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FIXED POINTS OF S^1-FIBRATIONS

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Let M be a S^1-fibration over a space B and $f\colon M \to M$ a map over B. We give some results when f can be deformed over B to a fixed point free map. When the fibration is principal then we compute $\check{H}^{n-1}(\mathrm{Fix}(f), k)$ where $n = \dim M$ and we find g homotopic to f over B which minimize the fixed points.

Introduction. In [1] or [2], A. Dold defined a fixed point index for fibre-preserving maps, i.e. for every map $f\colon U \subset E \to E$ which commutes with the projection $p\colon E \to B$ he defines an index $I(f)$ s.t. $I(f) \neq 0$ implies that every map g homotopic to f through a fibre-preserving homotopy has at least one fixed point. (We call fibre-preserving homotopy a homotopy over B). From [1] one can see that this index is not easy to compute even in the case where the fibration is

$$S^1 \times S^1 \xrightarrow{p} S^1,$$

p is the projection in the first coordinate and $f(x, y) = (x, xy)$. The purpose of this paper is to study the fixed point of a fibre-preserving map $f\colon M \to M$ where M is a S^1-fibration over a space B and M, B are compact manifolds without boundary.

The paper is divided in 3 parts: In Part I we give a criterion, in terms of the fundamental group, for f to be deformed over B to a fixed point free map. This is Proposition 1.3. Some corollaries of this result are given. In Part II we look at orientable S^1-fibrations. We give a lower bound for the number of Nielsen classes over B of f as well as the topological dimension of this class. This is Theorem 2.5.

In Part III we state the question of realizing a homotopy class over B by a map f s.t. $\mathrm{Fix}(f)$ = the set of fixed points of f is minimal in the sense we will describe. We will answer this question in the case where the fibration and the total space are orientable. This is Theorem 3.7.

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Part I. Detecting fixed points. Let

$$\begin{array}{ccc} S^1 & \to & M \\ & & \downarrow \\ & & B \end{array}$$

be a S^1-fibration over B where M, B are compact manifolds without boundary and $f\colon M \to M$ be a map over B i.e. $p \circ f = p$. For the study of the category of spaces over B see [**1**] and [**2**]. By a deformation over B we mean a fibre-preserving homotopy. Let $M \times_B M$ be the fibre square which we denote by $S(M)$ and Δ the diagonal in $S(M)$. Now we will recall Proposition 2.4 of [**3**].

PROPOSITION 1.1. *The map f can be deformed over B to a fixed point free map if and only if there is a map $h\colon M \to S(M) - \Delta$ which makes the diagram below commutative up to homotopy.*

$$\begin{tikzcd} & S(M) - \Delta \arrow[d, hook, "i"] \\ M \arrow[ur, "h"] \arrow[r, "(1,f)"'] & S(M) \end{tikzcd}$$

In [**3**], they consider fibrations $F \to M \to B$ where F is a manifold of dimension greater than or equal to three. Under this hypothesis they show that the homotopy fibre of the inclusion $i\colon M \times M - \Delta \to M \times M$ is at least 1-connected. Therefore by general obstruction theory we can always lift the map $(1, f)$ over the 2-skeleton of M and the obstructions to lift over the higher dimensional skeletons are cohomology classes. On the other hand if F is the circle S^1 or a 2-dimensional surface, different from S^2 or RP^2, the obstructions to lift $(1, f)$ are no longer cohomology classes. In these cases the problem of lifting $(1, f)$ can be treated in terms of Π_1. The case where F is a 2-dimensional surface is much more complicated than the case where $F = S^1$. We return to the case $F = S^1$.

Let $x_0 \in M$ be a base point of M and let us assume that $f(x_0) \neq x_0$. Denote $(x_0, f(x_0))$ the base point of $S(M) - \Delta$ and $S(M)$.

PROPOSITION 1.2. *The map h exists if and only if*

$$(1, f)_\#(\Pi_1(M, x_0)) = i_\#(\Pi_1(S(M) - \Delta; (x_0, f(x_0)))).$$

Proof. Let $\overline{M}$ be the covering space of $S(M)$ which corresponds to the subgroup $(1, f)_\#(\Pi_1(M, x_0))$. So we have the commutative diagram:

$$\begin{tikzcd} & & \overline{M} \arrow[d] \\ & S(M) - \Delta \arrow[dr, hook, "i"] & \\ M \arrow[uurr, bend left, "\bar{f}"] \arrow[rr, "(1,f)"'] & & S(M) \end{tikzcd}$$

where $\bar{f}$ is a lifting of $(1, f)$ which exists by elementary properties of covering spaces.

Now let us assume that

$$(1, f)_{\#}(\Pi_1(M, x_0)) = i_{\#}(\Pi_1(S(M) - \Delta; (x_0, f(x_0)))).$$

Then there is a map j: $S(M) - \Delta \to \overline{M}$ which is a lifting of i. By Proposition 2.1. of [3] we have

$$\Pi_i(S(M), S(M) - \Delta) \approx \Pi_i(S^1, S^1 - y_0) = 0, \qquad i > 1.$$

So j induces isomorphisms in all homotopy groups. Since $S(M) - \Delta$ and $\overline{M}$ are CW-complexes, there exists l: $\overline{M} \to S(M) - \Delta$ which is a homotopy inverse of j. Take $h = l \circ \bar{f}$.

Now suppose that h exists. Since $i \circ h$ is homotopic to $(1, f)$, it follows that

$$(1, f)_{\#}(\Pi_1(M, x_0)) \subset i_{\#}(\Pi_1(S(M) - \Delta, (x_0, f(x_0)))).$$

Let p_1: $S(M) \to M$ be the projection on the first coordinate. We have the fibration

$$\begin{array}{ccc} S^1 - \{x_0\} & \to & S(M) \\ & & \downarrow \\ & & M \end{array}$$

Therefore p_1: $\Pi_1(S(M)) \to \Pi_1(M)$ is an isomorphism. Since $p_1 \circ h \approx \mathrm{id}$ we have that $h_{\#}$ is an isomorphism and the equality $(1, f)_{\#}(\Pi_1(M, x_0)) = i_{\#}(\Pi_1(S(M) - \Delta, (x_0, f(x_0))))$ follows.

PROPOSITION 1.3. *A map f can be deformed over B to a fixed point free map if and only if*

$$(1, f)_{\#}(\Pi_1(M, x_0)) = i_{\#}(\Pi_1(S(M) - \Delta, (x_0, f(x_0)))).$$

Proof. This follows directly from Proposition 1.1. and 1.2.

Let us consider a fibre preserving map A: $M \to M$ whose restriction to each fibre is the antipodal map. Such a map exists because $M \to B$ is a locally trivial S^1-fibration. Without loss of generality let us assume that $f(x_0) = A(x_0)$.

PROPOSITION 1.4. *If* $\mathrm{im}(i_{\#}) = \mathrm{im}(1, f)_{\#}$ *then* $A_{\#} = f_{\#}$. *Conversely if* $\Pi_1(S^1) \to \Pi_1(M)$ *is injective or surjective then* $A_{\#} = f_{\#}$ *implies* $\mathrm{im}(i_{\#}) = \mathrm{im}(1, f)_{\#}$.

Proof. The map $(1, A)$: $M \to S(M)$ is a left inverse of p_1. Since $\Pi_1(S(M) - \Delta) \to \Pi_1(M)$ is an isomorphism, (see the proof of Proposition 2.2) it follows that

$$(1, A)_{\#}\Pi_1(M) \to \Pi_1(S(M) - \Delta)$$

is an isomorphism. Therefore $\operatorname{im}(i_{\#}) = \operatorname{im}(i \circ (1, A))_{\#}$. But

$$\operatorname{im}(i \circ (1, A))_{\#} = \operatorname{im}(1, f)_{\#}$$

is equivalent to $(i \circ (1, A))_{\#}(\alpha) = (1, f)_{\#}(\alpha)$ for every $\alpha \in \Pi_1(M)$. This implies $A_{\#}(\alpha) = f_{\#}(\alpha)$.

For the second part let us assume first that

$$\Pi_1(S^1) \to \Pi_1(M)$$

is injective. From the diagram below

$$\begin{tikzcd} \Pi_1(S^1) \arrow[d, "j_2"] & \Pi_1(S^1) \arrow[d, "j"] \\ \Pi_1(S(M)) \arrow[r, "p_2"] \arrow[d, "p_1"] & \Pi_1(M) \arrow[d, "p_1"] \\ \Pi_1(M) \arrow[r, "p"] & \Pi_1(B) \end{tikzcd}$$

we have

$$p_{1\#} \circ (1, A)_{\#}(\alpha) = p_{1\#}(1, f)_{\#}(\alpha) \Rightarrow [(1, A)_{\#}(\alpha) - (1, f_{\#}(\alpha)] = j_{2\#}(\beta)$$

for some $\beta \in \Pi_1(S^1)$. So

$$p_{2\#}[(1, A)_{\#}(\alpha) - (1, f)_{\#}(\alpha)] = A_{\#}(\alpha) - f_{\#}(\alpha) = j_{\#}(\beta) = 0.$$

Since $j_{\#}$ is injective we have $\beta = 0$. Therefore $(1, A)_{\#}(\alpha) = (1, f)_{\#}(\alpha)$ and the result follows. Finally let us assume that $j_{\#}$: $\Pi_1(S^1) \to \Pi_1(M)$ is surjective. We have the diagram:

$$\begin{tikzcd} S^1 \arrow[r, shift left, "(1, A|_{S^1})"] \arrow[r, shift right, "(1, f|_{Su1})"'] \arrow[d, "j"'] & S^1 \times S^1 \arrow[d, "j_1 \times j_2"] \\ M \arrow[r, shift left, "(1, A)"] \arrow[r, shift right, "(1, f)"'] & S(M) \end{tikzcd}$$

From the fact that $f_{\#} = A_{\#}$ and using the long exact sequence in homotopy of the fibration $S^1 \to M \to B$ we have that $(A|_{S^1})_{\#} = (f|_{S^1})_{\#}$. Given $\alpha \in \Pi_1(M)$, there exists $\beta \in \Pi_1(S^1)$ s.t. $j_{\#}(\beta) = \alpha$. So we have

$$\begin{aligned}(1, A)_{\#}(\alpha) + (1, f)_{\#}(\alpha) &= (1, A)_{\#}j_{\#}(\alpha) - (1, f)_{\#}j_{\#}(\alpha) \\ &= (j_1 \times j_2)_{\#}(1, A|_{S^1})_{\#}(\beta) - (1, f|_{S^1})(\alpha) = 0\end{aligned}$$

and the result follows.

COROLLARY 1.5. *Let $S^1 \to K \to S^1$ be the S^1-fibration where K is the Klein bottle. Then the 1_K: $K \to K$ 1_K = identity map cannot be deformed over B to a fixed point free map.*

COROLLARY 1.6. *Let*

$$\begin{tikzcd} f\colon B \times S^1 \arrow[rr] \arrow[dr, "p_1"'] & & B \times S^1 \arrow[dl, "p_1"] \\ & B & \end{tikzcd}$$

be a fibre-preserving map. Then f can be deformed over B to a fixed point free map if and only if $f = (1, g)$ where g: $B \times S^1 \to S^1$ is homotopic to p_2: $B \times S^1 \to S^1$ defined by $p_2(x, y) = y$.

Proof. The "if" part is clear. So let us assume that f can be deformed over B to a fixed point free map. By Proposition 1.4. we have $f_\# = A_\#$ and therefore $p_{2\#}f_\# = p_{2\#}A_\#$ or $(p_2 \circ f)_\# = (p_2 \circ A)_\#$. But this means that

$$(p_2 \circ f)^* = (p_2 \circ A)^*\colon H^1(S^1) \to H^1(B \times S^1)$$

and consequently $p_2 \circ f$ is homotopic to $p_2 \circ A$ which is homotopic to p_2. So the result follows.

REMARK. In general $f_\# = A_\#$ does not imply $\operatorname{im}(1, f)_\# = \operatorname{im}(i)_\#$. We can construct a counter-example with $B = S^1 \times S^1 \times S^2$ and the fibration is the induced fibration from the universal S^1-fibration by the map g: $S^1 \times S^1 \times S^2 \to K(Z, 2)$ which is represented by $\alpha_2 \otimes 1 + 2 \otimes \beta_2 \in H^2(S^\times S^1 \times S^2)$, α_2, β_2 being generators of $H^2(S^1 \times S^1)$, $H^2(S^2)$ respectively.

Part II. The homology of the fixed points set. We will start by recalling some results of [**4**].

Let $x, y \in \operatorname{Fix}(f)$, where f is a fibre-preserving map and $S^1 \to M \to B$ is a S^1-fibration. We say that x is equivalent to y over B if there is a path λ: $[0,1] \to M$ s.t. $\lambda(0) = x$, $\lambda(1) = y$ and λ is homotopic to $f(\lambda)$ rel$\{x, y\}$ over B.

DEFINITION 2.1. The equivalence classes are called the Nielsen classes of f over B.

PROPOSITION 2.2. *If M is compact then the number of Nielsen classes over B is finite.*

Proof. This is Lemma 2.1 of [**4**].

In [**4**] he also defines essential Nielsen classes and the Nielsen number of f. Now we will define a lower bound for the number of non-empty Nielsen classes of f over B for the case of a principal S^1-fibration. I believe it would be interesting to compare this number with the Nielsen number as defined in [**4**].

Let $S^1 \to M \to B$ be an orientable S^1-fibration and Θ: $S^1 \times M \to M$ the S^1-action. Given f: $M \to M$ a fibre-preserving map, there is a map θ_f: $M \to S^1$, which satisfies the equation $f(x) = \theta_f(x) \cdot x$, where $\theta_f(x) \cdot x$ means $\Theta(\theta_f(x), x)$.

PROPOSITION 2.3. *Given an orientable fibration and a map f then* $\mathrm{Fix}(f) = \theta_f^{-1}(1)$ *where* $1 \in S^1$.

Proof. Obvious.

Let $i(f)$ denote the number of elements of the group

$$\Pi_1(S^1)/\theta_{f\#}(\Pi_1(M)).$$

PROPOSITION 2.4. *If $i(f) = \infty$ then f can be deformed over B to a fixed point free map.*

Proof. If $i(f) = \infty$ then $\theta_{f\#}$ is the constant map. Therefore θ_f is homotopic to the constant map equal to $-1 \in S^1$. Therefore f is homotopic over B to the antipodal map A.

THEOREM 2.5. *Let $i(f) = r < \infty$. If g is homotopic to f over B then there exist at least r Nielsen classes $F_1, \ldots, F_r$ such that $\check{H}^{n-1}(F_i, K) \neq 0$ (Čech cohomology) where K is Z or $\mathbf{Z}_2$ depending on whether M is an orientable or a non-orientable n-dimensional compact manifold.*

Proof. Let us first assume that $r = 1$. Then we have the following commutative diagram

$$\begin{array}{ccccccc}
H_1(M) & \to & H_1(M, M - \mathrm{Fix}(f)) & \to & H_0(M - \mathrm{Fix}(f)) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H_1(S^1) & \to & H_1(S^1, S^1 - \{1\}) & \to & H_0(S^1 - \{1\}) & \to & \tilde{H}_0(S^1)
\end{array}$$

where the coefficients are in $\mathbf{Z}$ or $\mathbf{Z}_2$. Since $H_1(M) \to H_1(S^1)$ is surjective then $H_1(M, M - \mathrm{Fix}(f)) \neq 0$. So by Poincaré Duality it follows that $\check{H}^{n-1}(\mathrm{Fix}(f)) \neq 0$ and the result follows.

Now let $i(f) = r$ and $S^1 \xrightarrow{p_r} S^1$ be the r-fold covering map. There is a lifting $\overline{\Theta}_f$: $M \to S^1$ where

$$\operatorname{Fix}(f) = \bigcup_{k=0}^{r-1} \overline{\Theta}_f^{-1}(e^{2\Pi i K/r})$$

and we have that $\overline{\Theta}_{f\#}$: $\Pi_1(M) \to \Pi_1(S^1)$ is surjective. By the case $r = 1$ we have that

$$\check{H}^{n-1}\left(\overline{\Theta}_f^{-1}(e^{2\Pi i K'/r})\right) \neq 0.$$

It is easy to see that

$$x \in \overline{\Theta}_f^{-1}(e^{2\Pi i K/r}), \qquad y \in \Theta_f^{-1}(e^{2\Pi i K'/r})$$

and $K \neq K'$ then x and y do not belong to the same Nielsen class. Therefore the result follows.

Remark. (1) The Proposition 2.2. and Theorem 2.5. suggest what should be a function $g \in [f]$ over B s.t. $\operatorname{Fix}(g)$ is minimal. The definition will be given in Part III.

(2) I have not been able to extend the definition of this lower bound for non-orientable fibrations.

Part III. The realization problem. From now on let us assume that $S^1 \to M \to B$ is a principal S^1-fibration and M is a compact orientable n-manifold. Let f: $M \to M$ be a fibre-preserving map.

Definition 3.1. We say that $\operatorname{Fix}(g)$ is minimal, where g is homotopic to f over B, if $\operatorname{Fix}(g)$ is an $n-1$-submanifold with $i(f)$ connected components.

Proposition 3.2. *Given f: $M \to M$ we can find g homotopic to f over B such that* $\operatorname{Fix}(g)$ *is an $n - 1$-submanifold.*

Proof. Let θ_f: $M \to S^1$ be as defined in Part II. Now we can deform θ_f to a map $\overline{\Theta}$: $M \to S^1$ such that $1 \in S^1$ is a regular value. Therefore $g(x) = \overline{\Theta}(x)$. x is homotopic to f over B and $\operatorname{Fix}(g) = \overline{\Theta}^{-1}(1)$ is an $n - 1$-submanifold.

Proposition 3.3. *The homology class* $[\overline{\Theta}^{-1}(1)]$ *represented by the submanifolds* $\overline{\Theta}^{-1}(1)$ *is the Poincaré dual of the* 1*-dimensional cohomology class* $\overline{\Theta}^*(i_1)$ *where* $\overline{\Theta}^*$: $H^1(S^1, Z) \to H^1(M, \mathbf{Z})$ *and* i_1 *is the generator of* $H^1(S^1, Z)$.

Proof. See [**6**].

Recall that $H^1(M, \mathbf{Z})$ is a free abelian group and $H_{n-1}(M, \mathbf{Z})$ is also a free abelian group by Poincaré Duality.

PROPOSITION 3.4. *If* $\overline{\Theta}_{\#}\colon \pi_1(M) \to \pi_1(S^1)$ *is surjective then* $\overline{\Theta}^*(i_1)$ *is indivisible.*

Proof. Let $\overline{\Theta}^*(i_1) = \lambda\alpha$, $\lambda \in \mathbf{R}$, $\alpha \in H^1(M, \mathbf{Z})$.
So

$$\left\langle i_1, h\overline{\Theta}_{\#}(x)\right\rangle = \left\langle\overline{\Theta}^*(i_1), \quad h(x)\right\rangle = \langle\lambda\alpha, h(x)\rangle = \lambda\langle\alpha, h(x)\rangle$$

where h is the Hurewicz homomorphism, $x \in \Pi_1(M)$ and $\langle\ ,\ \rangle$ is the evaluation. Therefore $\operatorname{im}\overline{\Theta}_{\#} \subset \lambda \cdot \mathbf{Z}$. Since $\overline{\Theta}_{\#}$ is surjective we have $\lambda = 1$ and the result follows.

PROPOSITION 3.5 (*D. Sullivan*). *Given an indivisible homology class of* $H_{n-1}(M, \mathbf{Z})$ *then it can be represented by a connected* $n-1$*-submanifold.*

Proof. See [**5**] or the appendix.

THEOREM 3.6. *If* $\theta_f\colon \Pi_1(M) \to \Pi_1(S^1)$ *is surjective then* f *can be deformed over* B *to a map* g *s.t.* Fix(g) *is a connected* $n-1$*-submanifold.*

Proof. Since $\theta_{f\#}\colon \pi_1(M) \to \pi_1(S^1)$ is surjective, by Proposition 3.4. θ_f defines an $n-1$-homology class of M which is indivisible. By Proposition 3.5 there is a connected $n-1$-submanifold N which represents this class. Now let us take a tubular neighborhood of this submanifold. This neighborhood is homeomorphic to $N \times (-\varepsilon, \varepsilon)$. Then we define $\overline{\Theta}\colon M \to S^1$ such that $\overline{\Theta}^{-1}(1) = N$ and 1 is a regular value of $\overline{\Theta}$. By Proposition 3.3 $\overline{\Theta}$ is homotopic to θ_f and $g(x) = \overline{\Theta}(x) \cdot x$ is a function such as we are looking for.

Finally the main result.

THEOREM 3.7. *Given* $f\colon M \to M$ *there is a map* g *homotopic to* f *over* B *such that* Fix(f) *is minimal.*

Proof. Let $\tilde{\Theta}_f\colon M \to S^1$ be a lifting of θ_f i.e. $p_r \circ \tilde{\Theta}_f = \theta_f$ where p_r is the r-fold cover of S^1. By Theorem 3.6 $\tilde{\Theta}_f$ is homotopic to a map $\tilde{\theta}\colon M \to S^1$ such that $\tilde{\Theta}^{-1}(1)$ is a connected $n-1$-submanifold. Let $\phi\colon S^1 \to S^1$ be a diffeomorphism homotopic to the identity which sends the set $\{e^{2\Pi iK/r} \mid K = 0, 1, \ldots, r-1\}$ into a small neighbourhood of 1 whose points are regular values of $\tilde{\Theta}$. Let $\overline{\Theta} = p_r \circ \phi^{-1} \circ \tilde{\Theta}$. Then $g(x) = \overline{\Theta}(x) \cdot x$ is a function such as we are looking for.

Appendix. Now let us sketch the proof of Proposition 3.5. (This is due to Prof. D. Sullivan.)

Proof. Let M be a compact orientable manifold of dimension n and $N \subset M$ an $n-1$-compact embedded submanifold. Suppose $n \geq 3$ and N has more than 1 connected component. Call N_1, N_2 two components. Given $p \in N_1$, $q \in N_2$ there is a path λ in M such that $\lambda(0) = p$, $\lambda(1) = q$ since M is connected. We can assume that $\lambda[0,1] \cap N$ is a finite set $\{a_1, \ldots, a_t\}$ and $a_1 = p$, $a_t = q$. Let λ have the natural orientation. At each point a_i we have the intersection number of λ and N which is $+1$, -1 or 0. We can assume that the intersection number of a_i is either $+1$ or -1, otherwise we deform λ in such a way that a_i is not in the intersection. Now let us suppose that the total intersection number of λ and N is equal to zero. Then we can find 2 consecutive points, a_i, a_{i+1} such that one has intersection number $+1$ and the other has intersection number -1. Now we apply surgery, replacing two small discs around a_i, a_{i+1} by a tube around the arc from a_i to a_{i+1}. The new manifold represents the same homology class. Since $m \geq 3$ the following fact is true: if a_i, a_{i+1} belong to the same component of N then the new submanifold has the same number of components as N, otherwise the number of components decreases by one. Because the total intersection number is zero we can continue this process and end up with a submanifold N' with less components than N. If N' is not connected we apply the above procedure again until we get a connected submanifold.

Now let me show that it is always possible to connect one point of N_1 to a point of N_2 by a curve which has total intersection number zero. Let $p \in N_1$, $q \in N_2$ and λ a curve from p to q such that $\lambda[0,1] \cap N$ is finite. Call r the intersection number of λ and the submanifold N. Since $[N] \in H_{m-1}(M, \mathbf{Z})$ is indivisible by Poincaré Duality we can pass by g an embedded circle ϕ: $[0,1] \to M$ $\phi(0) = \phi(1) = g$ which has total intersection number $+1$ with N. Given a number s let $s \cdot \phi = \underbrace{\phi * \cdots * \phi}_{s}$ where $*$ is the composition of paths. Let T be a tubular neighborhood of $\phi[0,1]$. Since M is orientable then $T \approx D^{n-1} \times S^1$ where D^{n-1} is the $n-1$-disc. Now we can deform $s\phi$ to $\overline{\phi}_s$ in such a way that $\overline{\phi}_s([0,1])$ is an embedded circle. Finally let ϕ_s' be a small deformation of $\overline{\phi}_s$ such that $\phi_s'(1) = g' \neq g$ and $g' \in N_2$ and near g. Now consider the following curve $\lambda * \phi_s'$. Call $I_\lambda(g)$ and $I_{\phi_s'}(g)$ the intersection numbers of g as points of λ and ϕ_s'

respectively. If $I_\lambda(g) = -I_{\phi_s'}(g)$ then let $s = r - I_\lambda(g)$. If $I_\lambda(g) = I_{\phi_s'}(g)$ let $s = r$. Then we have that the total intersection number of $\lambda * \phi_s'$ is zero.

Now let $m = 2$. The fact that, in this case, an indivisible homology class can be represented by an embedded circle is classical and was known by Poincaré.

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