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# 4-FIELDS ON $(4k + 2)$-DIMENSIONAL MANIFOLDS

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**Let $M$ be a closed, connected, smooth and 2-connected** mod 2 **(i.e., $H_i(M, \mathbf{Z}_2) = 0$, $0 < i \leq 2$) manifold of dimension $n = 4k + 2$ with $k > 1$. We obtain some necessary and sufficient conditions for the span of an $n$-plane bundle $\eta$ over $M$ to be greater than or equal to 4. For instance for $k$ odd span $M \geq 4$ if and only if $\chi(M) = 0$. Some applications to immersion are given. In particular if $n = 2 + 2^l$, $l \geq 3$ and $w_4(M) = 0$ then $M$ immerses in $\mathbf{R}^{2n-4}$.**

**1. Introduction.** Let $M$ be a smooth manifold, assumed throughout the paper to be closed and connected and of dimension $n = 4k + 2$ with $k > 1$.

If $k > 2$ and $M$ is $(t - 2)$-connected mod 2 where $t = 5$ or 6, then Thomas in [**20**] gave necessary and sufficient conditions for span $M \geq t$. We shall give necessary and sufficient conditions for a 2-connected mod $2M$ to have span $\geq 4$.

*The Main Result.* Recall the Euler-Poincaré characteristic of $M$ is given by

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \operatorname{Rank} H_i(M; \mathbf{Z}),$$

where $n = \dim M = 4k + 2$. We state our main theorem as follows:

THEOREM 1.1. *Suppose $M$ is 2-connected* mod 2 *and* $\dim M = n \equiv 2 \bmod 4$ *and* $n \geq 10$.

(a) *If* $n \equiv 6 \bmod 8$ *then* $\operatorname{span}(M) \geq 4$ *if, and only if* $\chi(M) = 0$.

(b) *If* $n \equiv 10 \bmod 16$ *and* $w_4(M) = 0$ *then* $\operatorname{span}(M) \geq 4$ *if, and only if* $\chi(M) = 0$.

(c) *If* $n \equiv 2 \bmod 16$ *and* $w_4(M) = 0$ *then* $\operatorname{span}(M) \geq 4$ *if, and only if* $\delta w_{n-4}(M) = 0$ *and* $\chi(M) = 0$.

In Theorem 1.1 $\delta$ is the co-boundary operator associated with the sequence $0 \to Z \to Z \to Z_2 \to 0$.

*Notation.* Let $\mathrm{BSpin}_j$ be the classifying space of orientable $j$-plane bundles $\xi$ satisfying $w_2(\xi) = 0$. Let $\widetilde{\mathrm{BSO}_j}\langle 8\rangle$ (cf. [**13**]) be the classifying space for orientable $j$-plane bundles $\xi$ satisfying $w_2(\xi) = w_4(\xi) = 0$. Then

$\widetilde{\mathrm{BSO}_j}\langle 8\rangle$ fibres over $\mathrm{BSpin}_j$ with $k$-invariant $w_4 \in H^4(\mathrm{BSpin}_j;\ \mathbf{Z}_2)$. Throughout the remainder of the paper cohomology would be ordinary cohomology with coefficients in the mod 2 integers unless otherwise specified. We denote Eilenberg-MacLane spaces of type $(\mathbf{Z}_2, j)$ and $(\mathbf{Z}, j)$ by $K_j$ and $K_j^*$ respectively and their fundamental classes by $\iota_j$ and $\iota_j^*$ respectively.

**2. The $n$-MPT for the fibration $\pi$: $\mathrm{BSpin}_{n-4} \to \mathrm{BSpin}_n$.** We list the $k$-invariants for the modified Postnikov tower for the fibration $\pi$: $\mathrm{BSpin}_{n-4} \to \mathrm{BSpin}_n$ through dimension $n$ (abbreviated $n$-MPT see [**4**]). For the computation the reader can refer to Thomas [**17**]. Because of the fact that the indeterminacy $\mathrm{Indet}^n(k_3^2, M)$ is trivial, although our choice of $k_2^2$ and $k_3^2$ for $n \equiv 2 \bmod 8$ are not independent $k$-invariants, it does not affect our computation. Note that $\binom{n-4}{4} \equiv 1 \bmod 2 \Leftrightarrow (Sq^4 + w_4\,\cdot\,)w_{n-4} = w_n$.

TABLE 1. $k$ invariant for $\pi$

| | $k$-invariant | Dim | Defining relation |
|---|---|---|---|
| Stage 1 | $k_1^1$ | $n-3$ | $k_1^1 = \delta w_{n-4}$ |
| | $k_2^1$ | $n-2$ | $k_2^1 = w_{n-2}$ |
| Stage 2 | $k_1^2$ | $n-2$ | $Sq^2k_1^1 + Sq^1k_2^1 = 0$ |
| | $k_2^2$ | $n$ | $(Sq^4 + w_4)k_1^1 + \binom{n-4}{4}Sq^3k_2^1 = 0$ |
| | $k_3^2$ | $n$ | $(\delta Sq^2)k_2^1 = 0$ |
| Stage 3 | $k^3$ | $n$ | $Sq^2Sq^1k_1^2 + Sq^1k_2^2 = 0.$ |

We shall denote the $n$-MPT by

$$\begin{tikzcd}
 & & \mathrm{BSpin}_{n-4} \arrow[dll, "q_2"'] \arrow[dl, "q_1"] \arrow[d, "\pi"] \\
E_2 \arrow[r, "p_2"] & E_1 \arrow[r, "p_1"] & \mathrm{BSpin}_n
\end{tikzcd}$$

Since we shall be considering manifolds which are 2-connected mod 2, to realize $k_1^3$ we shall identify $(Sq^1k_1^2, k_2^2)$ in stage 2 instead of $(k_1^2, k_2^2)$. Let $E_1 \xrightarrow{p_1} \mathrm{BSpin}_n$ be the 1st stage $n$-MPT for the fibration. From the defining relation for $k_3^2$, the fact that $Sq^2w_{n-2} = w_n = \chi_n \bmod 2$ where $\chi_n$ is the Euler class for $\mathrm{BSpin}_n$, and the Peterson-Stein formula we deduce (via functional operation considerations). (See also [**6**, page 337].)

PROPOSITION 2.2.

$$k_3^2 = \frac{1}{2} p_1^* \chi_n$$

(*cf. Atiyah-Dupont* **[3]** *Theorem* 1.1 *page* 3.)

COROLLARY 2.3. *Suppose $\eta$ is an n-plane bundle over $M$. Suppose $\delta w_{n-4}(\eta) = 0$ and $w_{n-2}(\eta) = 0$. Then modulo zero indeterminacy $k_3^2(\eta) = 0$ if, and only if $\chi(\eta) = 0$, where $\chi(\eta)$ denotes the Euler class of $\eta$.*

**3. The case** $w_{n-4}(M) = 0$. Throughout this section we assume that $w_{n-4}(M) = 0$.

Consider the following relations:

$$\text{(3.1)} \quad \begin{cases} \tilde{\phi}_3\colon Sq^2Sq^2 + Sq^3\delta = 0 \quad \text{and} \\ \tilde{\phi}_4\colon \left(1 \otimes Sq^4 + \iota_4^* \otimes \rho_2\right)\delta + Sq^1\left(1 \otimes Sq^4 + \iota_4^* \otimes 1\right) \\ \qquad + \left(Sq^2Sq^1\right)Sq^2 = 0 \end{cases}$$

where $\iota_4^*$ is the fundamental class of $K(\mathbf{Z}, 4)$, $\rho_2$ is reduction mod 2, $\delta$ is the Bockstein operator associated with the exact sequence $0 \to Z \to Z \to Z_2 \to 0$. In (3.1), the tensor product is to be interpreted as for the Massey-Peterson algebra $\mathfrak{A}(K(\mathbf{Z}, 4))$ for the mod 2 steenrod algebra $\mathfrak{A}$. The multiplication for $\rho_2$ and $\delta$ is obvious. By abuse of notation and to save space we sometimes write $\alpha$ for $1 \otimes \alpha$ for $\alpha \in \overline{\mathfrak{A}} \cup \{\delta\}$. Consider the vector cohomology operation defined by (3.1). Its existence follows from the method of universal example as in Thomas **[18]**. Moreover it is easily seen that if we denote the operator by $(\tilde{\phi}_3, \tilde{\phi}_4)$ we have the following relation

$$\text{(3.2)} \qquad \Lambda_4\colon Sq^2\tilde{\phi}_3 + Sq^1\tilde{\phi}_4 = 0.$$

Hence we have a tertiary operation associated with the relation (3.2). Let us denote such an operation also by the symbol $\Lambda_4$. In the terminology of **[18]**, $(\tilde{\phi}_3, \tilde{\phi}_4)$ and $\Lambda_4$ are twisted cohomology operations.

Let $\zeta_j$: BSpin$_j \to K_4^*$ represent a generator of $H^4(\text{BSpin}_j; \mathbf{Z}) \approx \mathbf{Z}$. Then we have

THEOREM 3.3. *Let $j \geq 5$ and let $U_j$ be the Thom class of the universal spin j-plane bundle over* BSpin$_n$. *Then*

$$(0,0) \in (\tilde{\phi}_3, \tilde{\phi}_4)(U_j, \zeta_j) \quad \textit{and}$$

$$0 \in \Lambda_4(U_j, \zeta_j).$$

*Proof*. Since $H^3(\mathrm{BSpin}_j) \approx \{0\}$ and $H^4(\mathrm{BSpin}_j)$ is generated by the 4th mod 2 universal Stiefel-Whitney class $w_4$, trivially we can choose $(\tilde{\phi}_3, \tilde{\phi}_4)$ such that $(0,0) \in (\tilde{\phi}_3, \tilde{\phi}_4)(U_j, \zeta_j)$. If necessary we can replace $(\tilde{\phi}_3, \tilde{\phi}_4)$ by $(\tilde{\phi}_3, \tilde{\phi}_4 + Sq^4)$. Similarly we can choose the stable tertiary operation $\Lambda_4$ such that $0 \in \Lambda_4(U_j, \zeta_j)$.

Instead of writing $\zeta_j$, by abuse of notation we shall confuse $\zeta_j$ with the class $Q \in H^4(\mathrm{BSpin}_j; \mathbf{Z})$ which it represents. Notice that $2Q = P_1$ the first Pontrjagin class of the universal spin $j$-plane bundle over $\mathrm{BSpin}_j$.

Let $w_{n-4}$ be the $(n-4)$th mod 2 universal Stiefel-Whitney class considered as in $H^{n-4}(\mathrm{BSpin}_{n-4})$. Then $(\mathrm{Sq}^4 + Q \cdot )w_{n-4} = 0$, $Sq^2 w_{n-4} = 0$ and $\delta w_{n-4} = 0$. Thus an immediate corollary to Theorem 3.3 is

PROPOSITION 3.4.
(a) $(0,0) \in (\tilde{\phi}_3, \tilde{\phi}_4)(w_{n-2}, Q) \subset H^{n-1}(\mathrm{BSpin}_{n-4}) + H^n(\mathrm{BSpin}_{n-4})$.
(b) $0 \in \Lambda_4(w_{n-4}, Q) \subset H^n(\mathrm{BSpin}_{n-4})$.

Since $\pi^*$ maps $\mathrm{Indet}^{n-1,n}(\mathrm{BSpin}, (\tilde{\phi}_3, \tilde{\phi}_4))$ onto $\mathrm{Indet}^{n-1,n}(\mathrm{BSpin}_{n-4}, (\tilde{\phi}_3, \tilde{\phi}_4))$, $w_{n-4} \in H^{n-4}(\mathrm{BSpin}_n)$ is a generating class (see [**18**, §5]) for $(Sq^1 k_1^2, k_2^2)$. Thus by the generating class theorem [**18**, Theorem 5.9] we have

$$(3.5) \qquad \left(Sq^1 k_1^2, k_2^2\right) \in (\tilde{\phi}_3, \tilde{\phi}_4)\left(p_1^* w_{n-4}, p_1^* Q\right).$$

Consider the commutative diagram

$$\begin{array}{ccccc}
E_2 & \xrightarrow{P_1} & E_1 & \xrightarrow{(k_1^2, k_2^2, k_3^2)} & K_{n-2} \times K_n \times K_n^* \\
\downarrow f & & \downarrow \| & & \downarrow j \\
\tilde{E}_2 & \xrightarrow{\xi} & E_1 & \xrightarrow{(k_1^2, k_2^2)} & K_{n-2} \times K_n
\end{array}$$

where $j$ is the projection and $\xi$ is the principal fibration with $k$-invariant $(k_1^2, k_2^2)$ and $f$ is the natural map induced by the commutative right-hand square. Then there is a class $\tilde{k} \in H^n(\tilde{E}_2)$ associated with the relation $Sq^2 Sq^1 k_1^2 + Sq^1 k_2^2 = 0$ such that $f^* \tilde{k} = k^3$. Since $\mathrm{Ker}\, \pi^* \subset \mathrm{Ker}\, P_1^*$ in dimension $\leq n$, $q_1^*$ maps $\mathrm{Indet}^n(E_1, \Lambda_4, Q)$ onto $\mathrm{Indet}^n(\mathrm{BSpin}_{n-4}, \Lambda_4, Q)$. Thus we have by Proposition 3.4 and (3.5) the following

PROPOSITION 3.6. $w_{n-4} \in H^{n-4}(\mathrm{BSpin}_n)$ *is a generating class for* $\tilde{k}$. *Here* $\tilde{k}$ *is considered as a coset modulo* $\mathrm{Ker}\, \tilde{q}_1^* \cap \mathrm{Im}\, \xi^*$ *where* $\tilde{q}_1 = f \circ q_2$: $\mathrm{BSpin}_{n-4} \to \tilde{E}_2$.

By the connectivity condition on $M$, the $i$th Wu class is trivial unless $i \equiv 0\,(4)$. We can easily show with the help of $S$-duality that $\mathrm{Indet}^n(M, k^3) = \mathrm{Indet}^n(M, \Lambda_4, \eta^*Q)$ for any map $\eta$: $M \to \mathrm{BSpin}_n$ classifying a spin $n$-plane bundle over $M$.

PROPOSITION 3.7. *Suppose* $\eta$: $M \to \mathrm{BSpin}_n$ *is a map such that* $\eta^*(\delta w_{n-4}) = 0$, $0 \in \tilde{\phi}_4(\eta^* w_{n-4}, \eta^*(Q))$ *and* $\eta^*(\chi) = 0$, *then*

$$k^3(\eta) = \Lambda_4\big(\eta^* w_{n-4}, \eta^* Q\big).$$

*Proof*. Note that $\mathrm{Indet}^n(M, \tilde{k}) = \mathrm{Indet}^n(M, k^3)$. Since $M$ is 2-connected mod 2, $(k_1^2, k_2^2)(\eta) = (0, k_2^2)(\eta)$. Thus $(0, k_2^2)(\eta) = (0, \tilde{\phi}_4)(\eta^* w_{n-4}, \eta^* Q)$. Since $0 \in \tilde{\phi}_4(\eta^* w_{n-4}, \eta^* Q)$, $(0,0) \in (0, k_2^2)(\eta)$. Thus $\tilde{k}(\eta)$ is defined. Since $\eta^*(\chi) = 0$, then by Corollary 2.3 $k_3^2(\eta) = 0$ modulo zero indeterminacy. Therefore $k^3(\eta)$ is defined. By Proposition 3.6 and the generating class theorem, there exists an element $h$ in $H^n(E_1)$ such that $h \in \operatorname{Ker} q_1^*$ and

$$(\tilde{k} + h)(\eta) = \Lambda_4\big(\eta^* w_{n-4}, \eta^* Q\big).$$

Since $\operatorname{Ker} q_1^* \subset \operatorname{Ker} p_2^*$ through dimension $\leq n$ and $k_3^2(\eta) = 0$

$$k^3(\eta) = \big(f^*\tilde{k}\big)(\eta) = (\tilde{k} + h)(\eta) = \Lambda_4\big(\eta^* w_{n-4}, \eta^* Q\big).$$

For an $n$-plane bundle $\eta$ over $M$ with classifying map also denoted by $\eta$, let $w_j(\eta) = \eta^* w_j$ and $Q(\eta) = \eta^* Q$. We have from Proposition 3.7 the following

THEOREM 3.8. *Suppose* $\eta$ *is an n-plane bundle over M. Then* $\operatorname{span} \eta \geq 4$ *if, and only if* $\delta w_{n-4}(\eta) = 0$, $0 \in \tilde{\phi}_4(w_{n-4}(\eta), Q(\eta))$, $\chi(\eta) = 0$ *and* $0 \in \Lambda_4(w_{n-4}(\eta), Q(\eta))$

THEOREM 3.9. *Suppose M is 2-connected* mod 2 *and* $w_{n-4}(M) = 0$. *Then* $\operatorname{span}(M) \geq 4$ *if, and only if* $\chi(M) = 0$.

*Proof*. Immediate from Theorem 3.8.

**4. The case** $w_4(M) = 0$. In this section we shall assume that $w_4(M) = 0$.

Consider the following relations:

$$(4.1)\quad \begin{cases} \phi_1\colon Sq^3\big(\delta Sq^{n-4}\big) + Sq^2\big(Sq^2 Sq^{n-4}\big) = 0, \\ \phi_2\colon Sq^4\big(\delta Sq^{n-4}\big) + Sq^1\big(Sq^4 Sq^{n-4}\big) + Sq^2 Sq^1\big(Sq^2 Sq^{n-4}\big) = 0. \end{cases}$$

Choose stable secondary cohomology operation associated with $\phi_1$ and $\phi_2$ of Hughes-Thomas type [**5**], also denoted by the same symbols such that on the fundamental class $d_{n-4}$ of $D_{n-4}$, the principal bundle over $K_{n-4}$ with classifying map $(Sq^1\iota_{n-4}, Sq^2\iota_{n-4})$

$$0 \in \phi_1(d_{n-4}) \quad \text{and} \quad Sq^4 d_{n-4} \cup d_{n-4} \in \phi_2(d_{n-4}).$$

Moreover we can choose $(\phi_1, \phi_2)$ such that $(0,0) \in (\phi_1, \phi_2)(\iota_{n-5})$. By the Leray-Serre exact sequence for the universal example tower for $(\phi_1, \phi_2)$, we see that

$$\begin{aligned} \phi_1 &= \phi_3^* \circ Sq^{n-4} \text{ modulo } \{Sq^{n-1}, Sq^{n-2}Sq^1\} \quad \text{and} \\ \phi_2 &= \phi_4^* \circ Sq^{n-4} \text{ modulo } \{Sq^n, Sq^{n-1}Sq^1, Sq^{n-2}Sq^2\} \end{aligned}$$

where $\phi_3^*$ and $\phi_4^*$ are defined by the following relations

$$Sq^3\delta + Sq^2Sq^2 = 0 \quad \text{and} \quad Sq^4\delta + Sq^1Sq^4 + (Sq^2Sq^1)Sq^2 = 0.$$

Furthermore $(\phi_1, \phi_2)$ can be chosen in such a way that

$$\Omega\colon Sq^2\phi_1 + Sq^1\phi_2 = 0. \tag{4.2}$$

Consider now the fibration $\tilde{\pi}\colon \widetilde{\mathrm{BSO}}_{n-4}\langle 8\rangle \to \widetilde{\mathrm{BSO}}_n\langle 8\rangle$ where $\widetilde{\mathrm{BSO}}_j\langle 8\rangle$ is the classifying space for $n$-plane bundles $\xi$ satisfying $w_2(\xi) = w_4(\xi) = 0$. The $k$-invariants for the $n$-MPT is as defined before in Table 1. Then $(\phi_1, \phi_2)(T\tilde{\pi})^*U_n = s^4(\phi_3^*, \phi_4^*)(U_{n-4} \cup U_{n-4})$ where $s$ is the suspension homomorphism and $U_j$ is the Thom class of the universal bundle over $\widetilde{\mathrm{BSO}}_j\langle 8\rangle$. Therefore $(\phi_1, \phi_2)(T\tilde{\pi})^*U_n = 0$ modulo zero indeterminacy by a Cartan formula for $(\phi_3^*, \phi_4^*)$.

Now observe that $\tilde{\pi}^*\colon H^*(\widetilde{\mathrm{BSO}}_n\langle 8\rangle) \to H^*(\widetilde{\mathrm{BSO}}_{n-4}\langle 8\rangle)$ is an epimorphism in dimension $\leq n$ for $n \geq 30$ and $n \neq 34$. For $n < 30$ and $n = 34$ think of the $n$-MPT over $\widetilde{\mathrm{BSO}}_n\langle 8\rangle$ as the induced tower from the $n$-MPT over $\mathrm{BSO}_n$. With this in mind it can be easily verified that $(\delta w_{n-4}, w_{n-2})$ is admissible for $(Sq^1k_1^2, k_2^2)$ via $(\phi_1, \phi_2)$ [**12**, §3.2].

Let $E_2 \to E_1 \to \widetilde{\mathrm{BSO}}_n\langle 8\rangle$ be the Postnikov tower for $\tilde{\pi}$. Then by the admissible class theorem [**12**, Theorem 3.3] we have

THEOREM 4.3.

$$U(E_1)(Sq^1k_1^2, k_2^2) \in (\phi_1^*, \phi_2^*)U(E_1),$$

*where $U(E_i)$ is the Thom class of the bundle over $E_i$ induced from the universal n-plane bundle over $\widetilde{\mathrm{BSO}}_n\langle 8\rangle$ by the map $E_i \to \widetilde{\mathrm{BSO}}_n\langle 8\rangle$.*

From the relation (4.2) we can choose an operation associated with the relation (4.2) denoted by $\Omega$ such that on the fundamental class $b_{n-4}$ of $Y_{n-4}$, the principal bundle over $K_{n-4}^*$ with classifying map

$$(Sq^2\iota^*_{n-4}, Sq^4\iota^*_{n-4})$$

$$\tilde{\phi}^*_4(b_{n-4}) \cup (b_{n-4}) \in \Omega(b_{n-4}) \tag{4.4}$$

where $\tilde{\phi}^*_4$ is the secondary operation on integral classes associated with the relation

$$\tilde{\phi}^*_4\colon Sq^2Sq^3 + Sq^1Sq^4 = 0$$

and $K_j^*$ is an Eilenberg-MacLane space of type $(\mathbf{Z}, j)$ and $\iota_j^*$ its fundamental class. By the methods of **[12]** (see for example. **[12**, §4.20] we can easily derive (4.4). The details are left to the reader. Thus (4.4) and the admissible class theorem give us

THEOREM 4.5.

$$U(E_2)\cdot\left(k_1^3 + p_2^*p_1^*(w_{n-4}\theta_4)\right) \in \Omega(U(E_2)),$$

*where $\theta_4 \in H^4(\widetilde{\mathrm{BSO}_n}\langle 8\rangle)$ is defined by $\phi_4 U(\widetilde{\mathrm{BSO}_n}\langle 8\rangle) = U(\widetilde{\mathrm{BSO}_n}\langle 8\rangle)\cdot\theta_4$. Indeed by Proposition* 3.4 *of* **[12]** *treating $\widetilde{\mathrm{BSO}_n}\langle 8\rangle$ as a principal fibration over $BSO_n$ we see that $\phi_4(U(\widetilde{\mathrm{BSO}_n}\langle 8\rangle)) = U(\widetilde{\mathrm{BSO}_n}\langle 8\rangle)\cdot\theta_4$ where $\theta_4$ is such that $i^*\theta_4 = sq^1\iota_3$ where $i\colon K_3 \to \widetilde{\mathrm{BSO}_n}\langle 8\rangle$ is the inclusion of the fibre. Thus $\theta_4$ is a generator of $H^4(\widetilde{\mathrm{BSO}_n}\langle 8\rangle) \approx \mathbf{Z}_2$.*

REMARK. Notice that by a spectral sequence argument $q_1^*\colon H^*(E_1) \to H^*(\widetilde{\mathrm{BSO}_{n-4}}\langle 8\rangle)$ is an epimorphism through dimension $n$. Also

$$U(E_1)\cdot\left(\mathrm{Indet}^{n-1,n}(Sq^1k_1^2, k_2^2, E_1)\right) = \mathrm{Indet}^{2n-1,2n}(\phi_1, \phi_2, TE_1).$$

Hence we can apply the admissible class theorem.

Let $\xi$ be an $n$-plane bundle over $M$ such that $w_4(\xi) = 0$.

THEOREM 4.6. (a) *Suppose* $\mathrm{Indet}^n(k^3, M) \neq 0$. *Then* $\mathrm{span}(\xi) \geq 4$ *if, and only if $\delta w_{n-4}(\xi) = 0$, and $\chi(\xi) = 0$.*

(b) *Suppose* $\mathrm{Indet}^n(k^3, M) = 0$ *and $w_{n-4}(\xi)\theta_4(\xi) = 0$ where $\theta_4(\xi) = g^*\theta_4$, $g$ a classifying map into $\widetilde{\mathrm{BSO}_n}\langle 4\rangle$ for $\xi$. Suppose $\theta_4(\xi) = \theta_4(\nu)$, where $\nu$ is the normal bundle of $M$. Then* $\mathrm{span}(\xi) \geq 4$ *if, and only if $\delta w_{n-4}(\xi) = 0$, $\chi(\xi) = 0$, $\phi_2(U(\xi)) = 0$ and $\Omega(U(\xi)) = 0$ modulo zero indeterminacy.*

*Proof*. This follows from Theorem 4.5. The details are left to the reader.

**5. Evaluation on the manifold.** Let $g\colon M \times M \to T(M)$ be the map that collapses the complement of a tubular neighborhood of the diagonal to a point. Then let

$$\overline{U} = g^*(U(\tau)) \bmod 2 \in H^n(M \times M).$$

We want to give a decomposition of $\overline{U}$. Note that for any $x \in H^{n/2}(M)$, $x^2 = 0$. Thus $\mathbf{Z}_2$ rank of $H^{n/2}(M)$ is even. Suppose rank $H^{n/2}(M) = 2q$. Then we have the following.

PROPOSITION 5.1. *Suppose $H^{n/2}(M) \neq \{0\}$. There exists a basis $\{x_1, \ldots, x_q, y_1, \ldots, y_q\}$ for $H^{n/2}(M)$ and an integer $r \geq 0$ such that*

$$Sq^1 x_i = 0, \qquad i = 1, \ldots, q, \qquad Sq^1 y_{r+i} = 0, \qquad i = 1, \ldots, q-r,$$
$$Sq^1 y_i \neq 0, \qquad i = 1, \ldots, r,$$

*and $x_i y_j = \delta_{ij}\mu$ where $\delta_{ij}$ is the Kronecker function and $\mu \in H^n(M)$ is a generator. In particular $\{x_1, \ldots, x_r\} \subseteq Sq^1 H^{n/2-1}(M)$.*

*Proof*. First we remark that for $n = 4k + 2$ $\operatorname{Ker} Sq^1\colon H^{2k+1}(M) \to H^{2k+2}(M)$ is non-trivial unless $H^{2k+1}(M) = \{0\}$. For if $Sq^1 x \neq 0$ then for any $y \in H^{2k}(M)$ with $Sq^1 x \cdot y \neq 0$, $y$ satisfies $Sq^1 y \neq 0$ and $Sq^1 y \in H^{2k+1}(M) \cap \operatorname{Ker} Sq^1$. Choose generators

$$\{\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_{r+p}, \beta_1, \ldots, \beta_r, \beta_{r+1} \cdots \beta_{r+p}\}, \quad r + p = q$$

such that $\{\alpha_1, \ldots, \alpha_r\} \subseteq \operatorname{Im} Sq^1 \cap H^{2k+1}(M)$ and $\{\alpha_{r+1}, \ldots, \alpha_{r+p}\} \subseteq \operatorname{Cok} Sq^1 \cap \operatorname{Ker} Sq^1 \cap H^{2k+1}(M)$ and $\{\beta_1, \ldots, \beta_{r+p}\}$ are their corresponding duals (i.e. $\beta_i \cdot x = 0$ for all $x \in H^{2k+1}(M)$ and $x \neq \alpha_i$, $\beta_i \cdot \alpha_i \neq 0$). Notice this choice is possible by the above remark, for $Sq^1 x \neq 0$ and $x \in H^{2k+1}(M)$ implies that $x$ is dual to $Sq^1 y$ for some $y \in H^{2k}(M)$. Now $Sq^1 \beta_{r+i} = 0$, $1 \leq i \leq p$ for otherwise $\beta_{r+i}$ is dual to some $\alpha_i$, $1 \leq i \leq r$. Of course now letting $x_i = \alpha_i$, $y_i = \beta_i$ gives the required basis. Let

$$A = \sum_{i=0}^{2k} \sum_{l=1}^{n(i)} \alpha_i^l \otimes \beta_{n-i}^l + \sum_{i=1}^{q} x_i \otimes y_i$$

where $\dim H^i(M) = n(i)$ and $\{x_1, \ldots, x_q, y_1, \ldots, y_q\}$ are given by Proposition 4.1. Here $\alpha_i^k \cup \beta_{n-i}^j = \delta_{kj}\mu$. Then we have

THEOREM 5.2.
(i) $\overline{U} = A + tA$
(ii) $Sq^1 A = 0$
(iii) $A \cup tA = \hat{\chi}_2(M)\mu \otimes \mu$
*where*

$$\hat{\chi}_2(M) = \frac{1}{2}\left(\sum_{i=0}^{4k+2} \dim H^i(M)\right) \bmod 2 = \frac{1}{2}\chi(M) \bmod 2.$$

*Proof*. Assertion (i) follows from the fact that

$$\{\alpha_i^l, \beta_{n-i}^l\}_{i=1,\ldots,2k;l=1,\ldots,n(i)} \cup \{x_i, y_i\}_{i=1,\ldots,q}$$

is a basis for $H^*(M)$ and Milnor [**11**].

$Sq^1\overline{U} = 0$ and

$$Sq^1A = \sum_{i=0}^{2k}\left(\sum_{l=1}^{n(i)} Sq^1\alpha_i^l \otimes \beta_{n-i}^l + \alpha_i^l \otimes Sq^1\beta_{n-1}^l\right) + \sum_{i=1}^{r} x_i \otimes Sq^1y_i$$

is a sum of terms of bidegree $(j, n+1-j)$, $j \leq 2k+1$. Now $n+1-j = 4k+3-j \geq 4k+3-(2k+1) \geq 2k+2$. Therefore $Sq^1A + Sq^1tA = 0$ implies that $Sq^1A = Sq^1tA = 0$. Assertion (iii) is obvious.

PROPOSITION 5.3.
(i) $\delta Sq^{n-4}(A) = \delta w_{n-4}(M) \otimes \mu$
(ii) $Sq^4Sq^{n-4}(A) = 0$ *if* $w_4(M) = 0$.

*Proof*. (i)

$$Sq^{n-4}(A) = Sq^{4k-2}(A) = \sum_{l=1}^{n(2k)} Sq^{2k-2}\alpha_{2k}^l \otimes Sq^{2k}\beta_{2k+2}^l.$$

Now $Sq^{2k}\beta_{2k+2}^l = v_{2k} \cdot \beta_{2k+2}^l \neq 0$ if $\beta_{2k+2}^l$ is dual to $v_{2k}$ the $2k$th Wu class of $M$. We can choose for some $\alpha_{2k}^j$ to be $v_{2k}$. Thus $Sq^{2k}\beta_{2k+2}^l = 0$ for $l \neq j$. Thus

$$Sq^{n-4}(A) = Sq^{2k-2}v_{2k} \otimes \mu = w_{4k-2}(M) \otimes \mu = w_{n-4}(M) \otimes \mu,$$

and so $\delta Sq^{n-4}(A) = \delta w_{n-4}(M) \otimes \mu$.
(ii) is obvious.

PROPOSITION 5.4. *Suppose* $w_4(M) = 0$ *and* $\delta w_{n-4}(M) = 0$. *Then*
(i) $(\phi_1, \phi_2)$ *is defined on* $A$, *and*
(ii) *Modulo zero indeterminacy*,

$$(0, \phi_4^*(w_{n-4}(M) \otimes \mu)) = (\phi_1, \phi_2)(A).$$

*Hence*
(iii) $(0,0) = (\phi_1, \phi_2)(U(\tau))$.

*Proof*. Part (i) follows from 5.3. Part (iii) follows from Part (ii) since $g^*$ is injective. Note that $Sq^{n-2}Sq^2A = 0$ so that

$$\phi_2(A) = \phi_4^*Sq^{n-4}A = \phi_4^*(w_{n-4}(M) \otimes \mu).$$

Let $P \to K_n$ be a universal example tower for $(\phi_1, \phi_2)$. Consider $A$ as a map $A$: $M \times M \to K_n$. Since $\delta w_{n-4}(M) = 0$, $A$ has a lifting $\bar{A}$ to $P$. Let $m$: $P \times P \to P$ be the multiplication map. Then the map $h = m \circ (\bar{A}, \bar{A} \circ t)$ is a lifting of $A + t^*A$ regarded as a map $m \circ (A, A \circ t)$. Let $\phi$ be a representative for the operation $\phi_2$. Then $m^*\phi = 1 \otimes \phi + \phi \otimes 1$. Thus

$$h^*\phi = \bar{A}^*\phi + t^*\bar{A}^*\phi.$$

But $t^*$: $H^{2n}(M \times M) \to H^{2n}(M \times M)$ is an identity homomorphism. Therefore $h^*\phi = 0$.

Let $U$: $T(M) \to K_n$ represent the Thom class of the tangent bundle of $M$ reduced mod 2. Let $\bar{U}$: $T(\tau) \to P$ be any lifting of $U$ to $P$. Then $f = \bar{U} \circ g$ is a lifting of $A + t^*A$. Since $g^*$ is injective, $\phi_2(U(\tau))$ vanishes if and only if $g^*\phi_2(U(\tau)) = f^*(\phi) = 0$. Since $\mathrm{Indet}^{2n}(M \times M, \phi_2) = 0$, $h^*\phi = 0 \Rightarrow f^*(\phi) = 0$ since both $h$ and $f$ are liftings of $A + t^*A$. By the connectivity condition on $M$, this shows that $(\phi_1, \phi_2)(U(\tau)) = (0, 0)$. This completes the proof of Proposition 5.4.

Consider $\mathrm{Indet}^{2n}(\Omega, T(M))$. By the connectivity condition on $M$ $\mathrm{Indet}^{2n}(\Omega, T(M))$ is a sum of secondary operations defined below

$$\begin{aligned} &\mathrm{Indet}^{2n}(\Omega, T(M)) \\ &\quad = \left\{ \tilde{\phi}_4^*(x) + \zeta_3(y) \,|\, x \in H^{2n-4}(T(M); \mathbf{Z}),\ y \in H^{2n-3}(T(M)) \right\} \end{aligned}$$

where $\zeta_3$ is associated with

$$\zeta_3\colon Sq^2Sq^2 + Sq^1(Sq^2Sq^1) = 0.$$

By Atiyah-James duality the $S$-dual of $T(M)$ is the Thom space of the stable bundle $\alpha = -\tau - \tau$. Thus $\zeta_3$ is trivial on $H^{2n-3}(T(M))$. $\tilde{\phi}_4^*$ is also trivial on $H^{2n-4}(T(M);\ \mathbf{Z}))$ since $\tilde{\phi}_4^*(x) = \phi_4^*(x)$ and $\theta_4(\alpha) = 0$. Thus if $\mathrm{Indet}^n(k^3, M) = 0$ then $\mathrm{Indet}^{2n}(\Omega, T(M)) = \mathrm{Indet}^n(k^3, M) = \mathrm{Indet}^{2n}(\Omega, M \times M) = 0$.

THEOREM 5.5. *Suppose* $\delta w_{n-4}(M) = 0$ *and* $w_4(M) = 0$. *Suppose further that* $\mathrm{Indet}^n(k^3, M) = 0$. *Then*

$$\Omega(U(\tau)) = 0$$

*modulo zero indetermicacy.*

*Proof.* From Theorem 4.6 and the fact that $\mathrm{Indet}^n(k^3, M) = 0$, $\phi_4^*(w_{n-4}(M)) = 0$. Therefore $\Omega$ is defined on $A$ hence on $tA$. Thus $\Omega(A + tA) = \Omega(A) + t^*(\Omega A) = 0$ modulo zero indeterminacy.

5.6. *Proof of Theorem* 1.1.

1.1(a) follows from Theorem 3.9 since $w_{n-4}(M) = 0$ for $n \equiv 6 \bmod 8$. Similarly 1.1(b) follows from Theorem 3.9 since $n \equiv 10 \bmod 16$ and $w_4(M) = 0$ implies $w_{n-4}(M) = 0$. 1.1(c) follows from Theorem 4.6 and Theorem 5.5.

**6. Immersions of manifolds.** As an application of Theorem 3.8 and Theorem 4.6 we derive some immersion results. Note that for immersion we don't need the unstable $k$-invariants.

Suppose $M$ is a spin-manifold. Then by Massey **[9]** it can be easily shown that if $\dim m = n \equiv 2 \bmod 4$ then $\overline{w}_{n-2}(M) = 0$ and $\delta \overline{w}_{n-4}(M) = 0$. In particular if $\dim M = n \equiv 6 \bmod 8$, $\overline{w}_{n-4}(M) = 0$. Also if $\dim M = n \equiv 10 \bmod 16$ and $w_4(M) = 0$, then $\overline{w}_{n-4}(M) = 0$.

Thus using the proof of Theorem 3.8, letting $\eta$ be the stable normal bundle of $M$, we have:

THEOREM 6.1. *Suppose $M$ is 2-connected* mod 2 *and $n > 6$. If* $\dim M = n \equiv 6 \bmod 8$ *or if $n \equiv 10 \bmod 16$ and $w_4(M) = 0$, then $M$ immerses in* $R^{2n-4}$.

As an application of Theorem 4.6 bearing in mind that the condition $\chi(\xi) = 0$ does not apply to stable bundle we have:

THEOREM 6.2. *Suppose $M$ is 2-connected* mod 2, $\operatorname{Dim} M = n \equiv 2 \bmod 16$ *and $w_4(M) = 0$. Then $M$ immerses in $R^{2n-4}$.*

*Proof.* If $\operatorname{Indet}^n(k^3, M) \neq 0$, we have nothing to prove since $k^3(\nu)$ is defined and $0 \in k^3(\nu)$, where $\nu$ is the Spivak normal bundle. If $\operatorname{Indet}^n(k^3, M) = 0$, then $\tilde{\phi}_4^*$ is trivial on $H^{n-4}(M, \mathbf{Z})$. Since $\overline{w}_{n-4}(M)$ is an integral class, $\tilde{\phi}_4^*(\overline{w}_{n-4}(M)) = 0$ modulo zero indeterminacy. Therefore $\overline{w}_{n-4}(M) \cdot \theta_4(\nu) = 0$. Thus by Theorem 4.6(b) $M$ immerses in $\mathbf{R}^{2n-4}$ since $\phi_2(U(\nu)) = \Omega(U(\nu)) = 0$ being operation mapping into the top class of $T(\nu)$.

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