\langle 1, x\rangle$. Then $q \in I^n F \cap \operatorname{Ann}\langle 1, x\rangle = \langle 1, -x\rangle I^{n-1}F$, and so $q = \langle 1, -x\rangle p$, where p is anisotropic form in $I^{n-1}F$. We prove $q \in \langle 1, -x\rangle I^n F$ by induction on $\dim p$. If $\dim p \le 2^{n-1}$, then $\dim q \le 2^n$ and since $q \in I^{n+1}F$, we conclude $q = 0$ (by Arason-Pfister Haupsatz) and we are done. So assume $\dim p > 2^{n-1}$. Then by the hypothesis on $I^{n-1}F$, $p \cong a\phi \perp \theta$, where ϕ is a Pfister neighbor of some $\sigma \in P_n F$, $a \in \dot{F}$ and θ is an F-form. Thus $\sigma \cong \phi \perp \phi_1$ with $\dim \phi_1 < \dim \phi$ and for $p_1 = p \perp -a\sigma$ we have $p_1 = -a\phi_1 + \theta$ in WF and

$$\dim (p_1)_{an} \le \dim \phi_1 + \dim \theta < \dim \phi + \dim \theta = \dim p.$$

Now

$$\begin{aligned}\langle 1, -x\rangle p_1 &= \langle 1, -x\rangle p - a\langle 1, -x\rangle \sigma \\ &= q - a\langle 1, -x\rangle\sigma \in I^{n+1}F \cap \operatorname{Ann}\langle 1, x\rangle\end{aligned}$$

and so, by induction, $\langle 1, -x\rangle p_1 \in \langle 1, -x\rangle I^n$. Hence also $q = \langle 1, -x\rangle p_1 + a\langle 1, -x\rangle \sigma \in \langle 1, -x\rangle I^n F$, as required.

Recall that (1.21) and Theorem 1.22 imply our earlier statement in Proposition 1.18 (iii). The question now arises what other cases are covered by the Theorem. We answer this completely for $n = 3$ proving the following result.

PROPOSITION 1.23. *I^2F is a 3-Pfister neighbor ideal if and only if I^2F is linked (iff quaternion algebras form a subgroup in the Brauer group of F).*

Proof. If I^2F is linked, then by [7, Theorem 2.7], I^2F is strong 2-Pfister ideal in the sense of [**11**] and so by (1.20) it is 3-Pfister neighbor ideal.

To prove the converse, it is sufficient to show that if θ_1, $\theta_2 \in P_2F$, then the form $q = \theta_1' \perp \langle -1\rangle \theta_2'$ satisifes $q \equiv \phi \bmod I^3F$ for some $\phi \in P_2F$ (or even a 4-dimensional form in I^2F) by [**17**, Zusatz, p. 124]. If q is anisotropic 6-dimensional, up to a scalar, it contains a 5-dimensional Pfister neighbor of a 3-fold Pfister form. Since q lies in I^2F, going mod I^3F yields the result.

COROLLARY 1.24. *If I^2F is linked, then*

$$\operatorname{Sup} R^3F \subseteq \operatorname{Sup} R^4F.$$

This is an immediate consequence of 1.22 and 1.23. However, we will recapture and strengthen this result in the next section (see Corollary 2.6 and combine it with Proposition 1.16).

2. Amenable and linked fields. In this section we prove that basic question of whether $R^nF = \operatorname{Sup} R^nF$ for every $n \geq 1$, has an affirmative answer for every amenable and every linked field. We also compute the sets R^nF for all $\mathfrak{p}$-adic and all global fields.

Recall that a field F is said to be amenable, if for every finite set $\{\phi_1, \ldots, \phi_k\}$ of Pfister forms over F and for $K = F(\phi_1, \ldots, \phi_k)$, the iterated function field of $\phi_1, \ldots, \phi_k$ over F,

$$W(K/F) = \sum \phi_i \cdot WF,$$

where $W(K/F)$ is the kernel of the canonical ring homomorphism $WF \to WK$ (see [9] for details). We will use below corollary to a basic result of [9] on amenable fields.

LEMMA 2.1 ([9, *Cor.* 4.6]). *If F is amenable and ϕ is an anisotropic Pfister form over F, then for any integer $n \geq 1$,*

$$I^nF \cap \operatorname{Ann}\phi = \operatorname{Ann}\phi \cdot I^{n-1}F.$$

Using this we prove the following result.

THEOREM 2.2. *If F is an amenable field, then for every integer $n \geq 1$, $R^nF = \operatorname{Sup} R^nF$.*

Proof. We only need to show $R^nF \subseteq \operatorname{Sup} R^nF$. So suppose $x \in \dot{F}$ satisfies (C) of Theorem 1.3 and let $q \in I^nF \cap \operatorname{Ann}\langle 1, x\rangle$. If $x \notin -\dot{F}^2$, Lemma 2.1 applies, hence $q \in \operatorname{Ann}\langle 1, x\rangle \cdot I^{n-1}F$. Thus $q = \sum AB$ with $A \in \operatorname{Ann}\langle 1, x\rangle$ and $B \in I^{n-1}F$. By Witt Annihilator Theorem,

$$A = \sum a_i\langle\langle -c_i\rangle\rangle, \quad \text{where } a_i \in \dot{F} \text{ and } c_i \in D_F\langle 1, x\rangle.$$

We also have

$$B = \sum \pm \phi_j, \quad \text{where } \phi_j \in P_{n-1}F.$$

Since $\langle\langle -c_i \rangle\rangle \phi_j \in P_n F \cap \mathrm{Ann}\langle 1, x \rangle$ and x is n-rigid, we get $\langle\langle -c_i \rangle\rangle \phi_j \in \langle 1, -x \rangle P_{n-1} F$. It follows

$$AB = \sum \pm a_i \langle\langle -c_i \rangle\rangle \phi_j \in \langle 1, -x \rangle I^{n-1} F,$$

hence also $q \in \langle 1, -x \rangle I^{n-1} F$ as needed. If $x \in -\dot{F}^2 \cap R^n F$, we get $x \in \mathrm{Sup}\, R^n F$ by (1.11).

COROLLARY 2.3. *The conclusion of Theorem* 2.2 *holds for*

(i) *all global fields* (*of characteristic not two*),

(ii) *all fields of transcendence degree* ≤ 1 *over a real closed field*,

(iii) *all fields with the property that every anisotropic* 4-*dimensional form has determinant* 1 (*e.g., all* $\mathfrak{p}$-*adic fields*).

Proof. The fields in (i) and (ii) are amenable by [**10**, Cor. 6.4] and those in (iii) by [**9**, Theorem A.14].

Now we proceed to linked fields. A field F is said to be n-linked, if any pair of n-fold Pfister forms over F are linked (cf. [7]).

THEOREM 2.5. *Suppose F is an n-linked field, $n \geq 2$. Then $R^m F = \mathrm{Sup}\, R^m F$ for every $m \geq n$.*

Proof. Since for $n \geq 2$, F being n-linked implies that F is $(n+1)$-linked, it is sufficient to prove the Theorem in the case $m = n$. So assume $x \in R^n F$ and take an $q \in I^n F \cap \mathrm{Ann}\langle 1, x \rangle$. By [**5**, Lemma 4.4], there exists a representation

$$q = a_0 q_0 + a_1 q_1 + \cdots + a_r q_r,$$

where $a_i \in \dot{F}$ and $q_i \in P_{n+i} F$. We induct on r. If $r = 0$, $q_0 = a_0 q \in P_n F \cap \mathrm{Ann}\langle 1, x \rangle$, hence $q_0 \in \langle 1, -x \rangle P_{n-1} F$ by n-rigidity of x. Hence also $q \in \langle 1, -x \rangle I^{n-1} F$ as needed. Now assume $r > 0$. Then from $q \in \mathrm{Ann}\langle 1, x \rangle$ we get the equation

$$a_0 \langle 1, x \rangle q_0 + \cdots + a_{r-1} \langle 1, x \rangle q_{r-1} = -a_r \langle 1, x \rangle q_r \quad \text{in } WF.$$

By a dimension count it follows $\langle 1, x \rangle q_r$ is isotropic, hence hyperbolic. Thus $q_r \in P_{n+r} F \cap \mathrm{Ann}\langle 1, x \rangle = \langle 1, -x \rangle P_{n+r-1} F$, the latter by Proposition 1.16. Hence $a_r q_r \in \langle 1, -x \rangle I^{n-1} F$ and by induction, $q \in \langle 1, -x \rangle I^{n-1} F$ as needed.

COROLLARY 2.6. *If F is linked* (*i.e., F is* 2-*linked or, equivalently, the quaternion algebras form a subgroup in the Brauer group of F*), *then for every $n \geq 1$,*

$$R^n F = \mathrm{Sup}\, R^n F.$$

For a list of examples of linked fields including finite, $\mathfrak{p}$-adic and global fields, see [7].

Now that we have settled that n-rigid and super n-rigid elements coincide in the fields of number theory, we proceed to computing R^nF for F finite, $\mathfrak{p}$-adic and global field.

PROPOSITION 2.7. *Let F be a finite field (of characteristic not two). Then $R^1F = \dot{F}\setminus\dot{F}^2$ and $R^nF = \dot{F}$ for every $n \geq 2$.*

Proof. The first statement follows by inspecting the value groups of (two) forms $\langle 1, x\rangle$. Since F satisfies A_2, $R^2F = \dot{F}$ by (1.13), and the Proposition 1.16 gives the result for $n > 2$.

PROPOSITION 2.8. *Let F be a $\mathfrak{p}$-adic field (i.e., local with finite residue class field).*

(i) *If F is non-dyadic, then*

$$R^1F = \begin{cases} \dot{F}\setminus\dot{F}^2 & \text{if } -1 \in \dot{F}^2 \\ \dot{F}\setminus \pm\dot{F}^2 & \text{if } -1 \notin \dot{F}^2. \end{cases}$$

(ii) *If F is dyadic, then $R^1F = \varnothing$.*
(iii) *For any $\mathfrak{p}$-adic field F, $R^2F = \dot{F}\setminus\dot{F}^2$.*
(iv) *For any $\mathfrak{p}$-adic field F and for any integer $n \geq 3$, $R^nF = \dot{F}$.*

Proof. Assuming well-known facts about $\mathfrak{p}$-adic fields (cf. [13, Ch. VI]), we see that (i) and (ii) are routine. Since $I^3F = 0$ and the unique anisotropic 2-fold Pfister form ϕ over F satisfies $D_F(\phi') = \dot{F}\setminus\dot{F}^2$, we see that (iii) and (iv) also follows.

PROPOSITION 2.9. *Let F be a global field* (char $F \neq 2$). *Then*
(i) $R^1F = R^2F = \varnothing$.
(ii) $R^nF = \dot{F}$ *for* $n \geq 3$.

Proof. (i) Let $x \in \dot{F}$. By [16, 65:19] there exists a non-archimedean prime $\mathfrak{p}_1$ such that $x \in \dot{F}^2_{\mathfrak{p}_1}$. Let $\mathfrak{p}_2$ be another non-archimedean prime. By [16, 72:1], there exists $\phi \in P_2F$ which is anisotropic over $F_\mathfrak{p}$ for $\mathfrak{p} = \mathfrak{p}_1$ or $\mathfrak{p}_2$ but hyperbolic over all other localizations $F_\mathfrak{p}$. In particular, $\langle 1, x\rangle\phi$ is torsion in torsion-free I^3F hence hyperbolic. Now $-1 \notin D_{F_{\mathfrak{p}_1}}(\phi')$, so $\phi \notin \langle 1, -x\rangle IF$. This proves (i), since $R^1F \subseteq R^2F$.

(ii) If $\phi \in P_n F$, $n \geq 3$, then $\phi = 2\gamma$, $\gamma \in P_{n-1}F$ (cf. **13**, p. 172]). Suppose $\langle 1, x\rangle\phi = 0$. Then I^3F torsion-free implies that $\langle 1, x\rangle\gamma = 0$. Thus Lemma 1.5 produces y so that $\phi = 2\gamma = 2\langle\langle -x - y^2\rangle\rangle\theta = \langle\langle -x, -x - y^2\rangle\rangle\theta$, for some $\theta \in P_{n-2}F$ and (ii) is established.

REMARK (2.10). A part of the results in 2.7, 2.8 and 2.9 can be generalized to non-real n-linked fields. If F is non-real n-linked, then by [7, Cor. 2.5], $I^{n+2}F = 0$. It follows then that $R^{n+2}F = \operatorname{Sup} R^{n+2}F = \dot{F}$. In the case where F is linked and non-real, we get

$$R^nF = \dot{F} \quad \text{for every } n \geq 4.$$

3. **Abstract Witt rings.** We shall use the theory of abstract Witt rings as presented in [**15**]. For a Witt ring W we write G_W for the elementary 2-group of invertible elements in W generating additively W and IW for the ideal of even dimensional elements in W. The set of n-fold Pfister elements, i.e., those of the form $(1 + a_1) \cdot \cdots \cdot (1 + a_n)$, where $a_i \in G_W$ is denoted P_nW. For $\phi = b_1 + \cdots + b_n$ with $b_i \in G_W$ we will also use the form notation $\phi = \langle b, \ldots, b_n\rangle$ and call ϕ an n-dimensional form.

We say that $x \in G_W$ is n-rigid if

$$\text{(C)} \qquad P_nW \cap \operatorname{Ann}\langle 1, x\rangle = \langle 1, -x\rangle P_{n-1}W,$$

and $x \in G_W$ is said to be super n-rigid, if

$$\text{(D)} \qquad I^nW \cap \operatorname{Ann}\langle 1, x\rangle = \langle 1, -x\rangle I^{n-1}W.$$

The sets R^nW and $\operatorname{Sup} R^nW$ are introduced similarly as in the field case. Let us notice that the (ABCD)-Theorem 1.3 carries over to the abstract case. Proof of 1.3 requires three essential tools, all available in the abstract situation: Pure Subform Theorem (cf. [**15**, p. 45] or [**12**, Prop. 4.13]), Witt Annihilator Theorem (cf. [**15**, p. 85] or [**12**, Theorem 2.15]) and Theorem 2.1 of [**6**] (cf. [**12**, Prop. 4.13]). In fact, all the results of §1 through and including Proposition 1.18 (i) and (ii) carry over to abstract Witt rings. The remaining results require Arason-Pfister Hauptsatz (for I^nF with $n \geq 3$) or function fields which are not available in the abstract case.

Let $\mathscr{W}$ be the category of Witt rings. This category has direct products ([**15**, p. 99]) denoted by $\times$. If H is a finite elementary 2-group, then the group ring $W[H]$ is also an abstract Witt ring. An abstract Witt ring is said to be of elementary type if it is in the smallest subcategory $\mathscr{E}$ of $\mathscr{W}$ containing rings isomorphic to Witt rings of finite fields, $\mathfrak{p}$-adic fields, the reals, the complexes and closed under the direct products and finite group ring formation.

In this section we prove the following result.

THEOREM 3.1. *For every abstract Witt ring W of elementary type and every integer $n \geq 1$,*

$$R^n W = \operatorname{Sup} R^n W. \tag{3.1.1}$$

Recall that W is said to be reduced if $\operatorname{Nil} W = 0$. Since every finitely generated reduced Witt ring is of elementary type ([**15**, p. 165]) we get the following.

COROLLARY 3.2. *Every finitely generated reduced Witt ring W satifies* (3.1.1) *for every $n \geq 1$.*

COROLLARY 3.3. *If F is any pythagorean field with $\dot{F}/\dot{F}^2$ finite, then $R^n F = \operatorname{Sup} R^n F$ for every $n \geq 1$.*

Proof. This follows from 3.2 since Witt rings of pythagorean fields are reduced.

COROLLARY 3.4. *If F is any field with $|\dot{F}/\dot{F}^2| \leq 32$, then for every $n \geq 1$, $R^n F = \operatorname{Sup} R^n F$.*

Proof. From [**4**] and the earlier results cited there we know that every field F with the group of square classes of order not greater than 32 determines Witt ring of elementary type. Hence 3.1 applies.

COROLLARY 3.5. *For any field F with $|\dot{F}/\dot{F}^2| \leq 32$ and every $n \geq 1$, if F satisfies A_n, then $I^n F$ is torsion-free.*

Indeed, by 3.4, if $1 \in R^n F$, then $1 \in \operatorname{Sup} R^n F$.

To prove Theorem 3.1 we apply the usual method of handling elementary Witt rings. Thus we look at the behavior of rigid and super rigid elements under direct product and group ring operations and determine rigidities for basic indecomposable Witt rings. Putting together these will prove 3.1. The details are recorded in the following three propositions.

PROPOSITION 3.6. *Let (W_1, G_1) and (W_2, G_2) be abstract Witt rings and let $(W, G) = (W_1 \times W_2, G_1 \times G_2)$ be their product. Then for every $n \geq 1$:*

(i) $P_n W = P_n W_1 \times P_n W_2$.

(ii) $I^n W = I^n W_1 \times I^n W_2$.

(iii) $R^nW = R^nW_1 \times R^nW_2$ *for* $n \geq 2$.
(iv) $\operatorname{Sup} R^nW = \operatorname{Sup} R^nW_1 \times \operatorname{Sup} R^nW_2$ *for* $n \geq 2$.
(v) *If both* W_1 *and* W_2 *satisfy* (3.1.1), *then so does* W.

PROPOSITION 3.7. *Let* (S, G) *be an abstract Witt ring and* $\Delta = \{1, t\}$ *be a 2-element group. Let* $W = S[\Delta]$ *be the group ring. For* $n > 1$, *if* $R^{n-1}S = \operatorname{Sup} R^{n-1}S$ *and* $R^nS = \operatorname{Sup} R^nS$, *then* (3.1.1) *holds for* W.

PROPOSITION 3.8. *For every* $n \geq 1$, (3.1.1) *is satisfied for every abstract Witt ring* W *which is isomorphic to the Witt ring of a field from the following list*: $\mathbf{R}$, $\mathbf{C}$, *finite fields*, $\mathfrak{p}$*-adic fields*.

Since the result 3.8 has been established in §2, we will prove only 3.6 and 3.7.

Proof of 3.6. (i) and (ii) are immediate consequences of the definition of direct product. (iii) and (iv) are routine on using (C) and (D) and the following result on annihilators. If $q = (q_1, q_2) \in W = W_1 \times W_2$, then

$$\operatorname{Ann}_W q = \operatorname{Ann}_{W_1} q_1 \times \operatorname{Ann}_{W_2} q_2.$$

Here one inclusion is trivial and to prove " $\supseteq$ " one uses the fact that Witt ring does not contain zero divisors of odd dimension. Now (v) follows from (iii) and (iv).

Now we proceed to the case of group rings. So let (S, G_S) be a Witt ring and $\Delta = \{1, t\}$ be a 2-element group. Then $W = S[\Delta]$ is also a Witt ring and $G_W = G_S \times \Delta = G_S \cup tG_S$. In order to prove 3.7 we need the following lemma determining n-rigid elements in $W = S[\Delta]$.

LEMMA 3.9. *For every* $n > 1$, $R^nW = R^{n-1}S \cup tG_S$.

Proof. Since $tG_S \subseteq R^1W \subseteq R^nW$, we need only to prove that

$$R^nW \cap G_S = R^{n-1}S. \tag{3.9.1}$$

So let $x \in R^{n-1}S$ and let $\phi \in P_nW \cap \operatorname{Ann}\langle 1, x\rangle$. If $\phi \in P_nS$, we get $\phi \in \langle 1, -x\rangle P_{n-1}S$, by 1.16. If $\phi \notin P_nS$, we may write $\phi = \langle 1, yt\rangle\sigma$, $y \in G_S$, $\sigma \in P_{n-1}S$. Now $\langle 1, x\rangle\phi = 0$ implies $\langle 1, x\rangle\sigma = 0$ and it follows that

$$\phi = \langle 1, yt\rangle\sigma \in \langle 1, yt\rangle\langle 1, -x\rangle P_{n-2}S \subseteq \langle 1, -x\rangle P_{n-1}W.$$

Conversely, if $x \in R^nW \cap G_S$ and $\phi \in P_{n-1}S \cap \operatorname{Ann}\langle 1, x\rangle$ then $\langle 1, t\rangle\phi = \langle 1, -x\rangle\theta$ for some $\theta \in P_{n-1}W$ by hypothesis. If $\theta \in P_{n-1}S$, then $\phi = 0$ and we are done. Otherwise $\langle 1, -x\rangle\theta = \langle\langle a, -x\rangle\rangle\sigma - a\langle\langle t, -x\rangle\rangle\sigma$ for some $a \in G_S$ and $\sigma \in P_{n-2}S$ and it follows $\phi = \langle 1, -x\rangle\sigma$, as required.

Now we are ready to prove 3.7. So assume $n > 1$, $R^{n-i}S = \operatorname{Sup} R^{n-i}S$, $i = 0, 1$ and $x \in R^nW$. Let $q \in I^nW \cap \operatorname{Ann}\langle 1, x\rangle$. By [15, p. 113],

$$I^nW = I^nS \oplus \langle 1, -t\rangle I^{n-1}S,$$

hence $q = q_1 + \langle 1, -t\rangle q_2$, where $q_1 \in I^nS$ and $q_2 \in I^{n-1}S$. If $x \in G_S$, then $\langle 1, x\rangle q_1 = 0 = \langle 1, x\rangle q_2$ and we finish by (3.9.1), since $R^{n-1}S \subseteq R^nS$. So assume $x = ty$, $y \in G_S$. Now $\langle\langle -t, x\rangle\rangle = \langle\langle -y, x\rangle\rangle$, so

$$0 = \langle 1, x\rangle q = \langle 1, x\rangle q_1 + \langle\langle x, -y\rangle\rangle q_2 = \langle 1, x\rangle (q_1 + \langle\langle -y\rangle\rangle q_2).$$

But $q_1 + \langle\langle -y\rangle\rangle q_2 \in S$, so $q_1 + \langle\langle -y\rangle\rangle q_2 = 0$ and $q = y\langle\langle -yt\rangle\rangle q_2 \in \langle\langle -x\rangle\rangle I^{n-1}S$. This finishes the proof of 3.7.

Proof of Theorem 3.1. We induct on n. For $n = 1$ the result holds for every abstract Witt ring W (cf. (1.7)). So let $n > 1$. We prove (3.1.1) by induction on $|G_W|$. If $|G_W| \le 2$, then by Proposition 3.8 the result is true for every n. If W is an elementary Witt ring and $|G_W| > 2$, then either

(1) W is basic indecomposable, or
(2) $W = W_1 \times W_2$ with $|G_{W_i}| < |G_W|$, $i = 1, 2$, or
(3) $W = S[\Delta]$ with $|\Delta| = 2$.

In the first case W satisfies (3.1.1) by Proposition 3.8 (since W is the Witt ring of a $\mathfrak{p}$-adic field). In the second case, by induction on $|G_W|$, W_1 and W_2 satisfy (3.1.1), hence by Proposition 3.6 (v), so does W. In the third case, $|G_S| < |G_W|$, hence by induction on $|G_W|$ we have $R^nS = \operatorname{Sup} R^nS$ and by induction on n, $R^{n-1}S = \operatorname{Sup} R^{n-1}S$. Hence Proposition 3.7 applies and gives the result. This completes the proof of Theorem 3.1.

Remarks. (3.10). We can make the result in 3.9 complete by observing that

$$R^1W = R^1S \cup tG_S \quad \text{provided} \quad -1 \notin R^1S.$$

This combined with 3.9 leads to an interesting corollary:

$$R^2W = R^1W \quad \text{provided} \quad -1 \notin R^1S.$$

The excluded case $-1 \in R^1S$ can happen only when $|G_S| \le 2$ and it is easy to find R^1W by inspecting the value groups of binary forms over S.

(3.11) The results in 3.7, 3.9 and (3.10) apply to fields of formal power series $K = F((t))$. For example,

$$\text{if } |\dot{F}/\dot{F}^2| \ge 4, \quad \text{then } R^2K = R^1K = R^1F \cup t\dot{F}\dot{K}^2.$$

Finally, we discuss briefly the relations between n-rigid elements of an abstract Witt ring and the n-rigid elements in the sense of [**1**]. The notion of rigidity in the sense of [**1**] seems to be considerably stronger than ours. We compare the two in a case when the former is easily manageable. In what follows we use notation and terminology of [**1**].

PROPOSITION 3.12. *Let (W, G_W) be an abstract Witt ring, $-1 \neq -t \in G_W$ and $n \geq 2$. If $-t$ is n-rigid over S_0 (relative to $T_0 = 0$), then $-t$ is super n-rigid (i.e., $x = -t$ satisfies* (D)).

Proof. Let $q \in I^n W \cap \operatorname{Ann}_W\langle 1, -t\rangle$. By [**1**, Example 1.5.1], $-t$ is n-rigid over S_0 implies t is n-rigid over S_0, hence $I^n W = I^n S \oplus \langle 1, t\rangle I^{n-1} S$, by [**1**, Def. 1.4(ii)]. Thus $q = q + \langle 1, t\rangle q_2$ with some $q_1 \in I^n S$ and $q_2 \in I^{n-1} S$. It follows $0 = \langle 1, -t\rangle q = \langle 1, -t\rangle q_1$. Thus $q_1 \in I^n S \cap \operatorname{Ann}_W\langle 1, -t\rangle = 0$, by [**1**, Def. 1.4(iii)], and we get $q = \langle 1, t\rangle q_2 \in \langle 1, t\rangle I^{n-1} W$. This proves $-t \in \operatorname{Sup} R^n W$.

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