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DETERMINATION OF THE INTERTWINING OPERATORS FOR HOLOMORPHICALLY INDUCED REPRESENTATIONS OF HERMITIAN SYMMETRIC PAIRS

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Let X and Y be two holomorphically induced representations for an irreducible Hermitian symmetric pair (G, K) with integral highest weights. Then $\mathrm{Hom}(X, Y)$ equals either $\mathbf{C}$ or zero. In this article we give formulas for $\mathrm{Hom}(X, Y)$ when the infinitesmial character of X and Y is integral and either regular or semi-regular. One formula is given in terms of the highest weight of X and the other in terms of the highest weight of Y.

1. Introduction. Let X and Y be two holomorphically induced representations for an irreducible Hermitian symmetric pair (G, K) with integral highest weights. Then $\mathrm{Hom}(X, Y)$ equals either $\mathbf{C}$ or zero ([2], [7]). In this article we give formulas for $\mathrm{Hom}(X, Y)$ when the infinitesimal character of X and Y is integral and either regular or semi-regular. One formula is given in terms of the highest weight of X and the other in terms of the highest weight of Y. These formulas were established for the case of $\mathrm{SU}(p, q)$ by the first two authors [3].

This article complements the results of [7] which include formulas for the composition factors of the modules X and Y above when G is of classical type. With this in mind we shall suppose the reader is familiar with the notation and results of [7], especially sections eight through thirteen of that article. Since [7] does not include formulas for the composition factors when G is of exceptional type we will provide formulas in those cases. These formulas will be consistent with the formulas of [7] and will be derived with the help of [4].

Recently, H. Jakobsen has given a formula for $\mathrm{Hom}(X, Y)$ when one of these modules is induces from a one dimensional module [9]. Our formula for $\mathrm{Hom}(N_y, N_x)$ below follows from Jakobsen's work in the cases where either x is the identity element or y is the element of maximal length in $\mathcal{W}^{\mathfrak{m}}$. Otherwise our results are disjoint.

For references to other work on the description of $\mathrm{Hom}(X, Y)$ the reader should consult the introductions and bibliographies in [2] and [7].

2. Statement of results. In order to state our results we recall some of the notation of [7]. Let $\mathfrak{g}$ (resp. $\mathfrak{m}$) be the complexified Lie algebra of G (resp. K), and let $\mathfrak{p}$ be a maximal parabolic subalgebra of $\mathfrak{g}$ with Levi component $\mathfrak{m}$ and nilradical $\mathfrak{u}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{m}$ and let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{p}$. The Weyl group of $\mathfrak{g}$ (resp. $\mathfrak{m}$) is $\mathcal{W}$ (resp. $\mathcal{W}_{\mathfrak{m}}$) and we have $\mathcal{W} = \mathcal{W}_{\mathfrak{m}}\mathcal{W}^{\mathfrak{m}}$ where $\mathcal{W}^{\mathfrak{m}}$ is the set of minimal length left coset representations. The projection of $\mathcal{W}$ onto $\mathcal{W}^{\mathfrak{m}}$ given by the decomposition $\mathcal{W} = \mathcal{W}_{\mathfrak{m}}\mathcal{W}^{\mathfrak{m}}$ is denoted by $\omega \mapsto \overline{\omega}$. Let Δ, $\Delta(\mathfrak{m})$ and $\Delta(\mathfrak{u})$ be the sets of $\mathfrak{h}$-roots of $\mathfrak{g}$, $\mathfrak{m}$ and $\mathfrak{u}$ respectively. Put $2\rho = \sum_{\alpha \in \Delta^+} \alpha$.

For $\lambda \in \mathfrak{h}^*$ we denote by $M(\lambda)$ the $\mathfrak{g}$-Verma module of highest weight $\lambda - \rho$. Then $L(\lambda)$ and $N(\lambda)$ denote the simple quotient and maximal $\mathfrak{m}$-locally finite quotient of $M(\lambda)$ respectively (cf. [7]). For $x \in \mathcal{W}^{\mathfrak{m}}$ we denote by N_x and L_x the modules $N(x\rho)$ and $L(x\rho)$. We adopt the convention that if all roots are the same length, they are called short.

DEFINITION 2.1. Set $\mathcal{M} = \{(\gamma, \nu) \mid \gamma, \nu \in \Delta^+$ and either $\langle \gamma, \nu \rangle \neq 0$ or both γ and ν are long roots$\}$. Let $\mathcal{S}(\Delta^+)$ denote the collection of all subsets Ω of Δ^+ which satisfy the following conditions:

(a) If γ, ν are in Ω, $\gamma \neq \nu$, then $(\gamma, \nu) \notin \mathcal{M}$.

(b) If γ is in Ω and ξ is in Δ^+ with $\gamma \neq \xi$, $(\gamma, \xi) \in \mathcal{M}$ and $\xi \leq \gamma$ then there is a ζ in Ω with $\zeta \neq \gamma$, $(\zeta, \xi) \in \mathcal{M}$ and $\zeta \leq \gamma$.

We note that $\mathcal{S}(\Delta^+)$ is defined for any positive system of any root system. When there is no chance of confusion we will denote this set simply as $\mathcal{S}$.

Fix x in $\mathcal{W}^{\mathfrak{m}}$. Then $\mathcal{S}_x$ will denote the set of all Ω in $\mathcal{S}$ which satisfy $x\Omega \subseteq \Delta(\mathfrak{u}) \cup -\Delta(\mathfrak{u})$. By $\mathcal{E}_x$ we will denote the collection of Ω in $\mathcal{S}_x$ which satisfy the additional condition:

(c) If γ is in Ω then there is a ζ in Ω with $\gamma \leq \zeta$ and $x\zeta \in \Delta(\mathfrak{u})$.

For $\Omega \in \mathcal{S}_x$, put $\Omega^+ = \{\gamma \in \Omega \mid x\gamma \in \Delta(\mathfrak{u})\}$ and $\Omega^- = \{\gamma \in \Omega \mid x\gamma \in -\Delta(\mathfrak{u})\}$. We say that Ω is x-positive (resp. x-negative) if $\Omega = \Omega^+$ (resp. $\Omega = \Omega^-$). Note that the x-positive elements of $\mathcal{S}_x$ are automatically in $\mathcal{E}_x$. Let $r_\Omega = \prod_{\gamma \in \Omega^+} s_\gamma$ and $t_\Omega = \prod_{\gamma \in \Omega^-} s_\gamma$.

We can now state our main result.

PROPOSITION 2.2. *Let x and y be in $\mathcal{W}^{\mathfrak{m}}$. Then*:

(a)

$$\operatorname{Hom}(N_y, N_x) = \begin{cases} \mathbf{C} & \text{if } y = \overline{xr_\Omega} \text{ for some } x\text{-positive } \Omega \text{ in } \mathcal{S}_x, \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$\operatorname{Hom}(N_y, N_x) = \begin{cases} \mathbf{C} & \text{if } x = \overline{yt_\Omega} \text{ for some } y\text{-negative } \Omega \text{ in } \mathscr{S}_y, \\ 0 & \text{otherwise.} \end{cases}$$

We will prove only part (a) of (2.2). The proof of (b) is exactly the same and we omit the details.

In order for the reader to better relate (2.2) to the theory of composition factors we recall the following proposition from [7].

PROPOSITION 2.3. *Assume that* $\mathfrak{g}$ *is of classical type and let* x *and* y *be in* $\mathscr{W}^{\mathrm{m}}$. *Then*

(a) *The mappings* $\Omega \mapsto \Omega^+$ *and* $\Omega \mapsto \overline{xr_\Omega}$ *are injective when restricted to* $\mathscr{E}_x$.

(b) *The simple module* L_y *is a composition factor of* N_x *if and only if* $y = \overline{xr_\Omega}$ *for some* Ω *in* $\mathscr{E}_x$. *Moreover, the composition factors of* N_x *occur with multiplicity one.*

We will verify in section four that (2.3) also holds when $\mathfrak{g}$ is of exceptional type (cf. (4.3)).

3. The classical cases. Assume that $\mathfrak{g}$ is of classical type. Let notation be as in [7], sections eight through thirteen. There are five classical cases to consider which we denote by HS.i, $1 \leq i \leq 5$ [7, Table 8.1]. These correspond respectively to the cases: SU(p, q), SO($2n - 1, 2$), Sp($2n, \mathbf{R}$), SO($2n - 2, 2$) and SO*($2n$). In the cases of HS.2 and HS.4 the result can be obtained directly from [**6**, §7] and [**6**, §9] respectively. If $(\mathfrak{g}, \mathfrak{p})$ is of type HS.1, HS.3 or HS.5 with constant p equal to one [7, (8.1)] then the result follows from [**6**, §6]. So we may assume that $(\mathfrak{g}, \mathfrak{p})$ is of type HS.1, HS.3 or HS.5 with constant $p > 1$. We proceed by induction on the constant p. If α is a simple root we put $\mathscr{W}_\alpha = \{x \in \mathscr{W}^{\mathrm{m}} \mid x\alpha \in -\Delta(\mathfrak{u})\}$. Let ω_α be the fundamental weight corresponding to α. For x or xs_α in $\mathscr{W}_\alpha$, write N_x^α and L_x^α for $N(x(\rho - \omega_\alpha))$ and $L(x(\rho - \omega_\alpha))$ respectively.

LEMMA 3.1. *Let* α *be a simple root and let* $x, y \in \mathscr{W}_\alpha$. *Then*:

(a) $\operatorname{Hom}(N_y^\alpha, N_x^\alpha) \cong \mathbf{C}$ *or zero depending as* $y = \overline{xr_\Omega}$ *for some* x*-positive set* Ω *in* $\mathscr{S}_{x,\alpha}$ *or not.*

(b) $\operatorname{Hom}(N_y^\alpha, N_x^\alpha) \cong \mathbf{C}$ *or zero depending as* $x = \overline{yt_\Omega}$ *for some* y*-negative set* Ω *in* $\mathscr{S}_{y,\alpha}$ *or not.*

Proof. Suppose that α is a short root and let Λ_α: $\mathcal{O}' \to \mathcal{O}_\alpha$ be the equivalence of categories guaranteed by [7, (10.1) and (11.2)]. The inductive hypotheses assure that (2.2) holds in the category $\mathcal{O}'$. Thus (3.1) follows from the formulas of [7, (10.5) and (11.7)].

If β is the long simple root of HS.3 then let α be the adjacent short simple root. (3.1) holds for α; and so, by the formulas of [7, (12.15), (12.16) and (12.17)] and the equivalence of categories in [7, (12.14)], (3.1) must also hold for β. This completes (3.1).

In [2], an algorithm was given for computing $\mathrm{Hom}(N_y, N_x)$ inductively, using [7, Proposition 11.2]. This algorithm is the main ingredient in the remaining part of our proof. For x and y in $\mathcal{W}^{\mathfrak{m}}$ set $d(y,x) = \dim(\mathrm{Hom}(N_y, N_x))$. We know from [2] or [7] that $d(y,x) \leq 1$. If x has maximal length in $\mathcal{W}^{\mathfrak{m}}$ then $N_x = L_x$, there are no nonempty x-positive sets in $\mathcal{S}_x$ and $d(y,x) \neq 0$ if and only if $x = y$. Thus we may assume that x does not have maximal length.

Fix a nonempty x-positive set Ω in $\mathcal{E}_x$ and set $y = \overline{xr_\Omega}$. By [8, proof of (4.2)], if Ω is a set of simple roots then the standard map from N_y to N_x is nonzero and $d(y,x) = 1$. Thus we assume that Ω has some non-simple roots.

Choose γ to be any root in Ω that is minimal with respect to $\leq$ among the non-simple roots of Ω. We claim that there is short simple root β with

$$\langle \beta, \gamma^\vee \rangle = 1. \tag{3.2}$$

If γ is of the form $e_i - e_j$ then set $\beta = e_i - e_{i+1}$. If $\gamma = e_i + e_j$ with $i \leq j < n$ then set $\beta = e_j - e_{j+1}$. If $\gamma = e_i + e_n$, $1 \leq i < n-1$, then set $\beta = e_i - e_{i+1}$. Finally, suppose that $\gamma = e_{n-1} + e_n$. Since γ is not simple, $(\mathfrak{g}, \mathfrak{p})$ must be of type HS.3. Set $\lambda = 2e_n$. Then $(\lambda, \gamma) \in \mathcal{M}$ and $\lambda \leq \gamma$. Thus, by (2.1), there is an $\alpha \in \Omega$ with $\alpha \neq \gamma$ and $\alpha \leq \gamma$. This forces α to be $e_{n-1} - e_n$. This contradicts the assumption that Ω is in $\mathcal{E}_x$ since $x\alpha$ and $x\gamma$ cannot both be in $\Delta(\mathfrak{u}) \cup -\Delta(\mathfrak{u})$. Thus the final case does not occur. This proves the claim.

Let β be as in (3.2). Then $\beta \leq \gamma$; and so by (2.1), there is an α in Ω with $\alpha \leq \gamma$, $\alpha \neq \gamma$ and $(\alpha, \beta) \in \mathcal{M}$. Minimality of γ forces α to be simple. Thus, since β is a short root, α and β are adjacent simple roots.

LEMMA 3.3. *Let Ω, α, β and γ be as above. Then α and γ are the only roots in Ω not orthogonal to β.*

Proof. Since $x\alpha \in \Delta(\mathfrak{u})$ and $\langle x\alpha^\vee, x\beta \rangle = -1$, $x\beta \notin \Delta(\mathfrak{u})$. Similarly, since $x\gamma \in \Delta(\mathfrak{u})$ and $\langle x\gamma^\vee, x\beta \rangle = 1$, $x\beta \notin -\Delta(\mathfrak{u})$. Thus $x\beta \in \Delta(\mathfrak{m})$.

Let μ be the unique simple root in $\Delta(\mathfrak{u})$. Let ζ be any third root in Ω. Then, $x\zeta \in \Delta(\mathfrak{u})$. However, $xs_\alpha s_\zeta \beta = x\beta + x\alpha - \langle \beta, \zeta^\vee \rangle x\zeta$ cannot have μ-coefficient larger than 1; and so, $\langle \beta, \zeta^\vee \rangle \geq 0$. Similarly, $xs_\gamma s_\zeta \beta = x\beta - x\gamma - \langle \beta, \zeta^\vee \rangle x\zeta$ cannot have μ-coefficient less than -1; and so, $\langle \beta, \zeta^\vee \rangle \leq 0$. Thus $\langle \beta, \zeta \rangle = 0$. This proves the lemma.

From the lemma we see that $y\beta = \overline{xr_\Omega}\beta = mxs_\alpha s_\gamma \beta = m(xs_\alpha \beta - x\gamma)$ where $m \in \mathscr{W}_{\mathfrak{m}}$. Also, $xs_\alpha \beta = x(\alpha + \beta)$. Since $x\alpha$, $x\gamma \in \Delta(\mathfrak{u})$, we have exactly two possibilities,

$$xs_\alpha \beta \in \Delta(\mathfrak{m}) \quad \text{and} \quad y\beta \in -\Delta(\mathfrak{u})$$

or

$$xs_\alpha \beta \in \Delta(\mathfrak{u}) \quad \text{and} \quad y\beta \in \Delta(\mathfrak{m}).$$

In each of these cases, by [**2**, (2.3)], $d(y, xs_\alpha) = 0$. The short exact sequence

$$0 \to N_x \to \phi_\alpha N_x^\alpha \to N_{xs_\alpha} \to 0$$

gives rise to the long exact sequence

$$0 \to \mathrm{Hom}(N_y, N_x) \to \mathrm{Hom}(N_y, \phi_\alpha N_x^\alpha) \to \mathrm{Hom}(N_y, N_{xs_\alpha}) \to \cdots.$$

We have seen that the last term here is zero. Combining this with (3.1) and the adjoint property of ϕ_α and ψ_α we see $d(y, x) = \dim(\mathrm{Hom}(\psi_\alpha N_y, N_x^\alpha)) = \dim(\mathrm{Hom}(N_y^\alpha, N_x^\alpha)) = 1$.

Conversely, suppose now that $d(y, x) = 1$. Then by (2.3) there is an Ω in $\mathscr{E}_x$ with $y = \overline{xr_\Omega}$. We must show that Ω is x-positive. Let α be a simple root in Ω. Then $y\alpha \in -\Delta(\mathfrak{u})$, so there is a surjection $\phi_\alpha N_y^\alpha \to N_y \to 0$. This gives an injection $0 \to \mathrm{Hom}(N_y, N_x) \to \mathrm{Hom}(N_y^\alpha, N_x^\alpha)$. Thus by (3.1), $\Omega \setminus \{\alpha\}$ is an x-positive set in $\mathscr{S}_{x,\alpha}$. It remains only to show: $\alpha \in \Omega^+$. Suppose $\alpha \in \Omega^-$, i.e. $x\alpha \in -\Delta(\mathfrak{u})$. If μ is any other simple root in Ω then, as above, $\Omega \setminus \{\mu\}$ is x-positive and thus $x\alpha \in \Delta(\mathfrak{u})$. This contradiction assures that α is the unique simple root in Ω. Set $z = xs_\alpha$. Then Ω is a z-positive element of $\mathscr{E}_z$ and $y = \overline{z\prod_{\gamma \in \Omega} s_\gamma}$. But then, by the preceding paragraph, $d(y, z) = 1$ and $d(y, zs_\alpha) = 0$. However, $d(y, zs_\alpha) = d(y, x) = 1$. This contradiction shows that α must lie in Ω^+ and completes the proof of (2.2) in the classical cases.

4. The exceptional cases. Here we verify (2.2) when $\mathfrak{g}$ is of exceptional type E_6 or E_7. To make sense of (2.2) in the exceptional cases we must first prove a generalization of (2.3) for these cases.

For $x, y \in \mathscr{W}^{\mathfrak{m}}$ write (N_x, L_y) for the multiplicity of L_y as a composition factor of N_x. We write F_x for the set of all $y \in \mathscr{W}^{\mathfrak{m}}$ with $(N_y, L_x) = 1$. The labeled posets $\mathscr{W}^{\mathfrak{m}}$ for the two exceptional cases are given in Figures 4.1 and 4.2. These are consistent with the notation of [**4**]

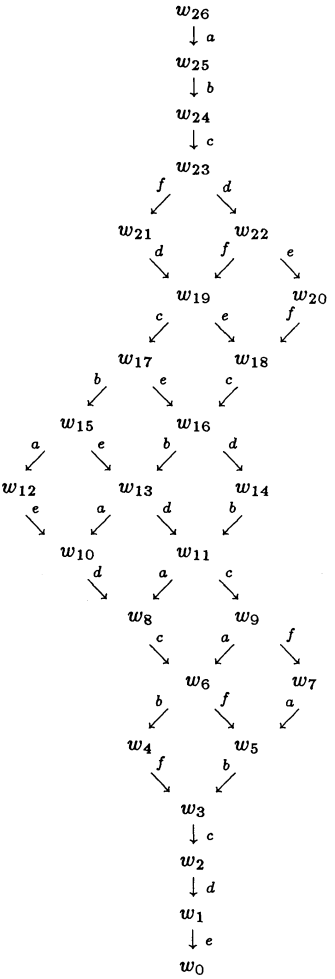


FIGURE 4.1. $\mathscr{W}^{\mathrm{m}}$ for E_6

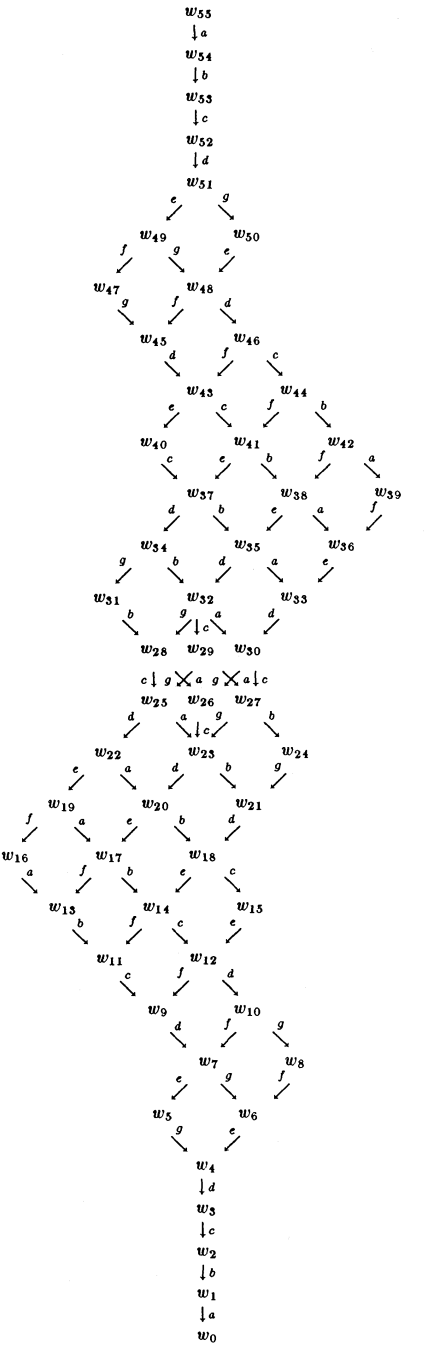
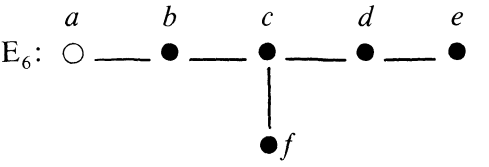


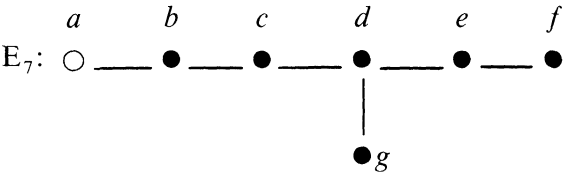
FIGURE 4.2. $\mathscr{W}^{\mathrm{m}}$ for E_7

and the labelings of the posets are consistent with the following labelings of the Dynkin diagrams:

(4.1)



(4.2)



In the labeled posets, the notation $w_i \xrightarrow{\alpha} w_j$ means that α is a simple root, $w_i = w_j s_\alpha$ and $w_j \in \mathscr{W}_\alpha$. We will use the Bourbaki convention for denoting roots. Simple roots will be denoted by the letter which labels them in the Dynkin diagram (4.1) or (4.2). In E_6,

$$\begin{pmatrix} r_a & r_b & r_c & r_d & r_e \\ & & r_f & & \end{pmatrix} = r_a a + r_b b + r_c c + r_d d + r_e e + r_f f.$$

In E_7,

$$\begin{pmatrix} r_a & r_b & r_c & r_d & r_e & r_f \\ & & & r_g & & \end{pmatrix} = r_a a + r_b b + r_c c + r_d d + r_e e + r_f f + r_g g.$$

Recall from [7, (8.3)] the definition of the orthogonal sets of noncompact positive roots Σ_x and Σ_x^+ for each $x \in \mathscr{W}^{\mathfrak{m}}$. For any orthogonal set Ω contained in $\Delta(\mathfrak{u})$ set $s_\Omega = \prod_{\gamma \in \Omega} s_\gamma$. Recall that if α is any simple root and $x \in \mathscr{W}_\alpha$ then $\Sigma_{x,\alpha} = \Sigma_x \setminus \{-x\alpha\}$ and $\Sigma^+_{x,\alpha} = \Sigma^+_{xs_\alpha} \setminus \{-x\alpha\}$. Recall also the definition of the sets $\mathscr{S}_{x,\alpha}$ from [7, (9.4)].

PROPOSITION 4.3. *Let* $(\mathfrak{g}, \mathfrak{p})$ *be of exceptional type* E_6 *or* E_7 *and let* x *and* y *be in* $\mathscr{W}^{\mathfrak{m}}$. *Then*

(a) $-x^{-1}\Sigma_x$ *is in* $\mathscr{S}_x$.

(b) $(N_y, L_x) = 1$ *if and only if* $y = \overline{s_\Omega x}$ *for some* $\Omega \subseteq \Sigma_x$.

(c) $(N_y, L_x) = 1$ *if and only if* $x = \overline{x r_\Omega}$ *for some* $\Omega \in \mathscr{E}_y$.

(d) ([**1**]) *The composition factors of* N_x *occur with multiplicity one.*

(e) $x^{-1}\Sigma_x^+$ *is in* $\mathscr{S}_x$.

Proof. We begin by proving parts (a) and (b). Choose $x \in \mathscr{W}_f$ (cf. (4.1) and (4.2)). In Tables 4.1 and 4.2 we have listed $-x^{-1}\Sigma_x$ and all those N_y in which L_x appears as a composition factor, i.e. F_x. The computation of $-x^{-1}\Sigma_x$ has been done directly and the composition factors are taken from [**4**]. We claim that for each $x \in \mathscr{W}_f$, $-x^{-1}\Sigma_x$ is in $\mathscr{S}_x$. It suffices only

TABLE 4.1

E_6 data for $w_j \in \mathcal{W}_f$. The sets $-w_j^{-1}\Sigma_{w_j}$ and $F_j = \{i \mid (N_{w_i}, L_{w_j}) = 1\}$.

j	$-w_j^{-1}\Sigma_{w_j}$	F_j
21	f	23,21
19	f, d	23,22,21,19
18	f, e	22,20,19,18
7	$f, \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ & & 1 & & \end{pmatrix}$	20,18,9,7
5	f, a	9,7,6,5
3	f, b	6,5,4,3

TABLE 4.2

Case E_7 data for $w_j \in \mathcal{W}_f$. The sets $-w_j^{-1}\Sigma_{w_j}$ and $F_j = \{i \mid (N_{w_i}, L_{w_j}) = 1\}$.

j	$-w_j^{-1}\Sigma_{w_j}$	F_j
47	f	49,47
45	f, g	49,48,47,45
43	f, d	48,46,45,43
41	f, c	46,44,43,41
38	f, b	44,42,41,38
36	f, a	42,39,38,36
16	$f, \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$	42,38,19,16
13	$f, a, \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$	42,39,38,36,19,17,16,13
11	$f, b, \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$	44,42,41,38,17,14,13,11
9	$f, c, \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$	46,44,43,41,14,12,11,9
7	$f, d, \begin{pmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$	48,46,45,43,12,10,9,7
6	$f, g, \begin{pmatrix} 1 & 2 & 3 & 4 & 2 & 1 \\ & & & 2 & & \end{pmatrix}$	49,48,47,45,10,8,7,6

to show that $-x^{-1}\Sigma_x$ satisfies (2.1b). Choose $\xi \in \Delta^+$ and $\gamma \in -x^{-1}\Sigma_x$ with $\xi \leq \gamma$, $\xi \neq \gamma$ and $\langle \xi, \gamma \rangle \neq 0$. Then γ cannot be simple; and so, by Tables 4.1 and 4.2, either $\mathfrak{g}$ is E_6 and $x = w_7$ or $\mathfrak{g}$ is E_7 and $x = w_j$ with $j \in \{6, 7, 9, 11, 13, 16\}$. A case by case check in these seven cases shows that (2.1b) holds. For each $x \in \mathcal{W}_f$ we see from Tables 4.1 and 4.2 that $\text{card}(F_x) = 2^{\text{card}(\Sigma_x)}$. Thus, to prove (4.3b) for $x \in \mathcal{W}_f$ it suffices to observe that for each $\Omega \subseteq \Sigma_x$, $\overline{s_\Omega x} \in F_x$. This computation can be verified from the tables.

Now fix any $x \in \mathcal{W}^{\mathfrak{m}}$. If $x = e$ then $\Sigma_x = \varnothing$ and $F_x = \{x\}$. So assume that $x \neq e$. Then there is a simple root α with $x \in \mathcal{W}_\alpha$. If $\alpha = f$

then we have already seen that (4.3a) and (4.3b) hold for x. So assume that $\alpha \neq f$. By a theorem of Vogan, [**10**], or more specifically [**7**, (10.1)], there is an equivalence of categories π: $\mathcal{O}_\alpha \to \mathcal{O}_f$. Define $w \in \mathcal{W}_f$ by $\pi(L_x) \cong L_w$. Then by (4.3a) and (4.3b) for w, $-w^{-1}\Sigma_{w,f} \in \mathcal{S}_{w,f}$ and $F_w \cap \mathcal{W}_f = \{\overline{s_\Omega w} | \Omega \subseteq \Sigma_{w,f}\}$. It follows from [**7**, (10.4) and (10.5)] that $-x^{-1}\Sigma_{x,\alpha} \in \mathcal{S}_{x,\alpha}$ and $F_x \cap \mathcal{W}_\alpha = \{\overline{s_\Omega x} | \Omega \subseteq \Sigma_{x,\alpha}\}$. Thus, by [**7**, (9.5)], $-x^{-1}\Sigma_x \in \mathcal{S}_x$. We also see, exactly as in [**7**, (12.4)], that $F_x = \{\overline{s_\Omega x} | \Omega \subseteq \Sigma_x\}$. This completes the proof of (4.3a) and (4.3b).

Part (c) follows from parts (a) and (b) and [**7**, (9.6)]. Part (d) comes from [**1**]. Finally, (e) follows from (a) and [**5**, (5.1)]. This proves (4.3).

The proof of (2.2) in the exceptional cases proceeds as in the classical cases. We need only establish (3.1) and (3.2) for the exceptional cases. These will be (4.5) and (4.6) (respectively) below. For any $x \in \mathcal{W}^{\mathrm{m}}$, put $H_x = \{ y \in \mathcal{W}^{\mathrm{m}} \mid \mathrm{Hom}(N_y, N_x) \cong \mathbf{C}\}$. Recall from [**1**] that we know $d(y, x) \leq 1$ for all $x, y \in \mathcal{W}^{\mathrm{m}}$. Moreover, from [**2**], the sets H_x are known explicitly. We wish only to show:

$$H_x = \{\overline{xr_\Omega} | \Omega \text{ an } x\text{-positive set in } \mathcal{S}_x\}. \tag{4.4}$$

LEMMA 4.5. *Let α be a simple root and let $x, y \in \mathcal{W}_\alpha$. Then* $\mathrm{Hom}(N_y^\alpha, N_x^\alpha) \cong \mathbf{C}$ *or zero depending as $y = \overline{xr_\Omega}$ for some x-positive set Ω in $\mathcal{S}_{x,\alpha}$ or not.*

Proof. In Tables 4.3 and 4.4 we have listed the x-positive sets in $\mathcal{S}_{x,f}$ and the sets $H_{x,f} = \{ y \in \mathcal{W}_f \mid \mathrm{Hom}(N_y^f, N_x^f) \neq 0\}$. The x-positive sets are found by direct computation while $H_{x,f}$ is gleaned from [**2**]. From the tables it follows by direct computation that (4.5) holds for $\alpha = f$. By [**7**, (10.1) and (10.5)], the lemma holds for all simple α.

TABLE 4.3

E_6 data for $w_j \in \mathcal{W}_f$. The w_j-positive sets in $\mathcal{S}_{w_j,f}$ and $H_{w_j,f} = \{ i \mid w_i \in \mathcal{W}_f, \mathrm{Hom}(N_{w_i}^f, N_{w_j}^f) \neq 0\}$.

j	w_j-positive sets in $\mathcal{S}_{w_j}$	$H_{w_j,f}$
21	$\emptyset$, $\{d\}$	21, 19
19	$\emptyset$, $\{e\}$	19, 18
18	$\emptyset$, $\{\begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ & & 1 & & \end{pmatrix}\}$	18, 7
7	$\emptyset$, $\{a\}$	7, 5
5	$\emptyset$, $\{b\}$	5,3
3	$\emptyset$	3

TABLE 4.4

E_7 data for $w_j \in \mathcal{W}_f$. The w_j-positive sets in $\mathcal{S}_{w_j,f}$ and $H_{w_j,f} = \{i \mid w_i \in \mathcal{W}_f, \operatorname{Hom}(N_{w_i}^f, N_{w_j}^f) \neq 0\}$.

j	w_j-positive sets in $\mathcal{S}_{w_j}$	$H_{w_j,f}$
47	$\emptyset$, $\{g\}$, $\{g, \begin{pmatrix} 1 & 2 & 3 & 4 & 2 & 1 \\ & & & 2 & & \end{pmatrix}\}$	47, 45, 6
45	$\emptyset$, $\{d\}$, $\{d, \begin{pmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ & & & 1 & & \end{pmatrix}\}$	45, 43, 7
43	$\emptyset$, $\{c\}$, $\{c, \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}\}$	43, 41, 9
41	$\emptyset$, $\{b\}$, $\{b, \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}\}$	41,38,11
38	$\emptyset$, $\{a\}$, $\{\gamma\}$, $\{a, \gamma\}$ where $\gamma = \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$	38, 36, 16, 13
36	$\emptyset$, $\{\begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}\}$	36, 13
16	$\emptyset$, $\{a\}$	16, 13
13	$\emptyset$, $\{b\}$	13, 11
11	$\emptyset$, $\{c\}$	11, 9
9	$\emptyset$, $\{d\}$	9, 7
7	$\emptyset$, $\{g\}$	7, 6
6	$\emptyset$	6

Let γ be any non-simple positive root. We now claim:

(4.6) There is a simple root β with $\langle \beta, \gamma^\vee \rangle = 1$.

Let S be the set of simple roots and identify S with the Dynkin diagram of $\mathfrak{g}$. Write $\gamma = \sum_{\mu \in S} r_\mu \mu$. Let $S_\gamma = \{\mu \in S \mid r_\mu \neq 0\}$. Then S_γ forms a connected Dynkin subdiagram of S. Suppose that γ satisfies the following condition.

(4.7) There exist two adjacent simple roots β and ζ with $r_\beta = r_\zeta = 1$ and β an extreme root of the diagram S_γ.

Then $\langle \beta, \gamma^\vee \rangle = 1$ and $\beta \leq \gamma$, as required. Thus we are reduced to considering only those roots γ which do not satisfy (4.7). These roots, together with appropriate β, are listed in Tables 4.5 and 4.6. In Table 4.6, which gives the necessary roots for E_7, we have only included those roots for which $r_a \neq 0$. The E_7 roots for which $r_a = 0$ can be thought of as roots for E_6 and are then included in Table 4.5. The tables complete the proof of (4.6).

TABLE 4.5

E_6 roots γ which do not satisfy (4.7) and corresponding β with $\langle \beta, \gamma^\vee \rangle = 1$.

γ	β
$\begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ & & 1 & & \end{pmatrix}$	c
$\begin{pmatrix} 1 & 2 & 2 & 1 & 0 \\ & & 1 & & \end{pmatrix}$	b
$\begin{pmatrix} 0 & 1 & 2 & 2 & 1 \\ & & 1 & & \end{pmatrix}$	d
$\begin{pmatrix} 1 & 2 & 2 & 2 & 1 \\ & & 1 & & \end{pmatrix}$	b
$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ & & 1 & & \end{pmatrix}$	c
$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \end{pmatrix}$	f

TABLE 4.6

E_7 roots γ with $r_\alpha \geq 1$ and which do not satisfy (4.7) and corresponding β with $\langle \beta, \gamma^\vee \rangle = 1$.

γ	β
$\begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 0 \\ & & & 1 & & \end{pmatrix}$	b
$\begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$	b
$\begin{pmatrix} 1 & 2 & 2 & 3 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$	b
$\begin{pmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ & & & 1 & & \end{pmatrix}$	c
$\begin{pmatrix} 1 & 2 & 2 & 3 & 2 & 1 \\ & & & 2 & & \end{pmatrix}$	b
$\begin{pmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ & & & 2 & & \end{pmatrix}$	c
$\begin{pmatrix} 1 & 2 & 3 & 4 & 2 & 1 \\ & & & 2 & & \end{pmatrix}$	d
$\begin{pmatrix} 1 & 2 & 3 & 4 & 3 & 1 \\ & & & 2 & & \end{pmatrix}$	e
$\begin{pmatrix} 1 & 2 & 3 & 4 & 3 & 2 \\ & & & 2 & & \end{pmatrix}$	f

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