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**CONTINUATION OF BOUNDED HOLOMORPHIC FUNCTIONS
FROM CERTAIN SUBVARIETIES TO WEAKLY
PSEUDOCONVEX DOMAINS**

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CONTINUATION OF BOUNDED HOLOMORPHIC FUNCTIONS FROM CERTAIN SUBVARIETIES TO WEAKLY PSEUDOCONVEX DOMAINS

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Let D be a weakly pseudoconvex domain in C^n with C^∞ -boundary and V be a subvariety in D which intersects ∂D transversally. If ∂V is nonsingular and consists of strictly pseudoconvex boundary points of D , then any bounded holomorphic function in V can be extended to a bounded holomorphic function in D .

1. Introduction. Let Ω be an open set in some complex manifold. We denote by $H^\infty(\Omega)$ the space of all bounded holomorphic functions in Ω and by $A(\Omega)$ the space of all holomorphic functions in Ω which are continuous in $\bar{\Omega}$. Let G be a bounded strictly pseudoconvex domain in C^n with C^2 -boundary and \tilde{M} be a submanifold in a neighborhood of \bar{G} which intersects ∂G transversally. Let $M = \tilde{M} \cap G$. Then Henkin [5] proved the following.

FUNDAMENTAL THEOREM. *There exists a continuous linear operator*

$$E: H^\infty(M) \rightarrow H^\infty(G) \text{ satisfying } Ef|_M = f.$$

Moreover $Ef \in A(G)$ if $f \in A(M)$.

In the present paper we shall extend the above results to the weakly pseudoconvex case. Let D be a bounded weakly pseudoconvex domain in C^n with C^∞ -boundary. Let \tilde{V} be a subvariety in a neighborhood \tilde{D} of \bar{D} which intersects ∂D transversally. Let $V = \tilde{V} \cap D$ and $D = \{z \in \tilde{D}: \rho(z) < 0\}$. Suppose that \tilde{V} is written in the form

$$\tilde{V} = \{z \in \tilde{D}: h_1(z) = \cdots = h_p(z) = 0\},$$

where h_1, \dots, h_p are holomorphic in \tilde{D} and $\partial h_1 \wedge \cdots \wedge \partial h_p \neq 0$ on $\partial D \cap \tilde{V}$. In addition, we assume that ∂V consists of strictly pseudoconvex boundary points of D . In this setting we shall show the following:

THEOREM 1. *There exists a continuous linear operator*

$$E: H^\infty(V) \rightarrow H^\infty(D) \text{ satisfying } Ef|_V = f.$$

Moreover $Ef \in A(D)$ if $f \in A(V)$.

In the case when $p = 1$, the above theorem is nothing but the result of Adachi [1].

I wish to thank Professor N. Kerzman who gave me the validity of Proposition 3. I also wish to thank the referee for several helpful suggestions.

2. Some results. Let

$$s(\zeta, z) = (s_1(\zeta, z), \dots, s_n(\zeta, z)): \partial D \times D \rightarrow C^n$$

be a C^∞ function that satisfies

$$\langle s, \zeta - z \rangle = \sum_{j=1}^n s_j(\zeta_j - z_j) \neq 0 \quad \text{for } (\zeta, z) \in \partial D \times D.$$

Then Hatziafratis [3] proved the following theorem.

THEOREM 2. *For $f \in A(V)$ and $z \in V$ we have the integral formula*

$$f(z) = \int_{\partial V} f(\zeta) \frac{K(\zeta, z)}{\langle s, \zeta - z \rangle^{n-p}}$$

where $K(\zeta, z)$ is a $C^\infty(n-p, n-p-1)$ -form on $\partial D \times D$. Moreover, if $s_1(\zeta, z), \dots, s_n(\zeta, z)$ are holomorphic in z , then $K(\zeta, z)$ is also holomorphic in z .

Let G be a bounded strictly pseudoconvex domain in C^n with C^∞ boundary. According to the construction of Henkin [4], there exist a neighborhood U of \bar{G} , a neighborhood V of ∂G , and a C^∞ function $\Phi: V \times U \rightarrow C$ such that for each $\zeta \in V$, $\Phi(\zeta, z)$ is holomorphic in U and such that $\Phi(\zeta, z) = 0$ implies $\zeta = z$. Moreover, Φ admits a division

$$\Phi(\zeta, z) = \sum_{j=1}^n P_j(\zeta, z)(\zeta_j - z_j)$$

where $P_j: V \times U \rightarrow C$ of class C^∞ and holomorphic in the second variable. In addition, if we set

$$\begin{aligned} T(\zeta, z) &= 2 \sum_{i=1}^n \frac{\partial \rho}{\partial z_i}(\zeta)(z_i - \zeta_i) \\ &\quad + \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial z_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j) \end{aligned}$$

then there exists a positive constant r such that

$$\Phi(\zeta, z) = T(\zeta, z)G(\zeta, z) \quad \text{for } \{(\zeta, z) \in V \times U: |\zeta - z| < r\} = S_r,$$

where $G(\zeta, z)$ is a non-vanishing C^∞ function in S_r .

Now we have the following proposition using the techniques of the proof of Fornaess Imbedding theorem [2].

PROPOSITION 1. *Let D be a bounded weakly pseudoconvex domain in C^n with C^∞ boundary. Let K be a compact subset of ∂D and consist of strictly pseudoconvex boundary points of D . Then there exists a strictly pseudoconvex domain \hat{D} in C^n with C^∞ boundary such that $\hat{D} \supset D$ and $\partial \hat{D}$ coincides with ∂D near K .*

In view of Proposition 1, if we can get the extension F to \hat{D} , then $F|_D$ is the required function. Therefore we may assume that D is a strictly pseudoconvex domain. Let $\{\varepsilon_\nu\}$ be a sequence of positive numbers which converges to 0. Let $D_\nu = \{z \in D: \rho(z) < -\varepsilon_\nu\}$, $V_\nu = V \cap D_\nu$, and $n - p = k$. If $f \in H^\infty(V)$, then by Hatziafratis [3], we have, for large ν and $z \in V_\nu$,

$$f(z) = \int_{\partial V} f(\zeta) \frac{K(\zeta, z)}{\Phi(\zeta, z)}$$

where $K(\zeta, z)$ is a $C^\infty(k, k-1)$ -form depending holomorphically on z . We set for $z \in D_\nu$,

$$H_\nu(z) = \int_{\partial V_\nu} f(\zeta) \frac{K(\zeta, z)}{\Phi(\zeta, z)^k}.$$

Then we have the following proposition which is proved by the same argument as the proof of lemma 1 in Adachi [1].

PROPOSITION 2. *For $z \in \bar{D} \setminus \partial V$, $H(z) = \lim_{\nu \rightarrow \infty} H_\nu(z)$ exists. $H(z)$ is holomorphic in D and $H(z) = f(z)$ for $z \in V$.*

Let $z^0 \in \partial V$ and $S_{z^0, \sigma} = \{z: |z - z^0| < \sigma\}$. Then there exist a constant $\sigma_1 > 0$ and a biholomorphic change of coordinates on a neighborhood of z^0 such that ρ is strictly convex in a neighborhood of

$$\bar{D} \cap S_{z^0, \sigma_1}, \quad V \cap S_{z^0, \sigma_1} = \{z \in S_{z^0, \sigma_1}: z_{p+1} = \cdots = z_n = 0\},$$

and $(\partial\rho/\partial z_i)(z^0) \neq 0$ for some i ($1 \leq i \leq p$). Without loss of generality we may assume that $(\partial\rho/\partial z_1)(z^0) \neq 0$. Let $z \in S_{z^0, \sigma_1}$. We consider the system of equations for $\zeta^0 = (\zeta_1^0, \dots, \zeta_n^0)$ of the following form:

$$(1) \quad \begin{cases} \sum_{i=1}^n \frac{\partial \rho}{\partial \zeta_i}(\zeta^0)(\zeta_i^0 - z_i) = 0, \\ \zeta_i^0 = z_i \quad (i = 2, \dots, p), \quad \zeta_{p+1}^0 = \dots = \zeta_n^0 = 0. \end{cases}$$

LEMMA 1. *There exist positive constants σ_2 ($< \sigma_1$), γ_1 and γ_2 , depending only on D and V , such that for any $\sigma \leq \sigma_2$ and any $z \in S_{z^0, \sigma/2}$ there exists a unique solution $\zeta^0 = \zeta^0(z)$ of the system (1) which belongs to the set $S_{z^0, \sigma} \cap \tilde{V}$. Here the point $\zeta^0 = \zeta^0(z)$ has the following properties:*

- (2) $|z - \zeta^0|^2 \leq [\rho(z) - \rho(\zeta^0)]/\gamma_1$
- (3) $|z - \zeta^0|^2 \geq |z_{p+1}|^2 + \dots + |z_n|^2 \geq \gamma_2[\rho(z) - \rho(\zeta^0)]$.

Proof. From the system (1), we have

$$\zeta_1 = z_1 - \sum_{i=p+1}^n \frac{\partial \rho}{\partial z_i}(\zeta) z_i \left(\frac{\partial \rho(\zeta)}{\partial z_1} \right)^{-1}.$$

We set

$$a_i(\zeta) = -\frac{\partial \rho}{\partial z_i}(\zeta) \left(\frac{\partial \rho}{\partial z_1}(\zeta) \right)^{-1},$$

then $a_i(\zeta)$ is C^∞ in a neighborhood of z^0 . We set by recurrence that

$$\zeta_1^{(1)} = z_1, \quad \zeta^{(j)} = (\zeta_1^{(j)}, z_2, \dots, z_p, 0, \dots, 0)$$

$$\zeta_1^{(j)} = z_1 + \sum_{i=p+1}^n a_i(\zeta^{(j-1)}) z_i.$$

Then

$$\begin{aligned} |\zeta_1^{(j)} - \zeta_1^{(j-1)}| &\leq \sum_{i=p+1}^n |\nabla a_i| |\zeta_1^{(j-1)} - \zeta_1^{(j-2)}| |z_i| \\ &\leq \frac{1}{2} |\zeta_1^{(j-1)} - \zeta_1^{(j-2)}|. \end{aligned}$$

Therefore $\{\zeta^{(j)}\}$ converges. $\lim_{j \rightarrow \infty} \zeta^{(j)} = \zeta^0$ is the solution of the system

(1). The strict convexity of the function ρ and the equation (1) imply

$$(4) \quad \rho(\zeta^0) - \rho(z) + \gamma_1 |\zeta^0 - z|^2 \leq 0,$$

$$(5) \quad \rho(\zeta^0) - \rho(z) + \gamma_2' |\zeta^0 - z|^2 \geq 0.$$

From the inequality (4), we have the inequality (2). From the system (1), we have

$$\begin{aligned} |\xi^0 - z|^2 &= |z_{p+1}|^2 + \cdots + |z_n|^2 + |\xi_1^0 - z_1|^2 \\ &\leq |z_{p+1}|^2 + \cdots + |z_n|^2 + \left(\sum_{i=p+1}^n |a_i(\xi^0)| |z_i| \right)^2 \\ &\leq \gamma_2'' (|z_{p+1}|^2 + \cdots + |z_n|^2). \end{aligned}$$

Together with the inequality (5), we have

$$|\xi^0 - z|^2 \geq |z_{p+1}|^2 + \cdots + |z_n|^2 \geq \gamma_2 (\rho(z) - \rho(\xi^0)).$$

This completes the proof of Lemma 1.

3. Proof of Theorem 1. At first we prove that if $f \in H^\infty(V)$, then $H(z) \in H^\infty(D)$. Let $z \in S_{z^0, \sigma_1} \cap D_\nu$. We set

$$\tilde{H}_\nu(z) = \int_{\partial V_\nu \cap S_{z^0, \sigma_1}} \frac{f(\xi) K(\xi, z)}{\Phi(\xi, z)^k}.$$

It is sufficient to show that

$$|\tilde{H}_\nu(z)| \leq \gamma_3 \sup_{\xi \in V} |f(\xi)|.$$

LEMMA 2. *Let $f(z) \in H^\infty(V)$. Then for any point $z^0 \in \partial V$ and any point $z \in \partial(S_{z^0, \sigma} \cap D_\nu) \cap \partial V_\nu$, ($\sigma < \sigma_2/2$), we have*

$$\left| \frac{d\tilde{H}_\nu(\xi^0 + \lambda(z - \xi^0))}{d\lambda} \right|_{\lambda=1} \leq \gamma_4 \sup_{\xi \in V} |f(\xi)|.$$

Proof of Lemma 2. We set $\varepsilon = (|z_{p+1}|^2 + \cdots + |z_n|^2)^{1/2}$, where $z = (z_1, \dots, z_n) \in \partial(S_{z^0, \sigma} \cap D_\nu) \cap \partial V_\nu$. By Lemma 1, we have

$$\varepsilon \leq |\xi^0 - z| \leq \left\{ \frac{\rho(z) - \rho(\xi^0)}{\gamma_1} \right\}^{1/2} \leq \frac{\varepsilon}{(\gamma_1 \gamma_2)^{1/2}}.$$

Since

$$\sum_{i=1}^n \frac{\partial \rho}{\partial \xi_i}(\xi^0)(\xi_i^0 - z_i) = 0,$$

it follows that

$$\left| \sum_{i=1}^n \frac{\partial \Phi}{\partial z_i}(\zeta, z)(\zeta_i^0 - z_i) \right| \leq \gamma_5 \varepsilon(|\zeta - z| + \varepsilon).$$

On the other hand, we have

$$\begin{aligned} \frac{d\tilde{H}_\nu(z + \lambda w)}{d\lambda} \Big|_{\lambda=0} &= \int_{\partial V_\nu \cap S_{z^0, \sigma_1}} \frac{f(\zeta) \sum_{j=1}^n \frac{\partial K}{\partial z_j}(\zeta, z) w_j}{\Phi(\zeta, z)^k} \\ &\quad - \int_{\partial V_\nu \cap S_{z^0, \sigma_1}} \frac{f(\zeta) K(\zeta, z) k \sum_{j=1}^n \frac{\partial \Phi}{\partial z_j}(\zeta, z) w_j}{\Phi(\zeta, z)^{k+1}} \end{aligned}$$

Therefore we have

$$\begin{aligned} \left| \frac{d\tilde{H}_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \Big|_{\lambda=1} \right| &\leq \gamma_6 \int_{\partial V_\nu \cap S_{z^0, \sigma_1}} \frac{\varepsilon |f(\zeta)| d\lambda}{|\Phi(\zeta, z)|^k} \\ &\quad + \gamma_7 \int_{\partial V_\nu \cap S_{z^0, \sigma_1}} \frac{|f(\zeta)| \varepsilon(|\zeta - z| + \varepsilon) d\lambda}{|\Phi(\zeta, z)|^{k+1}} \end{aligned}$$

where $d\lambda$ is surface measure on ∂V_ν . We can choose coordinates $(\eta_1(\zeta), \dots, \eta_n(\zeta))$ in S_{z^0, σ_2} such that

$$\eta_1(\zeta) = \rho(\zeta) - \rho(z) + i \operatorname{Im} \Phi(\zeta, z).$$

Then we have

$$|\Phi(\zeta, z)| \geq \gamma_8 \left[\left(t_1 + |\zeta - z|^2 \right)^2 + t_2^2 \right]^{1/2}.$$

By the estimates of Henkin [5], we have

$$\left| \frac{d\tilde{H}_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \Big|_{\lambda=1} \right| \leq \gamma_9 \sup_{\zeta \in V} |f(\zeta)|.$$

This completes the proof of Lemma 2.

By the same method as the proof of Henkin [5] (cf. Adachi [1]), we can prove that

$$\sup_{z \in D} |H(z)| \leq \gamma_{10} \sup_{\zeta \in V} |f(\zeta)|.$$

The next step is to show that if $f \in A(V)$, the $H(z) \in A(D)$. In order to prove this statement, we need the following modified version of N. Kerzman [6]. In the Theorem 1.4.1' of Kerzman, V is a manifold. But the proof of the theorem is applicable in our case.

PROPOSITION 3. *Let $f \in A(V)$. Then there exists a sequence $\{f_k\}$ of holomorphic functions in a neighborhood of \bar{V} in \tilde{V} such that $\|f_k - f\|_V \rightarrow 0$ when $k \rightarrow \infty$.*

From Proposition 3 we can suppose that f is holomorphic in \bar{V}' ($V \subset V' \subset \bar{V}' \subset \tilde{V}$). Let $z^0 \in \partial V$ and let $z \in S_{z^0, \sigma/2} \cap (\bar{D}_\nu | \partial V_\nu)$. By using Stokes' formula, we have

$$\begin{aligned} H_\nu(z) &= \int_{\partial V'} \frac{f(\zeta) K(\zeta, z)}{\Phi(\zeta, z)^k} - \int_{(V' - V_\nu) \cap S_{z^0, 2\sigma}} f(\zeta) \bar{\partial}_\zeta \left(\frac{K(\zeta, z)}{\Phi(\zeta, z)^k} \right) \\ &\quad - \int_{(V' - V_\nu) | S_{z^0, 2\sigma}} f(\zeta) \bar{\partial}_\zeta \left(\frac{K(\zeta, z)}{\Phi(\zeta, z)^k} \right). \end{aligned}$$

Therefore it is sufficient to show that

$$F_\nu(z) = \int_{(V' - V_\nu) \cap S_{z^0, 2\sigma}} f(\zeta) \bar{\partial}_\zeta \left(\frac{K(\zeta, z)}{\Phi(\zeta, z)^k} \right)$$

is continuous at z^0 . In order to prove this fact, we need the following.

LEMMA 3. *Let $z \in S_{z^0, \sigma/2} \cap (\bar{D}_\nu | \partial V_\nu)$. Then it follows that*

$$(6) \quad \left| \frac{dF_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \leq \gamma_{11} \epsilon |\log \epsilon| \sup_{\zeta \in V} |f(\zeta)|,$$

where $\zeta^0 = \zeta^0(z)$ is the solution of the system (1).

Proof of Lemma 3. We can write

$$\begin{aligned} F_\nu(z) &= \int_{(V' - V_\nu) \cap S_{z^0, 2\sigma}} f(\zeta) \frac{A(\zeta, z)}{\Phi(\zeta, z)^k} \\ &\quad + \int_{(V' - V_\nu) \cap S_{z^0, 2\sigma}} \frac{f(\zeta) \sum_{j=1}^n (\zeta_j - z_j) B_j(\zeta, z)}{\Phi(\zeta, z)^{k+1}} \end{aligned}$$

where $A(\zeta, z)$ and $B_j(\zeta, z)$ are (k, k) -forms which are smooth in (ζ, z) and holomorphic in z . Therefore

$$\begin{aligned} & \left| \frac{dF_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \\ & \leq \gamma_{12} \int_{(V' - V_\nu) \cap S_{z^0, 2\sigma}} |f(\zeta)| \frac{\varepsilon d\lambda}{|\Phi(\zeta, z)|^{k+1}} \\ & \quad + \gamma_{13} \int_{(V' - V_\nu) \cap S_{z^0, 2\sigma}} \frac{|f(\zeta)| |\zeta - z| \varepsilon (|\zeta - z| + \varepsilon) d\lambda}{|\Phi(\zeta, z)|^{k+2}}. \end{aligned}$$

By applying the estimates of Henkin [5], we have the inequality (6). This completes the proof of Lemma 3.

Using the method of Henkin [5], we can prove

$$|F_\nu(z) - F_\nu(z^0)| \leq \gamma_{14} \sigma |\log \sigma| \sup_{\zeta \in V'} |f(\zeta)| + \sigma \sup_{\zeta \in V'} |\operatorname{grad} f(\zeta)|.$$

Therefore $F_\nu(z)$ is continuous at z^0 . This completes the proof of Theorem 1.

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