# UNITARY EQUIVALENCE OF INVARIANT SUBSPACES IN THE POLYDISK

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Invariant subspaces $M$ and $N$ of $H^2(T^n)$ are called unitarily equivalent if $M = \psi N$ for a unimodular function $\psi$ on $T^n$. In this note, it is given a complete characterization of pairs of invariant subspaces $M$ and $N$ of $H^2(T^n)$ such that $M = \phi N$ for an inner function $\phi$. This is a generalization of Agrawal, Clark and Douglas' results. As an application, if $M$ is an invariant subspace of $H^2(T^n)$ and if $M$ is unitarily equivalent to $S(f)$, an invariant subspace generated by an outer function $f$, then $M = \phi S(f)$ for some inner function $\phi$.

It is well known that Beurling [4] showed that every invariant subspace $M$ of $H^2(T)$ can be written by $M = \psi H^2(T)$ for some inner function $\psi$. Although it is easy to see that a Beurling-type characterization is not possible for invariant subspaces of $H^2(T^n)$, $n \geq 2$, it is very difficult to determine all invariant subspaces of $H^2(T^n)$ for $n \geq 2$. In [3], Ahern and Clark studied an invariant subspace which has finite codimension in $H^2(T^n)$. These invariant subspaces are not Beurling-type. Recently Agrawal, Clark and Douglas [2] studied pairs of invariant subspaces of $H^2(T^n)$ which are unitarily equivalent. Here two invariant subspaces $M_1$ and $M_2$ are called *unitarily equivalent* if there is a unimodular function $\psi$ on $T^n$ with $M_2 = \psi M_1$. In [1, Corollary 3], they showed that distinct invariant subspaces having finite codimensions in $H^2(T^n)$ are not unitarily equivalent. In [9], Rudin gives two examples of unitarily equivalent invariant subspaces of $H^2(T^2)$ answering problems posed in [2]. In [6], Nakazi gives a characterization of invariant subspaces $M$ of $L^2(T^2)$ with $M = FH^2(T^2)$ for some unimodular function $F$. From the view point of the Beurling theorem, it is interesting to characterize pairs of unitarily equivalent subspaces $M_1$ and $M_2$ of $H^2(T^n)$ such that $M_2 = \psi M_1$ for some inner function $\psi$. In [2], they give some sufficient conditions of these pairs. One of these conditions is $M_2 \subset M_1$.

In §2, we shall show a theorem which contains Schneider's lemma as a corollary (Corollary 1). Also our theorem gives us a complete characterization of pairs of invariant subspaces $M_1$ and $M_2$ of $H^2(T^n)$ such that $M_2 = \psi M_1$ for some inner function $\psi$ (Corollary 2). Of course this

theorem covers Propositions 1, 2, 3, and 4 in [2]. In §3, we shall study invariant subspaces which are unitarily equivalent to the one generated by an outer function.

**1. Notations and Theorems.** For a positive integer $n$, let $T^n$ denote the cartesian product of $n$ unit circles. The usual Lebesgue spaces, with respect to the normalized Haar measure $m_n$ on $T^n$, are denoted by $L^p(T^n)$, $1 \leq p \leq \infty$. Let $H^p(T^n)$ be the space of all $f$ in $L^p(T^n)$ whose Fourier transforms

$$\hat{f}(\alpha) = \int_{T^n} f(z) \bar{z}_1^{\alpha_1} \bar{z}_2^{\alpha_2} \cdots \bar{z}_n^{\alpha_n} \, dm_n(z)$$

vanish outside $(Z_+)^n$, the $n$-fold product of nonnegative integers. A function $\psi$ in $L^\infty(T^n)$ and $H^\infty(T^n)$ is called *unimodular* and *inner* if $|\psi| = 1$ a.e. $dm_n$, respectively. A closed subspace $M$ of $L^2(T^n)$ is called *invariant* if $z_i M \subset M$ for every $i = 1, 2, \ldots, n$. We note that if $M$ is an invariant subspace then $H^\infty(T^n)M \subset M$. A function $f$ in $H^2(T^n)$ is called *outer* if

$$\log|f(0)| = \int_{T^n} |f(z)| \, dm_n(z).$$

We denote by $S(f)$ the invariant subspace generated by a function $f$ in $L^2(T^n)$. [8] is a convenient reference for the function theory in the polydisk.

To state our theorem, we use the following notations. Let $H_k$ and $\mathcal{H}_k$ denote the closure in $L^2(T^n)$ of the algebra generated by

$$\{1, z_i; \, i = 1, 2, \ldots, n\} \cup \{\bar{z}_k\} \quad \text{and}$$
$$\{1, z_i, \bar{z}_i \colon i = 1, 2, \ldots, n\} \setminus \{\bar{z}_k\},$$

respectively. Let $L_k^p$ denote the closure in $L^p(T^n)$, weak*-closure if $p = \infty$, of the algebra generated by

$$\{1, z_i, \bar{z}_i \colon i = 1, 2, \ldots, n\} \setminus \{z_k, \bar{z}_k\}.$$

Then $H_k$ and $\mathcal{H}_k$ are invariant subspaces, $\cap_{k=1}^n \mathcal{H}_k = H^2(T^n)$, and $\mathcal{H}_k$ coincides with the closed linear span of $\{z_k^m L_k^2; \, m = 0, 1, 2, \ldots\}$.

For an invariant subspace $M$ (generally not closed), let $(M)_k$ denote the closure of $L_k^\infty M$ in $L^2(T^n)$. Then $(M)_k$ is an invariant subspace and $L_k^\infty (M)_k = (M)_k$. We note $(H^2(T^n))_k = \mathcal{H}_k$. A closed subspace $N$ of $L^2(T^n)$ is called *reducing* if $z_i N = N$ for every $i = 1, 2, \ldots, n$. If $N$ is reducing, then $L^\infty(T^n)N = N$, hence $N = \chi_U L^2(T^n)$, where $\chi_U$ is a characteristic function for a Borel subset $U$ of $T^n$. We note that $\mathcal{H}_k$ does not contain any reducing subspaces.

Our main results are

THEOREM 1. *Let $M_1$ be an invariant subspace of $H^2(T^n)$ and $\phi \in L^\infty(T^n)$. Let $M_2$ denote the closure of $\phi M_1$ in $L^2(T^n)$. Then $\phi \in H^\infty(T^n)$ if and only if $(M_2)_k \subset (M_1)_k$ for every $k = 1, 2, \ldots, n$.*

COROLLARY 7. *Let $f \in H^2(T^n)$ be an outer function, and $M$ be an invariant subspace of $H^2(T^n)$ which is unitarily equivalent to $S(f)$. Then $M = \psi S(f)$ for some inner function $\psi$.*

## 2. Proof of Theorem 1 and its applications.

The following lemma is a corollary of the Merrill and Lal theorem [5] (see Remark after Lemma 1). In this case, we can prove it directly. For the sake of completeness, we give its proof.

LEMMA 1. *Let $M$ be an invariant subspace of $H^2(T^n)$. Then for each $k = 1, 2, \ldots, n$, $(M)_k = F_k \mathcal{H}_k$ for a unimodular function $F_k$ in $\mathcal{H}_k$.*

*Proof.* Let fix $k$. Since $M \subset H^2(T^n)$, $(M)_k \subset \mathcal{H}_k$. Hence $\bigcap_{i=1}^\infty z_k^i (M)_k = \{0\}$. Put

$$N = (M)_k \ominus z_k(M)_k.$$

Then $N \neq \{0\}$. Since $L_k^\infty (M)_k = (M)_k$, $L_k^\infty N = N$. Thus we have

$$(1) \qquad (M)_k = N \oplus z_k N \oplus z_k^2 N \oplus \cdots.$$

Let $g \in N$. Since $g \perp g z_k^i$ for $i = 1, 2, \ldots$, we get

$$\int_{T^n} |g|^2 z_k^i \, dm_n = 0$$

for every nonzero integer $i$. This implies $|g| \in L_k^2$. Since $|f| > 0$ a.e. $dm_n$ for $f \in H^2(T^n)$, by (1) there exists $g_0$ in $N$ such that $|g_0| > 0$ a.e. $dm_n$. Put $g_0 = F|g_0|$, where $F$ is unimodular. Since $L_k^\infty N = N$, $N \supset L_k^\infty g_0 = FL_k^\infty|g_0|$. Since $L_k^\infty|g_0|$ is dense in $L_k^2$, we have $FL_k^2 \subset N$.

To show $FL_k^2 = N$, let $g \in N$. Since $F \in N$,

$$F z_k^i \perp g z_k^j$$

for every $i, j \geq 0$ with $i \neq j$. Hence

$$\int_{T^n} \bar{F} g z_k^p \, dm_n = 0$$

for every nonzero integer $p$. Thus $\bar{F}g \in L_k^2$, so that $g \in FL_k^2$. Conse-
quently $FL_k^2 = N$. By (1),

$$(M)_k = FL_k^2 \oplus Fz_k L_k^2 \oplus Fz_k^2 L_k^2 \oplus \cdots = F\mathscr{H}_k.$$

Since $F \in (M)_k \subset \mathscr{H}_k$, this completes the proof.

REMARK. In [5], they showed the following (in more general form); if $M$ is an invariant subspace of $L^2(T^n)$ with $z_i M = M$ for $i = 1, 2, \ldots, n$ with $i \neq k$, then $M = \chi_U F\mathscr{H}_k \oplus \chi_V L^2(T^n)$, where $F$ is unimodular. In this case, it is easy to see that $M = F\mathscr{H}_k$ if and only if $M$ has no reducing subspaces and there is a function $f$ in $M$ with $|f| > 0$ a.e. $dm_n$. This fact is essentially pointed out, for the case $n = 2$, by Nakazi (see [6, Theorem 6]). Using this fact, we can also prove Lemma 1.

*Proof of Theorem 1.* Let $M_1$ be an invariant subspace of $H^2(T^n)$, $\phi \in L^\infty(T^n)$ and $M_2$ be the closure of $\phi M_1$ in $L^2(T^n)$. By Lemma 1, $(M_1)_k = F_k \mathscr{H}_k$ for some unimodular function $F_k$ for $k = 1, 2, \ldots, n$.
First suppose that $(M_1)_k \supset (M_2)_k$ for $k = 1, 2, \ldots, n$. Then

$$F_k \mathscr{H}_k = (M_1)_k \supset (M_2)_k \supset \phi(M_1)_k = \phi F_k \mathscr{H}_k.$$

Hence $\phi \mathscr{H}_k \subset \mathscr{H}_k$, so that $\phi \in \bigcap_{k=1}^n \mathscr{H}_k = H^2(T^n)$. Thus $\phi \in H^\infty(T^n)$.
Next suppose $\phi \in H^\infty(T^n)$. We note that $(M_2)_k$ coincides with the closure of $\phi(M_1)_k$ in $L^2(T^n)$. Since $\phi \mathscr{H}_k \subset \mathscr{H}_k$, we have

$$\phi(M_1)_k = \phi F_k \mathscr{H}_k \subset F\mathscr{H}_k = (M_1)_k.$$

Thus $(M_2)_k \subset (M_1)_k$. This completes the proof.

The following corollary is proved in [2, Proposition 3] using an idea of Schneider [10]. We can prove this as an application of our theorem.

COROLLARY 1. *Let* $\phi \in L^\infty(T^n)$ *and* $f \in H^2(T^n)$ *such that* $f \neq 0$ *and* $\phi^m f \in H^2(T^n)$ *for* $m = 1, 2, \ldots$ *. Then* $\phi \in H^\infty(T^n)$.

*Proof.* Let $M_1$ denote the invariant subspace of $H^2(T^n)$ generated by $\{\phi^m f; \ m = 1, 2, \ldots\}$. Let $M_2$ denote the closure of $\phi M_1$ in $L^2(T^n)$. Then $M_2 \subset M_1 \subset H^2(T^n)$, so that $(M_2)_k \subset (M_1)_k$ for $k = 1, 2, \ldots, n$. By Theorem 1, $\phi \in H^\infty(T^n)$.

The following is a direct corollary of our theorem. This answers the question posed in the introduction.

COROLLARY 2. *Let $M_1$ and $M_2$ be unitarily equivalent invariant subspaces of $H^2(T^n)$. Put $M_2 = \psi M_1$, where $\psi$ is unimodular. Then $\psi$ is inner if and only if $(M_1)_k \supset (M_2)_k$ for every $k = 1, 2, \ldots, n$.*

COROLLARY 3. *Let $M_1$ and $M_2$ be invariant subspaces of $H^2(T^n)$ such that $(M_1)_k = (M_2)_k$ for $k = 1, 2, \ldots, n$. Then $M_1$ is unitarily equivalent to $M_2$ if and only if $M_1 = M_2$.*

*Proof.* Suppose that $M_2 = \psi M_1$ and $\psi$ is unimodular. By Corollary 2, $\psi$ and $\bar{\psi}$ are inner. Hence $\psi$ is constant, so that $M_1 = M_2$.

COROLLARY 4. *Let $M_1$ be an invariant subspace of $H^2(T^n)$ such that $(M_1)_k = \mathcal{H}_k$ for $k = 1, 2, \ldots, n$. If $M_2$ is an invariant subspace of $H^2(T^n)$ with $M_2 = \psi M_1$, where $\psi$ is unimodular, then $\psi$ is inner.*

*Proof.* Since $M_2 \subset H^2(T^n)$,
$$(M_2)_k \subset \left( H^2(T^n) \right)_k = \mathcal{H}_k = (M_1)_k.$$
By Corollary 2, $\psi$ is inner.

An invariant subspace $M$ of $H^2(T^n)$ has *full range* if the closed linear span of $\{ \bar{z}_k^m M; \; m = 1, 2, \ldots \}$ coincides with $H_k$ for $k = 1, 2, \ldots, n$ (see [**2**, p. 5]).

By the following lemma, we can consider that Corollary 4 is a generalization of both Propositions 1 and 2 in [**2**].

LEMMA 2. *Let $M$ be one of the following invariant subspaces of $H^2(T^n)$.*
(1) *$M$ has full range.*
(2) *$M$ contains a nonzero function independent of $z_k$ for each $k = 1, 2, \ldots, n$.*
*Then $(M)_k = \mathcal{H}_k$ for $k = 1, 2, \ldots, n$.*

*Proof.* (1) Suppose that $M$ has full range. Then by the definitions, $H_i \subset (M)_k$ for $i \neq k$. Since $\mathcal{H}_k$ coincides with the linear span of $\{ H_i; \; i = 1, 2, \ldots, n$ and $i \neq k \}$, we get $\mathcal{H}_k \subset (M)_k$, so that $\mathcal{H}_k = (M)_k$.

(2) Suppose that $f_k \in M$ is a nonzero function independent of $z_k$. Then
$$(M)_k = \text{the closure of } L_k^\infty M \text{ in } L^2(T^n)$$
$$\supset \text{the closure of } L_k^\infty f_k \text{ in } L^2(T^n) = L_k^2,$$
the last equality follows from $|f_k| > 0$ a.e. $dm$. Since $z_k(M)_k \subset (M)_k$, we get $\mathcal{H}_k \subset (M)_k$, so that $\mathcal{H}_k = (M)_k$.

The following example shows that Corollary 4 is not covered by the work of Agrawal, Clark and Douglas [2].

EXAMPLE. For cases $n \geq 3$, there is an invariant subspace $M$ of $H^2(T^n)$ such that
(a) $M$ does not contain a function independent of $z_k$,
(b) $M$ does not have full range, and
(c) $(M)_k = \mathcal{H}_k$ for $k = 1, 2, \ldots, n$.
We shall show the existence of $M$ as above for $n = 3$. Let $\{\psi_i\}_{i=0}^{\infty}$ be a sequence of nonconstant inner functions in $H^{\infty}(T)$ satisfying the following conditions.
  (i) $\psi_i H^2(T) \subsetneqq \psi_{i+1} H^2(T)$ for every $i$, and
  (ii) $\bigcup_{i=0}^{\infty} \psi_i H^2(T)$ is dense in $H^2(T)$.
Let $M$ denote the invariant subspace of $H^2(T^3)$ generated by

$$\bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty} z_1^i z_2^j \psi_j(z_3) H^2(T^3).$$

Then every nonzero function in $M$ is not independent of $z_3$. Hence $M$ satisfies (a). By (i), $\psi_0(z_3) H^2(T^3) \subsetneqq H^2(T^3)$. Hence by the definition of $M$, the linear span of $\{\bar{z}_1^m M; \ m = 1, 2, \ldots\}$ does not contain $H^2(T^3)$, because it does not contain nonconstant functions. Thus $M$ satisfies (b). By (ii), $(M)_3 = \mathcal{H}_3$. Since the linear span of $\{z_3^k \psi_j(z_3); \ k$ is an integer$\}$ coincides with $L^2(T)$, we have $(M)_k = \mathcal{H}_k$ for $k = 1, 2$. Thus $M$ satisfies (c).

COROLLARY 5 [2, Proposition 4]. *Let $M$ and $M_1$ be invariant subspaces of $H^2(T^n)$ such that $M \supset M_1$ and $M_1$ has finite codimension in $M$. If $M_2$ is an invariant subspace of $M$ with $M_2 = \psi M_1$, where $\psi$ is unimodular, then $\psi$ is inner.*

*Proof.* Since $M \ominus M_1$ has finite dimension, it is easy to see $(M)_k = (M_1)_k$ for $k = 1, 2, \ldots, n$. Since $M \supset M_2$, $(M_2)_k \subset (M)_k = (M_1)_k$. By Corollary 2, $\psi$ is inner.

COROLLARY 6. *Let $M_1$ and $M_2$ be invariant subspaces of $H^2(T^n)$. Suppose that both of $M_1 \ominus M_2$ and $M_2 \ominus M_1$ have finite dimensions. Then $M_1$ and $M_2$ are unitarily equivalent if and only if $M_1 = M_2$.*

*Proof.* Let $M$ denote the invariant subspace generated by $M_1$ and $M_2$. Then $M_1$ and $M_2$ have finite codimensions in $M$. Put $M_2 = \psi M_1$ for some unimodular function $\psi$. By Corollary 5, $\psi$ is constant, so that $M_1 = M_2$.

**3. Outer functions.** Rudin [7] showed the following.

(i) If $S(f) = H^2(T^n)$ and $f \in H^2(T^n)$, then $f$ is outer.

(ii) There is an outer function $f$ such that $S(f) \neq H^2(T^n)$.
If $M$ is an invariant subspace of $H^2(T^n)$ such that $M$ is unitarily equivalent to $H^2(T^n)$, then $M = \psi H^2(T^n)$ for some inner function $\psi$ [2, Corollary 1]. In this section, we shall show that the above assertion is true if $H^2(T^n)$ is replaced by $S(f)$ for outer functions $f$.

THEOREM 2. *Let $f \in H^2(T^n)$ be an outer function. Then $(S(f))_k = \mathscr{H}_k$ for every $k = 1, 2, \ldots, n$.*

By Corollary 4, we get

COROLLARY 7. *Let $f \in H^2(T^n)$ be an outer function and let $M$ be an invariant subspace of $H^2(T^n)$. If $M$ is unitarily equivalent to $S(f)$, then $M = \psi S(f)$ for some inner function $\psi$.*

*Proof of Theorem 2.* Let $f \in H^2(T^n)$ be an outer function. Without loss of generality, we may assume $k = n$. By Lemma 1, $(S(f))_n = F_n \mathscr{H}_n$ for some unimodular function $F_n$ in $\mathscr{H}_n$. We shall show that $F_n$ is independent of $z_n$. We can write $f = F_n h$, where $h \in \mathscr{H}_n$. Write

$$z = (z', z_n) \in T^n, \quad \text{where } z' \in T^{n-1}.$$

Since $f$, $F_n$ and $h$ are contained in $\mathscr{H}_n$, there is a Borel subset $E$ of $T^{n-1}$ with $m_{n-1}(E) = 1$ such that for every fixed $z' \in E$,

$$(2) \qquad f(z', z_n), \ F_n(z', z_n), \ h(z', z_n) \in H^2(T)$$

and $F_n(z', z_n)$ is inner. Since $f(z', 0) \in H^2(T^{n-1})$,

$$\log|f(0)| = \log\left| \int_{T^{n-1}} f(z', 0) \, dm_{n-1}(z') \right|$$

$$\leq \int_{T^{n-1}} \log|f(z', 0)| \, dm_{n-1}(z') \quad \text{by [8, p. 47]}.$$

Hence, by our assumption,

$$\int_{T^{n-1}} \left\{ \log|f(z', 0)| - \int_T \log|f(z', z_n)| \, dm_1(z_n) \right\} dm_{n-1}(z') \geq 0.$$

Since $\log|f(z', 0)| \leq \int_T \log|f(z', z_n)| \, dm_1(z_n)$ for $z' \in E$,

$$\log|f(z', 0)| = \int_T \log|f(z', z_n)| \, dm_1(z_n) \quad \text{a.e. } z' \in E.$$

Thus $f(z', z_n)$ is outer for a.e. $z' \in E$. Since $f = F_n h$, for a.e. fixed $z' \in E$, we have

$$f(z', z_n) = F_n(z', z_n) h(z', z_n) \quad \text{a.e. } z_n \in T.$$

By (2), an inner function $F_n(z', z_n)$ is constant for a.e. $z' \in E$. Then for nonzero integers $i$,

$$\int_{T^n} F_n(z) z_n^i \, dm_n(z) = \int_{T^{n-1}} dm_{n-1}(z') \int_T F_n(z', z_n) z_n^i \, dm_1(z_n) = 0.$$

This implies that $F_n(z)$ is independent of $z_n$. Hence $F_n$ is invertible in $\mathscr{H}_n$, so that we get $(S(f))_n = \mathscr{H}_n$. This completes the proof.

## REFERENCES

[1]   O. P. Agrawal, *Invariant subspaces of shift operator for the quarter plane*, PhD Thesis, SUNY Stony Brook, 1983.

[2]   O. P. Agrawal, D. N. Clark and R. G. Douglas, *Invariant subspaces in the polydisk*, Pacific J. Math., **121** (1986), 1–11.

[3]   P. R. Ahern and D. N. Clark, *Invariant subspaces and analytic continuation in several variables*, J. Math. Mech., **19** (1970), 963–969.

[4]   A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math., **81** (1949), 239–255.

[5]   S. Merrill and N. Lal, *Characterization of certain invariant subspaces of $H^p$ and $L^p$ spaces derived from logmodular algebras*, Pacific J. Math., **30** (1969), 463–474.

[6]   T. Nakazi, *Certain invariant subspaces of $H^2$ and $L^2$ on a bidisc*, Preprint.

[7]   W. Rudin, *Proof of a conjecture of Helson*, Bull. Amer. Math. Soc., **74** (1968), 727–729.

[8]   _____, *Function Theory in Polydiscs*, Benjamin, New York, 1969.

[9]   _____, *Invariant subspaces of $H^2$ on a torus*, J. Functional Anal., **61** (1985), 378–384.

[10]  R. B. Schneider, *Isometries of $H^p(U^n)$*, Canad. J. Math., **25** (1973), 92–95.

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