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CONTINUATION OF BOUNDED HOLOMORPHIC FUNCTIONS FROM CERTAIN SUBVARIETIES TO WEAKLY PSEUDOCONVEX DOMAINS

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Let D be a weakly pseudoconvex domain in C^n with C^∞-boundary and V be a subvariety in D which intersects ∂D transversally. If ∂V is nonsingular and consists of strictly pseudoconvex boundary points of D, then any bounded holomorphic function in V can be extended to a bounded holomorphic function in D.

1. Introduction. Let Ω be an open set in some complex manifold. We denote by $H^\infty(\Omega)$ the space of all bounded holomorphic functions in Ω and by $A(\Omega)$ the space of all holomorphic functions in Ω which are continuous in $\overline{\Omega}$. Let G be a bounded strictly pseudoconvex domain in C^n with C^2-boundary and $\tilde{M}$ be a submanifold in a neighborhood of $\overline{G}$ which intersects ∂G transversally. Let $M = \tilde{M} \cap G$. Then Henkin [5] proved the following.

FUNDAMENTAL THEOREM. *There exists a continuous linear operator*

$$E\colon H^\infty(M) \to H^\infty(G) \text{ satisfying } Ef\,|_M = f.$$

Moreover $Ef \in A(G)$ iff $f \in A(M)$.

In the present paper we shall extend the above results to the weakly pseudoconvex case. Let D be a bounded weakly pseudoconvex domain in C^n with C^∞-boundary. Let $\tilde{V}$ be a subvariety in a neighborhood $\tilde{D}$ of $\overline{D}$ which intersects ∂D transversally. Let $V = \tilde{V} \cap D$ and $D = \{z \in \tilde{D}\colon \rho(z) < 0\}$. Suppose that $\tilde{V}$ is written in the form

$$\tilde{V} = \left\{z \in \tilde{D}\colon h_1(z) = \cdots = h_p(z) = 0\right\},$$

where $h_1, \dots, h_p$ are holomorphic in $\tilde{D}$ and $\partial h_1 \wedge \cdots \wedge \partial h_p \neq 0$ on $\partial D \cap \tilde{V}$. In addition, we assume that ∂V consists of strictly pseudoconvex boundary points of D. In this setting we shall show the following:

THEOREM 1. *There exists a continuous linear operator*

$$E\colon H^\infty(V) \to H^\infty(D) \text{ satisfying } Ef\,|_V = f.$$

Moreover $Ef \in A(D)$ iff $f \in A(V)$.

In the case when $p = 1$, the above theorem is nothing but the result of Adachi [1].

I wish to thank Professor N. Kerzman who gave me the validity of Proposition 3. I also wish to thank the referee for several helpful suggestions.

2. Some results. Let

$$s(\zeta, z) = (s_1(\zeta, z), \ldots, s_n(\zeta, z)) \colon \partial D \times D \to C^n$$

be a C^∞ function that satisfies

$$\langle s, \zeta - z\rangle = \sum_{j=1}^{n} s_j(\zeta_j - z_j) \neq 0 \quad \text{for } (\zeta, z) \in \partial D \times D.$$

Then Hatziafratis [3] proved the following theorem.

THEOREM 2. *For $f \in A(V)$ and $z \in V$ we have the integral formula*

$$f(z) = \int_{\partial V} f(\zeta) \frac{K(\zeta, z)}{\langle s, \zeta - z\rangle^{n-p}}$$

where $K(\zeta, z)$ is a $C^\infty(n-p, n-p-1)$-form on $\partial D \times D$. Moreover, if $s_1(\zeta, z), \ldots, s_n(\zeta, z)$ are holomorphic in z, then $K(\zeta, z)$ is also holomorphic in z.

Let G be a bounded strictly pseudoconvex domain in C^n with C^∞ boundary. According to the construction of Henkin [4], there exist a neighborhood U of $\overline{G}$, a neighborhood V of ∂G, and a C^∞ function Φ: $V \times U \to C$ such that for each $\zeta \in V$, $\Phi(\zeta, z)$ is holomorphic in U and such that $\Phi(\zeta, z) = 0$ implies $\zeta = z$. Moreover, Φ admits a division

$$\Phi(\zeta, z) = \sum_{j=1}^{n} P_j(\zeta, z)(\zeta_j - z_j)$$

where P_j: $V \times U \to C$ of class C^∞ and holomorphic in the second variable. In addition, if we set

$$T(\zeta, z) = 2\sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i}(\zeta)(z_i - \zeta_i) + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j)$$

then there exists a positive constant r such that

$$\Phi(\zeta, z) = T(\zeta, z)G(\zeta, z) \quad \text{for } \{(\zeta, z) \in V \times U: |\zeta - z| < r\} = S_r$$

where $G(\zeta, z)$ is a non-vanishing C^∞ function in S_r.

Now we have the following proposition using the techniques of the proof of Fornaess Imbedding theorem [**2**].

PROPOSITION 1. *Let D be a bounded weakly pseudoconvex domain in C^n with C^∞ boundary. Let K be a compact subset of ∂D and consist of strictly pseudoconvex boundary points of D. Then there exists a strictly pseudoconvex domain $\hat{D}$ in C^n with C^∞ boundary such that $\hat{D} \supset D$ and $\partial\hat{D}$ coincides with ∂D near K.*

In view of Proposition 1, if we can get the extension F to $\hat{D}$, then $F|_D$ is the required function. Therefore we may assume that D is a strictly pseudoconvex domain. Let $\{\varepsilon_\nu\}$ be a sequence of positive numbers which converges to 0. Let $D_\nu = \{z \in D: \rho(z) < -\varepsilon_\nu\}$, $V_\nu = V \cap D_\nu$; and $n - p = k$. If $f \in H^\infty(V)$, then by Hatziafratis [**3**], we have, for large ν and $z \in V_\nu$,

$$f(z) = \int_{\partial V} f(\zeta)\frac{K(\zeta, z)}{\Phi(\zeta, z)}$$

where $K(\zeta, z)$ is a $C^\infty(k, k-1)$-form depending holomorphically on z. We set for $z \in D_\nu$

$$H_\nu(z) = \int_{\partial V_\nu} f(\zeta)\frac{K(\zeta, z)}{\Phi(\zeta, z)^k}.$$

Then we have the following proposition which is proved by the same argument as the proof of lemma 1 in Adachi [**1**].

PROPOSITION 2. *For $z \in \overline{D}|\partial V$, $H(z) = \lim_{\nu\to\infty} H_\nu(z)$ exists. $H(z)$ is holomorphic in D and $H(z) = f(z)$ for $z \in V$.*

Let $z^0 \in \partial V$ and $S_{z^0,\sigma} = \{z: |z - z^0| < \sigma\}$. Then there exist a constant $\sigma_1 > 0$ and a biholomorphic change of coordinates on a neighborhood of z^0 such that ρ is strictly convex in a neighborhood of

$$\overline{D} \cap S_{z^0,\sigma_1}, \qquad V \cap S_{z^0,\sigma_1} = \{z \in S_{z^0,\sigma_1}: z_{p+1} = \cdots = z_n = 0\},$$

and $(\partial\rho/\partial z_i)(z^0) \neq 0$ for some i $(1 \leq i \leq p)$. Without loss of generality we may assume that $(\partial\rho/\partial z_1)(z^0) \neq 0$. Let $z \in S_{z_0^0,\sigma_1}$. We consider the system of equations for $\zeta^0 = (\zeta_1^0, \ldots, \zeta_n^0)$ of the following form:

$$(1) \qquad \begin{cases} \sum_{i=1}^{n} \frac{\partial\rho}{\partial\zeta_i}(\zeta^0)(\zeta_i^0 - z_i) = 0, \\ \zeta_i^0 = z_i \quad (i = 2, \ldots, p), \qquad \zeta_{p+1}^0 = \cdots = \zeta_n^0 = 0. \end{cases}$$

LEMMA 1. *There exist positive constants* σ_2 $(< \sigma_1)$, γ_1 *and* γ_2, *depending only on* D *and* V, *such that for any* $\sigma \leq \sigma_2$ *and any* $z \in S_{z^0,\sigma/2}$ *there exists a unique solution* $\zeta^0 = \zeta^0(z)$ *of the system* (1) *which belongs to the set* $S_{z^0,\sigma} \cap \tilde{V}$. *Here the point* $\zeta^0 = \zeta^0(z)$ *has the following properties*:

(2) $|z - \zeta^0|^2 \leq [\rho(z) - \rho(\zeta^0)]/\gamma_1$

(3) $|z - \zeta^0|^2 \geq |z_{p+1}|^2 + \cdots + |z_n|^2 \geq \gamma_2[\rho(z) - \rho(\zeta^0)]$.

Proof. From the system (1), we have

$$\zeta_1 = z_1 - \sum_{i=p+1}^{n} \frac{\partial\rho}{\partial z_i}(\zeta) z_i \left(\frac{\partial\rho(\zeta)}{\partial z_1} \right)^{-1}.$$

We set

$$a_i(\zeta) = -\frac{\partial\rho}{\partial z_i}(\zeta)\left(\frac{\partial\rho}{\partial z_1}(\zeta) \right)^{-1},$$

then $a_i(\zeta)$ is C^∞ in a neighborhood of z^0. We set by recurrence that

$$\zeta_1^{(1)} = z_1, \ \zeta^{(j)} = \left(\zeta_1^{(j)}, z_2, \ldots, z_p, 0, \ldots, 0 \right)$$

$$\zeta_1^{(j)} = z_1 + \sum_{i=p+1}^{n} a_i\left(\zeta^{(j-1)}\right) z_i.$$

Then

$$\left|\zeta_1^{(j)} - \zeta_1^{(j-1)}\right| \leq \sum_{i=p+1}^{n} |\nabla a_i| \left|\zeta_1^{(j-1)} - \zeta_1^{(j-2)}\right| |z_i|$$

$$\leq \frac{1}{2}\left|\zeta_1^{(j-1)} - \zeta_1^{(j-2)}\right|.$$

Therefore $\{\zeta^{(j)}\}$ converges. $\lim_{j\to\infty} \zeta^{(j)} = \zeta^0$ is the solution of the system (1). The strict convexity of the function ρ and the equation (1) imply

(4) $\rho(\zeta^0) - \rho(z) + \gamma_1|\zeta^0 - z|^2 \leq 0$,

(5) $\rho(\zeta^0) - \rho(z) + \gamma_2'|\zeta^0 - z|^2 \geq 0$.

From the inequality (4), we have the inequality (2). From the system (1), we have

$$\begin{aligned}|\zeta^0 - z|^2 &= |z_{p+1}|^2 + \cdots + |z_n|^2 + |\zeta_1^0 - z_1|^2 \\ &\le |z_{p+1}|^2 + \cdots + |z_n|^2 + \left(\sum_{i=p+1}^{n} |a_i(\zeta^0)|\,|z_i|\right)^2 \\ &\le \gamma_2''\left(|z_{p+1}|^2 + \cdots + |z_n|^2\right).\end{aligned}$$

Together with the inequality (5), we have

$$|\zeta^0 - z|^2 \ge |z_{p+1}|^2 + \cdots + |z_n|^2 \ge \gamma_2\left(\rho(z) - \rho(\zeta^0)\right).$$

This completes the proof of Lemma 1.

3. Proof of Theorem 1. At first we prove that if $f \in H^\infty(V)$, then $H(z) \in H^\infty(D)$. Let $z \in S_{z^0,\sigma_1} \cap D_\nu$. We set

$$\tilde{H}_\nu(z) = \int_{\partial V_\nu \cap S_{z^0,\sigma_1}} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^k}.$$

It is sufficient to show that

$$|\tilde{H}_\nu(z)| \le \gamma_3 \sup_{\zeta \in V} |f(\zeta)|.$$

LEMMA 2. *Let $f(z) \in H^\infty(V)$. Then for any point $z^0 \in \partial V$ and any point $z \in \partial(S_{z^0,\sigma} \cap D_\nu) \,|\, \partial V_\nu$, $(\sigma < \sigma_2/2)$, we have*

$$\left|\left.\frac{d\tilde{H}_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda}\right|_{\lambda=1}\right| \le \gamma_4 \sup_{\zeta \in V} |f(\zeta)|.$$

Proof of Lemma 2. We set $\varepsilon = (|z_{p+1}|^2 + \cdots + |z_n|^2)^{1/2}$, where $z = (z_1, \ldots, z_n) \in \partial(S_{z^0,\sigma} \cap D_\nu)|\partial V_\nu$. By Lemma 1, we have

$$\varepsilon \le |\zeta^0 - z| \le \left\{\frac{\rho(z) - \rho(\zeta^0)}{\gamma_1}\right\}^{1/2} \le \frac{\varepsilon}{(\gamma_1\gamma_2)^{1/2}}.$$

Since

$$\sum_{i=1}^{n} \frac{\partial \rho}{\partial \zeta_i}(\zeta^0)(\zeta_i^0 - z_i) = 0,$$

it follows that

$$\left|\sum_{i=1}^{n}\frac{\partial\Phi}{\partial z_i}(\zeta,z)(\zeta_i^0-z_i)\right|\le\gamma_5\varepsilon\big(|\zeta-z|+\varepsilon\big).$$

On the other hand, we have

$$\left.\frac{d\tilde{H}_\nu(z+\lambda w)}{d\lambda}\right|_{\lambda=0}=\int_{\partial V_\nu\cap S_{z^0,\sigma_1}}\frac{f(\zeta)\sum_{j=1}^{n}\frac{\partial K}{\partial z_j}(\zeta,z)w_j}{\Phi(\zeta,z)^k}$$

$$-\int_{\partial V_\nu\cap S_{z^0,\sigma_1}}\frac{f(\zeta)K(\zeta,z)k\sum_{j=1}^{n}\frac{\partial\Phi}{\partial z_j}(\zeta,z)w_j}{\Phi(\zeta,z)^{k+1}}$$

Therefore we have

$$\left|\left.\frac{d\tilde{H}_\nu(\zeta^0+\lambda(z-\zeta^0))}{d\lambda}\right|_{\lambda=1}\right|\le\gamma_6\int_{\partial V_\nu\cap S_{z^0,\sigma_1}}\frac{\varepsilon|f(\zeta)|\,d\lambda}{|\Phi(\zeta,z)|^k}$$

$$+\gamma_7\int_{\partial V_\nu\cap S_{z^0,\sigma_1}}\frac{|f(\zeta)|\varepsilon\big(|\zeta-z|+\varepsilon\big)\,d\lambda}{|\Phi(\zeta,z)|^{k+1}}$$

where $d\lambda$ is surface measure on ∂V_ν. We can choose coordinates $(\eta_1(\zeta),\dots,\eta_n(\zeta))$ in S_{z^0,σ_2} such that

$$\eta_1(\zeta)=\rho(\zeta)-\rho(z)+i\operatorname{Im}\Phi(\zeta,z).$$

Then we have

$$|\Phi(\zeta,z)|\ge\gamma_8\Big[\big(t_1+|\zeta-z|^2\big)^2+t_2^2\Big]^{1/2}.$$

By the estimates of Henkin [5], we have

$$\left|\left.\frac{d\tilde{H}_\nu(\zeta^0+\lambda(z-\zeta^0))}{d\lambda}\right|_{\lambda=1}\right|\le\gamma_9\sup_{\zeta\in V}|f(\zeta)|.$$

This completes the proof of Lemma 2.

By the same method as the proof of Henkin [5] (cf. Adachi [1]), we can prove that

$$\sup_{z\in D}|H(z)|\le\gamma_{10}\sup_{\zeta\in V}|f(\zeta)|.$$

The next step is to show that if $f \in A(V)$, the $H(z) \in A(D)$. In order to prove this statement, we need the following modified version of N. Kerzman [**6**]. In the Theorem 1.4.1′ of Kerzman, V is a manifold. But the proof of the theorem is applicable in our case.

PROPOSITION 3. *Let $f \in A(V)$. Then there exists a sequence $\{f_k\}$ of holomorphic functions in a neighborhood of $\overline{V}$ in $\tilde{V}$ such that $\|f_k - f\|_V \to 0$ when $k \to \infty$.*

From Proposition 3 we can suppose that f is holomorphic in $\overline{V}'$ ($V \subset V' \subset \overline{V}' \subset \tilde{V}$). Let $z^0 \in \partial V$ and let $z \in S_{z^0,\sigma/2} \cap (\overline{D}_\nu | \partial V_\nu)$. By using Stokes' formula, we have

$$H_\nu(z) = \int_{\partial V'} \frac{f(\zeta)K(\zeta,z)}{\Phi(\zeta,z)^k} - \int_{(V'-V_\nu)\cap S_{z^0,2\sigma}} f(\zeta)\bar{\partial}_\zeta\left(\frac{K(\zeta,z)}{\Phi(\zeta,z)^k}\right)$$

$$-\int_{(V'-V_\nu)|S_{z_0,2\sigma}} f(\zeta)\bar{\partial}_\zeta\left(\frac{K(\zeta,z)}{\Phi(\zeta,z)^k}\right).$$

Therefore it is sufficient to show that

$$F_\nu(z) = \int_{(V'-V_\nu)\cap S_{z^0,2\sigma}} f(\zeta)\bar{\partial}_\zeta\left(\frac{K(\zeta,z)}{\Phi(\zeta,z)^k}\right)$$

is continuous at z^0. In order to prove this fact, we need the following.

LEMMA 3. *Let $z \in S_{z^0,\sigma/2} \cap (\overline{D}_\nu | \partial V_\nu)$. Then it follows that*

$$\left|\left.\frac{dF_\nu(\zeta^0 + \lambda(z-\zeta^0))}{d\lambda}\right|_{\lambda=1}\right| \le \gamma_{11}\varepsilon|\log\varepsilon| \sup_{\zeta\in V}|f(\zeta)|, \tag{6}$$

where $\zeta^0 = \zeta^0(z)$ is the solution of the system (1).

Proof of Lemma 3. We can write

$$F_\nu(z) = \int_{(V'-V_\nu)\cap S_{z^0,2\sigma}} f(\zeta)\frac{A(\zeta,z)}{\Phi(\zeta,z)^k}$$

$$+\int_{(V'-V_\nu)\cap S_{z^0,2\sigma}} \frac{f(\zeta)\sum_{j=1}^{n}(\zeta_j - z_j)B_j(\zeta,z)}{\Phi(\zeta,z)^{k+1}}$$

where $A(\zeta, z)$ and $B_j(\zeta, z)$ are (k, k)-forms which are smooth in (ζ, z) and holomorphic in z. Therefore

$$\left| \frac{dF_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \bigg|_{\lambda=1} \right|$$

$$\leq \gamma_{12} \int_{(V' - V_\nu) \cap S_{z^0, 2\sigma}} |f(\zeta)| \frac{\varepsilon \, d\lambda}{|\Phi(\zeta, z)|^{k+1}}$$

$$+ \gamma_{13} \int_{(V' - V_\nu) \cap S_{z^0, 2\sigma}} \frac{|f(\zeta)| \, |\zeta - z| \varepsilon(|\zeta - z| + \varepsilon) \, d\lambda}{|\Phi(\zeta, z)|^{k+2}}.$$

By applying the estimates of Henkin [5], we have the inequaltiy (6). This completes the proof of Lemma 3.

Using the method of Henkin [5], we can prove

$$|F_\nu(z) - F_\nu(z^0)| \leq \gamma_{14} \sigma |\log \sigma| \sup_{\zeta \in V'} |f(\zeta)| + \sigma \sup_{\zeta \in V'} |\operatorname{grad} f(\zeta)|.$$

Therefore $F_\nu(z)$ is continuous at z^0. This completes the proof of Theorem 1.

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