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TRANSITIVE ISOMETRY GROUPS WITH NONCOMPACT ISOTROPY

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Let G be a connected Lie group acting effectively and transitively by isometries on a riemannian manifold M . Then G is a Lie subgroup of the full isometry group, which is not necessarily closed. In this paper we study the structure of the closure of G in $I(M)$ and illustrate the results with examples, with non-compact isotropy, where the closure is described explicitly.

Introduction. If M is a riemannian manifold the isotropy group in $I(M)$ is compact; hence a homogeneous riemannian M can always be represented as a quotient G'/H' with H' compact.

Assume now that G is a connected Lie group acting effectively and transitively by isometries on M . Then G is a Lie subgroup of $I(M)$ which will be closed in $I(M)$ if and only if the isotropy subgroup H is compact.

In this paper we study in detail the closure of G in $I(M)$. Also if G is any connected Lie group and H a closed subgroup we compare three standard conditions on H which ensure that G/H admits a riemannian invariant structure. The rest of the paper is devoted to illustrate the fact that it is rather common to have transitive, effective, non-closed Lie subgroups of $I(M)$, hence the isotropy subgroup is non-compact. This situation arises quite frequently, even when M is compact (Lemma 1.4). Also, any connected semisimple Lie group with infinite center admits a closed non-compact subgroup H such that G acts effectively on G/H and G/H carries a G -invariant riemannian structure. In this case, that is, when G is semisimple, we give an upper bound for the dimension of \overline{G}_L and provide examples showing that these bounds are sharp (see (2.4), Proposition 2.3 and Remark 2.4).

In [DMW] the use of \overline{G}_L proved to be convenient in the study of bounded isometries on a riemannian manifold acted on transitively and effectively by a semisimple Lie group without local compact factors. Also, some examples where G_L is not closed in $I(M)$ (G semisimple) are given in [DMW] Example (3.10). The authors would like to thank J. A. Wolf for very useful comments on a first version of this paper and, in particular, for suggesting a simpler proof of Proposition 2.2.

1. Closure of G . Let G be a connected Lie group with Lie algebra \mathfrak{g} and let H be a closed subgroup of G with Lie algebra \mathfrak{h} . We will say that H satisfies condition (1) (respectively (2), (3)) if

(1) $\overline{\text{Ad}_G(H)}$ is compact in $\text{GL}(\mathfrak{g})$, the closure with respect to the $\text{GL}(\mathfrak{g})$ topology;

(2) $\text{cl}(\text{Ad}_G(H))$ is compact in $\text{Ad}(G)$, where cl denotes closure with respect to the $\text{Ad}(G)$ topology;

(3) \mathfrak{h} is a compactly embedded subalgebra of \mathfrak{g} (or equivalently see [H, p. 130]) $\text{Ad}_G(H_0)$ is compact in $\text{Ad}(G)$, where H_0 denotes the identity component of H .

Along this section we will analyze the relationship among the different conditions (1), (2), (3).

Condition (1). It is well known (cf. [Ch E]. Ch. III) that if G acts effectively on G/H then H satisfies (1) if and only if G/H admits a G -invariant riemannian structure. Furthermore for each such structure, G is a Lie subgroup of the full isometry group $I(M)$, $M = G/H$, via $x \rightarrow L_x$, L_x the left translation on $M = G/H$. If S is a Lie subgroup of G , denote $S_L = \{L_x | x \in S\}$. The next theorem describes the closure of G_L in $I(M)$.

1.1. THEOREM. *Let G be a connected Lie group acting effectively on $M = G/H$, H a subgroup of G satisfying (1). Then*

$$\overline{G}_L = \{L_x \circ \varphi : x \in G, \varphi \in \overline{H}_L\}.$$

In particular, G_L is closed if and only if H is compact. Moreover

- (i) H_L is isomorphic to $\overline{\text{Ad}(H)}$ (as Lie groups).
- (ii) *There is an exact sequence*

$$\{1\} \rightarrow \{(x, \text{Ad}(x^{-1})) : x \in H\} \rightarrow G \rtimes \overline{\text{Ad}(H)} \rightarrow \overline{G}_L \rightarrow \{1\}$$

where \rtimes denotes semidirect product.

Proof. If L_{x_n} converges in $I(M)$ then $x_n H \rightarrow x^* H$ in G/H , for some x^* in G . Hence there is a sequence h_n in H such that $x_n h_n \rightarrow x^*$ in G . Now $L_{h_n^{-1}} \in H_L$ and since H_L is contained in the isotropy subgroup in $I(M)$, $L_{h_n^{-1}} \rightarrow \phi$, $\phi \in \overline{H}_L$. As a consequence $L_{x_n} = L_{x_n h_n} \circ L_{h_n^{-1}}$ converges to $L_{x^*} \circ \phi$ and the first assertion follows.

Clearly, if H is compact, $\overline{H}_L = H_L \subset G_L$. Conversely if G_L is closed in $I(M)$ then H_L is compact in G_L since H_L is the intersection of G_L with the isotropy subgroup in $I(M)$.

Next we prove (i). Let \mathfrak{m} be an $\overline{\text{Ad}(H)}$ invariant complement of \mathfrak{h} in \mathfrak{g} where \mathfrak{h} denotes the Lie algebra of H . It is not hard to show that the canonical map $\overline{H}_L \rightarrow \overline{\text{Ad}(H)}|_{\mathfrak{m}}$ sending $\varphi \rightarrow (d\varphi)_{eH}$ is an isomorphism of Lie groups. Hence (i) will follow once we prove that $\overline{\text{Ad}(H)}$ and $\overline{\text{Ad}(H)}|_{\mathfrak{m}}$ are isomorphic.

Let $\pi: \overline{\text{Ad}(H)} \rightarrow \overline{\text{Ad}(H)}|_{\mathfrak{m}}$ denote the restriction homomorphism. By compactness of $\overline{\text{Ad}(H)}$ it follows that π is onto. If T is in $\overline{\text{Ad}(H)}$ and $T|_{\mathfrak{m}}$ is the identity, then $T|_{\mathfrak{m}_1}$ is the identity for any \mathfrak{m}_1 , an $\overline{\text{Ad}(H)}$ invariant complement of \mathfrak{h} . Hence we may take \mathfrak{m}_1 to be the orthogonal complement of \mathfrak{h} with respect to an $\overline{\text{Ad}(H)}$ invariant inner product and consequently \mathfrak{m}_1 will satisfy $[\mathfrak{h}, \mathfrak{m}_1] \subset \mathfrak{m}_1$. If $X \in \mathfrak{h}$, $Y \in \mathfrak{m}_1$

$$[TX - X, Y] = [TX, Y] - [X, Y] = T[X, Y] - [X, Y] = 0.$$

Hence $\text{Ad}(\exp(TX - X))$ is the identity in \mathfrak{m}_1 or equivalently $(dL_{\exp(TX - X)})_{eH}$ is the identity in G/H . Since we are assuming that the action of G on G/H is effective it follows that $\exp(TX - X) = e$ for all X in \mathfrak{h} , hence T is the identity on \mathfrak{g} .

Now (ii) follows immediately from the description of \overline{G}_L , after observing that the kernel of the Lie group homomorphism $G \times \overline{H}_L \rightarrow \overline{G}_L$, $(x, \varphi) \rightarrow L_x \circ \varphi$ consists of pairs $(x, L_{x^{-1}})$ where x is in H . \square

Let G be a connected Lie group and H a subgroup of G satisfying (1). Set

$$\text{Aut}(G, H) = \{ \varphi \in \text{Aut}(G): \varphi(H) = H, (d\varphi)_e \in \overline{\text{Ad}(H)} \}.$$

If $\varphi \in \text{Aut}(G, H)$, $\overline{\varphi}: G/H \rightarrow G/H$ will denote the induced diffeomorphism, $\overline{\varphi}(xH) = \varphi(x)H$. If G acts effectively on $M = G/H$ then, for any G -invariant riemannian structure on M , $\overline{\varphi}$ is an isometry of M . Furthermore, if $(d\varphi)_e = \lim \text{Ad}(h_n)$, the sequence $L_{h_n} \rightarrow \psi$, ψ in \overline{H}_L (by taking a subsequence if necessary), hence $(dL_{h_n})_{eH} \rightarrow (d\psi)_{eH}$, that is, $\overline{\varphi} = \psi$ since they both fix eH .

1.2. PROPOSITION. *Assume the hypotheses of Theorem 1.1 hold. Then the correspondence $\varphi \rightarrow \overline{\varphi}$, $\text{Aut}(G, H) \rightarrow \overline{H}_L$ is a group isomorphism.*

Proof. It is clearly a homomorphism. To check it is surjective assume $\psi = \lim L_{h_n}$, $h_n \in H$. By taking a subsequence, if necessary, we have that $\text{Ad}(h_n) \rightarrow T$, T an automorphism of \mathfrak{g} . If ϕ denotes the lifting of T to \tilde{G} , \tilde{G} the simply connected covering group of G , then ϕ is the identity restricted to the center of \tilde{G} . In fact, let \tilde{h}_n in \tilde{G} be such that $d(I_{\tilde{h}_n})_e = \text{Ad}(h_n)$. Then $I_{\tilde{h}_n}(\exp X) = \exp \text{Ad}(h_n)X \rightarrow \exp TX = \phi(\exp X)$ and since

\tilde{G} is connected I_{h_n} converges pointwise to ϕ ; the assertion follows. As a consequence ϕ induces an automorphism $\varphi: G \rightarrow G$ lifting T and since I_{h_n} converges pointwise to φ then $\varphi(H) \subset H$. By repeating the above argument with T^{-1} we conclude that $\varphi \in \text{Aut}(G, H)$. Also from $I_{h_n}(x) \rightarrow \varphi(x)$, x in G , it follows that $L_{h_n}(xH) \rightarrow \bar{\varphi}(xH)$ and consequently $\bar{\varphi} = \psi$.

Assume $\bar{\varphi}$ is the identity on G/H . Then $y^{-1}\varphi(y) \in H$, for all y in G . Hence

$$\left. \frac{d}{dt} \right|_{t=0} \exp(-tY)\varphi(\exp tY) = -Y + (d\varphi)_e Y \in \mathfrak{h}$$

for all Y in \mathfrak{g} . As a consequence $(d\varphi)_e|_{\mathfrak{m}}$ is the identity, \mathfrak{m} an $\text{Ad}(H)$ invariant complement of \mathfrak{h} . Hence $(d\varphi)_e$ is the identity on \mathfrak{g} (see the proof of (i), Theorem 1.1) or equivalently, $\varphi = \text{Id}$. \square

Condition (2). Let $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$ be the adjoint representation, Z the center of G , $\text{ad}: \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$ the derivative of Ad . Since $\text{Ad}(G)$, the connected Lie subgroup of $\text{Aut}(\mathfrak{g})$ with Lie algebra $\text{ad}(\mathfrak{g})$, is isomorphic to G/Z , condition (2) is equivalent to

$$(2') \quad \pi(\overline{ZH}) \text{ compact in } G/Z,$$

where $\pi: G \rightarrow G/Z$ is the standard projection. Furthermore, since $\text{Aut}(\mathfrak{g})$ carries the relative topology of $\text{Gl}(\mathfrak{g})$, Condition (2) implies Condition (1) (the converse does not hold in general, see Example 1.4 below). For subgroups H of G satisfying (2), one can give a simple description of \overline{G}_L , in terms of G and the closure of ZH in G (see also [DMW], (3.5)).

1.3. PROPOSITION. *Let G be a connected Lie group, acting effectively on $M = G/H$, H satisfying (2). Then $\overline{G}_L = \{L_x \circ R_{y^{-1}}: x \in G, y \in \overline{ZH}\}$ and there is an exact sequence*

$$\{1\} \rightarrow S \rightarrow G \times \overline{ZH} \rightarrow \overline{G}_L \rightarrow \{1\}$$

where $S = \{(z, zh) | z \in Z, h \in H\}$. Furthermore, if H is not compact, \overline{ZH}/H is an abelian Lie subgroup of positive dimension of $Z(\overline{G}_L)$, the center of \overline{G}_L .

Proof. We observe first that for any G -invariant structure on $M = G/H$, if $y \in \overline{ZH}$ the map R_y is in the closure of G in $I(M)$, where $R_y(zH) = zyH$. In fact, since \overline{ZH} is contained in $N(H)$, the normalizer of H , R_y is a diffeomorphism. Furthermore the condition $y = \lim z_n h_n$ says

that $xyH = \lim L_{z_n}(xH)$, hence $R_y = \lim L_{z_n}$ in $I(M)([H])$, p. 203, Lemma 2.4).

The map $\alpha: G \times \overline{ZH} \rightarrow \overline{G}_L$, $\alpha(x, y) = L_x \circ R_{y^{-1}}$ is a Lie group homomorphism. To check that α is surjective it is enough (by Theorem 1.1) to consider φ in \overline{H}_L , $\varphi = \lim L_{h_n}$. Since H satisfies (2'), there exists a subsequence h_{n_j} such that $\pi(h_{n_j}) \rightarrow \pi(u)$, $u \in \overline{ZH}$. Hence $z_{n_j}h_{n_j} \rightarrow u$ for some z_{n_j} in Z and $L_{h_n}(xH) = h_n z_n x z_n^{-1} h_n^{-1} H \rightarrow L_u \circ R_{u^{-1}}(xH)$, that is, $\varphi = \alpha(u, u)$. Let (x, y) in $\ker \alpha$. Then $z^{-1}xzy^{-1} \in H$ for all z in G . Since $yx^{-1} \in H$ (taking $z = e$) it follows that $L_x \circ R_{x^{-1}} \in \text{Aut}(G, H)$ induces the identity on G/H . By Proposition 1.2, $L_x \circ R_{x^{-1}}$ is the identity on G and the description of $S = \ker \alpha$ follows.

It is not hard to check that \overline{ZH}/H is an abelian Lie group which is a Lie subgroup of $Z(\overline{G}_L)$. Assume $\dim \overline{ZH}/H = 0$. The map

$$\beta: Z \times H \rightarrow \overline{ZH}, \quad (z, h) \rightarrow zh$$

is a one-to-one Lie group homomorphism. By regularity $\beta(Z \times H) = ZH$ is open in \overline{ZH} , hence closed. Now if ZH is closed and H satisfies (2') it follows that H is compact. This completes the proof. \square

We will next show that, even the case M compact riemannian, it is quite common to have a representation with non-compact isotropy subgroup. This question was posed to us by Cristian Sanchez.

1.4. LEMMA. *Let G be a compact, connected semisimple Lie group, H a closed subgroup of G such that G acts effectively on G/H . Assume that $M = G/H$ carries a G -invariant riemannian structure such that $\text{rank}(U) \geq \text{rank}(H) + 2$, where $U = \{x \in N(H): R_x \text{ is an isometry}\}$. Then M can be represented as a quotient G'/H' , where G' acts effectively by isometries on G'/H' and H' is a closed noncompact subgroup. Furthermore, if $\phi: G' \rightarrow I(M)$ is the inclusion as a Lie subgroup, then $\phi(G') = G_L \cdot S_R$ where S is a torus in U .*

Proof. Let \mathfrak{u} denote the Lie algebra of U and let T be a torus in U such that $\dim(T) \geq 2$ and $\mathfrak{t} \cap \mathfrak{h} = 0$ (here \mathfrak{t} is the Lie algebra of T). Fix $X \in \mathfrak{t}$ so that $t \rightarrow \exp(tX)$ is one-to-one.

Set $G' = G \times \mathbf{R}$; let $\phi: G' \rightarrow I(M)$ be given by $\phi(g, t) = L_g \circ R_{\exp(-tX)}$. We assert that ϕ is a monomorphism. Indeed, $(g, t) \in \ker(\phi)$ if and only if $x^{-1}gx \exp(-tX) \in H$, for any x in G . Thus $g = h \exp(tX) \in U$, for some h in H and $\text{Ad}(g)(Z) - Z \in \mathfrak{h}$, for any $Z \in \mathfrak{g}$. Arguing as in the proof of Proposition 1.2 one concludes that $\text{Ad}(g) = \text{Id}$, or $g \in Z(G)$. On the other hand, the one-parameter group generated by X

intersects H in the identity element, since $T \cap H$ is a finite group. This easily implies that $h = \exp(tX) = e$, or $t = 0$. Hence M is isometric to G'/H' where $H' = \{(h \exp(tX), t): h \in H, t \in \mathbf{R}\}$.

Finally, by Proposition 1.3 it is easy to check that if S denotes the closure of $\{\exp(tX): t \in \mathbf{R}\}$, then

$$\overline{G'_L} = G_L \cdot (S \times \{0\})_R.$$

From this it easily follows that $\overline{\phi(G')}$ is as asserted in the lemma. This completes the proof. \square

1.5. REMARK. (i) As a particular case in the lemma we may take $H = \{e\}$. That is, $M = G$ is any compact, semisimple, connected Lie group endowed with a left invariant metric such that $\text{rank}(U) \geq 2$. For instance, if $\text{rank}(G) \geq 2$ and we use a bi-invariant metric, then G is a riemannian symmetric space admitting a representation G'/H' , with H' non-compact.

(ii) The proof above only uses that $Z(G)$ is finite and that H is compact. Hence the lemma remains valid under these assumptions on G and H , only.

We recall that in a variety of cases Conditions (1) and (2) are equivalent. For instance when $\text{Ad}(G)$ is closed in $\text{GL}(\mathfrak{g})$ since both topologies coincide. This happens for G semisimple since any derivation of \mathfrak{g} is inner. Also $\text{Ad}(G)$ is closed if G is reductive. It may fail to be so when G is solvable as the next example shows.

1.6. EXAMPLE. We provide here an example of a subgroup H of a connected Lie group G , satisfying (1) and not satisfying (2). Moreover since G acts effectively on G/H , we apply Theorem 1.1 to compute the closure of G in $I(M)$, $M = G/H$.

If $t_i \in S^1$, $z_i \in \mathbf{C}$ for $1 \leq i \leq n+1$, denote

$$s(t_1, \dots, t_{n+1}) = \begin{vmatrix} t_1 & & & & \\ & \ddots & & & \\ & & t_{n+1} & & \\ & & & & 1 \end{vmatrix},$$

$$n(z_1, \dots, z_{n+1}) = \begin{vmatrix} 1 & & & z_1 \\ & \ddots & & \vdots \\ & & 1 & z_{n+1} \\ & & & 1 \end{vmatrix}$$

and set

$$S = \{s(t_1, \dots, t_{n+1}) : t_i \in S^1\}, \quad N = \{n(z_1, \dots, z_{n+1}) : z_i \in \mathbb{C}^{n+1}\},$$

$G_1 = NS$. Since

$$s(t_1, \dots, t_{n+1})n(z_1, \dots, z_{n+1})s(t_1, \dots, t_{n+1})^{-1} = n(t_1z_1, \dots, t_{n+1}z_{n+1})$$

then it is clear that G_1 is a closed subgroup of $\mathrm{GL}(n+2, \mathbb{C})$ isomorphic to the semidirect product $N \rtimes S$ where S acts on N by

$$s(t_1, \dots, t_{n+1}) \cdot (n(z_1, \dots, z_{n+1})) = n(t_1z_1, \dots, t_{n+1}z_{n+1}).$$

We note that if $g = ns$, g commutes with N if and only if s commutes with N , that is, $s = 1$. Hence, if G_2 is a Lie subgroup with $N \subset G_2$ and if, for each j , there is $s(t_1, \dots, t_{n+1})$ in $S \cap G_2$ with $t_j \neq 1$, then $Z(G_2) = 1$.

Now let

$$\phi: \mathbb{R}^{n+1} \rightarrow G_1, \quad \phi(x_1, \dots, x_{n+1}) = s(e^{2\pi i x_1}, \dots, e^{2\pi i x_{n+1}}).$$

Let \mathbf{V} be an n -dimensional subspace of \mathbb{R}^{n+1} and let $v \in \mathbf{V}$ be such that ϕ is injective when restricted to \mathbf{V} and $\mathbf{R}v + \sum_1^{n+1} \mathbf{Z}e_j$ is dense in \mathbb{R}^{n+1} , $\{e_1, \dots, e_{n+1}\}$ the standard basis of \mathbb{R}^{n+1} ; (see appendix, Lemma A.4).

Set $G = N\phi(\mathbf{V})$, $H = \phi(\mathbf{R}v)$. Then G is a normal dense Lie subgroup of G_1 and H is dense in S . On the other hand, H is a closed Lie subgroup of G . Now, by the choice of H , it is not hard to see that $\overline{\mathrm{Ad}(H)} = \mathrm{Ad}_G(S)$ and this is, in turn, isomorphic to T^{n+1} . Hence (1) holds. However, given that $Z = \{e\}$, and H is not compact, (2) is not satisfied.

One checks that $\overline{\mathrm{Ad}(H)} = \{I_s : s \in S\}$. From this and the fact that G acts effectively on G/H , it follows (see 1.1 and 1.2) that

$$\overline{G}_L = \{L_x \circ \bar{I}_s : x \in G, s \in S\}.$$

By a calculation one checks that $Z(\overline{G}_L) = \{e\}$ in contrast to the case when (2) is satisfied (see Proposition 1.3).

1.7. REMARK. Clearly a connected subgroup H of G satisfying condition (3) will satisfy (2) (then (1)). Now if G acts effectively on G/H , the conditions $\mathrm{Ad}_G(H)$ compact and H compact are equivalent. Hence if H satisfies (3), G_L will be closed in $I(M)$. As 1.4, 1.5, and 1.6 show, it is not true in general that if G/H is riemannian homogeneous then H satisfies (3). On the other hand, we see later (see Remark 2.3) that if G is any simply connected semisimple Lie group with infinite center there exists H a non-compact, Lie subgroup satisfying (2) with G acting effectively on G/H . Such an H will not satisfy (3).

As a final observation we describe a way to construct new examples of pairs (G, H) , H satisfying (1), H non-compact.

1.8. LEMMA. *Let (G, H) be a pair of a connected Lie group G and a closed subgroup H so that (1) holds. Let (σ, \mathbf{V}) be a finite dimensional representation of G such that H acts unitarily. Set $G_\sigma = \mathbf{V} \rtimes_\sigma G$. Assume that $\ker(\sigma) \cap \ker(\text{action of } G \text{ on } G/H) = \{e\}$. Then G_σ acts effectively on G_σ/H and H satisfies (1). Furthermore, if $\ker(\sigma) \cap Z(G) = \{e\}$, G acts without non-zero fixed vectors and H is not compact, then (2) will not be satisfied.*

Proof. Clearly H is closed in G_σ and $\text{Ad}_{G_\sigma}(H) \subset O(\mathbf{V}) \times \text{Ad}_G(H)$, hence (1) holds. On the other hand

$$(1.9) \quad \begin{aligned} (0, g_1) \cdot (v, g) \cdot (0, g_1^{-1}) &= (\sigma(g_1)v, g_1 g g_1^{-1}), \\ (v_1, 1) \cdot (v, g) \cdot (-v_1, 1) &= (v + v_1 - \sigma(g)v_1, g). \end{aligned}$$

Thus

$$\bigcap_{x \in G_\sigma} x H x^{-1} = \left(\bigcap_{x \in G} x H x^{-1} \right) \cap \ker \sigma = \{e\},$$

by the assumption. Hence the action is effective. On the other hand, by (2.5)

$$Z(G_\sigma) = \{(g_0, v_0); g_0 \in Z(G) \cap \ker(\sigma), \sigma(g)v_0 = v_0 \text{ for any } g \in G\}$$

and under the assumptions $Z(G_\sigma) = \{e\}$, hence the second assertion is clear. \square

1.9. REMARKS. (a) We note that Example 1.6 is a particular instance of the construction in the lemma with $(G, H) = (\mathbf{R}^n, \mathbf{R})$. In this case, the representation of \mathbf{R}^n in \mathbf{C}^{n+1} is faithful and unitary and the action of G on G/H is far from being effective but the action of G_σ on G_σ/H is so. This procedure could naturally be iterated by taking a finite dimensional representation (τ, \mathbf{W}) of G_σ , unitary on H , thus getting a new pair $((G_\sigma)_\tau, H)$.

(b) As another illustration of the lemma, let G be semisimple and H a closed non-compact subgroup such that (2) holds. Let (σ, \mathbf{V}) be a non-trivial, finite dimensional, irreducible representation of G . We have that $Z(G_\sigma) = \{(0, z) | z \in Z \cap \ker \sigma\}$. Hence

$$p_\sigma(\overline{Z(G_\sigma)H}) = \{0\} \times p(\overline{(Z \cap \ker \sigma)H})$$

is compact, where $p_\sigma: G_\sigma \rightarrow G_\sigma/Z(G_\sigma)$ is the canonical projection. Hence (2) holds and if, for instance, G acts effectively or σ is faithful the action of G_σ is effective.

2. Non compact isotropy. In this section we consider the case when G is semisimple. As observed in §1, for H a closed subgroup of G Conditions (1) and (2) are equivalent. The first result we prove is a characterization of connected non-compact subgroups H satisfying (2). We also give a way to compute \overline{G}_L explicitly, in the light of Proposition 1.2 and a classical result (see Theorem A.1 in the appendix). We also construct families of examples where $\dim \overline{G}_L - \dim G_L$ is the maximum possible.

Let then G be a connected semisimple Lie group. If $G = K \cdot \exp(\mathfrak{p})$ is a Cartan decomposition, K is compact if and only if Z , the center of G , is finite. Denote by \mathfrak{k} the Lie algebra of K and let $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$. Then $\mathfrak{k} = z_{\mathfrak{k}} \oplus \mathfrak{k}'$, $z_{\mathfrak{k}}$ the center of \mathfrak{k} . Denote by Z_K and K' , respectively, the connected Lie subgroups of K corresponding to $z_{\mathfrak{k}}$ and \mathfrak{k}' . Then K' is compact and if furthermore, G is simply connected, K is isomorphic to the direct product of Z_K and K' . In this case Z_K is a vector group.

2.1. PROPOSITION. *Let G be a simply connected semisimple Lie group and H a closed connected subgroup such that (2) is satisfied. Then there is a Cartan decomposition $G = K \exp(\mathfrak{p})$ such that*

$$(2.1) \quad H = \exp(\mathbf{V}) \cdot (H \cap K')$$

where V is a vector subspace of the center of \mathfrak{h} , $\exp: \mathbf{V} \rightarrow \exp \mathbf{V}$ is an isomorphism onto a closed subgroup and $\exp(\mathbf{V}) \cap (H \cap K') = 1$. Furthermore \overline{ZH} can be explicitly computed.

Proof. Let $p: G \rightarrow G/Z$ be the canonical projection. Since $p(\overline{ZH}) = \overline{p(H)}$ is compact, by a theorem of Iwasawa, there is a maximal compact subgroup K_0 of G/Z such that $p(\overline{ZH}) \subset K_0$. Thus $\overline{ZH} \subset K = p^{-1}(K_0)$. Let $K = Z_K \cdot K'$ as above. If $Z_k = \{e\}$, then K is compact, hence H , and (2.1) is clear. We thus assume from now on that $z_{\mathfrak{k}} \neq 0$. Now, \mathfrak{h} admits an $\text{Ad}(H)$ -invariant inner product, hence $\mathfrak{h} = z_0 \oplus \mathfrak{h}'$ where z_0 is the center of \mathfrak{h} and $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$. Let H' (resp. Z_H) be the connected Lie subgroup of H corresponding to \mathfrak{h}' (resp. z_0). Then H' is compact and $H' \subset K'$. Write $z_0 = \mathbf{V} \oplus z'$ with $z' = z_0 \cap \mathfrak{k}'$ and \mathbf{V} a vector subspace of z_0 . Fix $\{(U_i, Y_i): U_i \in z_{\mathfrak{k}}, Y_i \in \mathfrak{k}', 1 \leq i \leq m\}$ a basis of \mathbf{V} . Then the set $\{U_i: 1 \leq i \leq m\}$ is linearly independent since $\mathbf{V} \cap \mathfrak{k}' = \{0\}$. Also $[Y_i, Y_j] = 0$, for all i, j . Assume now that

$$\exp\left(\sum_1^m t_i(U_i, Y_i)\right) = \exp\left(\sum_1^m t_i(U_i, 0)\right) \exp\left(\sum t_i(0, Y_i)\right) \in K'.$$

Since $\exp(\sum_1^m t_i U_i) = 1$, then, $t_i = 0$ for $1 \leq i \leq m$. Hence, $\exp|_{\mathbf{V}}$ is one-to-one and $\exp(\mathbf{V}) \cap K' = \{1\}$. This implies that $Z_H \cap K' = \exp(z')$, that is, $Z_H = \exp \mathbf{V} \cdot (Z_H \cap K')$.

Let now $v_n \in \mathbf{V}$ be so that $\exp(v_n) \rightarrow u \in G$. Then $u \in Z_H$, hence $u = \exp(v)u'$, $u' \in Z_H \cap K'$. Write

$$v_n v^{-1} = \sum_1^m t_{i,n} (U_i, Y_i).$$

Since $\exp(v_n v^{-1}) \rightarrow u' \in K'$ it follows that $\exp(\sum_1^m t_{i,n} U_i) \rightarrow 1$ in Z_K . Hence $t_{i,n} \rightarrow 0$, for $1 \leq i \leq m$ (again we use that U_1, \dots, U_m are linearly independent). Therefore $u' = 1$ and $\exp(\mathbf{V})$ is closed in G . This completes the proof of the first assertion. We now describe how to compute \overline{ZH} .

Fix $Z' \subset Z \cap Z_K$, a subgroup such that $|Z/Z'| < \infty$ and let Z'' be a system of representatives of Z/Z' . Then

$$(2.2) \quad \overline{ZH} = \overline{Z'H \cdot Z''} = \overline{Z' \cdot \exp \mathbf{V} \cdot H \cap K' \cdot Z''}.$$

Let $\{(U_i, Y_i) | 1 \leq i \leq m\}$ be as above. Also let $\mathfrak{s} = \sum_1^m \mathbf{R} Y_i$, $T' = \exp(\mathfrak{s})$ and let \mathfrak{t}' be the Lie algebra of T' . Choose $\{X_j; 1 \leq j \leq d\}$ a basis of \mathfrak{t}' so that $\sum \mathbf{Z} X_j = \ker(\exp|_{\mathfrak{t}'})$ and $\{V_i; 1 \leq i \leq c\}$ a basis of $\mathfrak{z}_{\mathfrak{t}}$ such that $Z' = \exp(\sum \mathbf{Z} V_i)$.

Thus

$$(2.3) \quad \overline{Z' \exp(\mathbf{V})} = \exp \left(\overline{\sum_1^c \mathbf{Z} V_i + \sum_1^d \mathbf{Z} X_j + \mathbf{V}} \right)$$

and it then follows from (2.2) and (2.3) that \overline{ZH} can be computed (at least in principle) by means of Theorem A.1 in the appendix. \square

2.2. REMARK. Assume that G is semisimple and the hypotheses of Proposition 1.3 hold. Then, given any G -invariant riemannian metric on G/H .

$$(2.4) \quad \dim \overline{G}_L \leq \dim G + \text{rank}(K) - \text{rank}(H)$$

where $G = K \cdot \exp(\mathfrak{p})$ is a Cartan decomposition of G . In fact, since H satisfies (2), \overline{G}_L is given by Proposition 1.3, hence

$$\dim \overline{G}_L = \dim G + \dim \overline{ZH}/H.$$

Now $\overline{ZH} \subset K = p^{-1}(K_0)$, K_0 a maximal compact subgroup of G/Z , $p(\overline{ZH}) \subset K_0$. Let \mathfrak{h} denote the Lie algebra of \overline{ZH} . It is clear that \mathfrak{h} is an ideal of \mathfrak{g} and there is an abelian ideal α of \mathfrak{g} with $\mathfrak{h} = \alpha \oplus \mathfrak{h} \subset \mathfrak{k}$. Hence $\dim \overline{G}_L = \dim G + \dim \alpha \leq \dim G + \text{rank}(K) - \text{rank}(H)$ as asserted.

We next provide a family of examples where $\dim \bar{G}_L - \dim G_L$ is the maximum possible. This will show that the bound in (2.4) is sharp.

2.2. PROPOSITION. *Let G be a simply connected semisimple Lie group such that the center of each simple factor is infinite. Then there exists a discrete subgroup $H \simeq \mathbf{Z}$ such that $\bar{Z}H = T$, a compact Cartan subgroup, and G acts effectively on G/H . Furthermore if G is not locally isomorphic to $\mathrm{SL}(2, \mathbf{R})$, there exists a closed subgroup $H' \simeq \mathbf{R}$ with $\bar{Z}H' = T$ and G acting effectively on G/H' . Then for any G -invariant riemannian metric on G/H (resp. G/H')*

$$\dim \bar{G}_L = \dim G + \mathrm{rank}(K) \quad (\text{resp. } \dim \bar{G}_L = \dim G + \mathrm{rank}(K) - 1).$$

Furthermore $\bar{G}_L = I(G/H)_0$ (resp. $I(G/H')_0$).

Proof. Let T be a Cartan subgroup, $T \subset K$ and let \mathfrak{t} denote the Lie algebra of T . Let $p: G \rightarrow G/Z$ be the canonical projection. Then T/Z is a torus. Fix X in \mathfrak{t} so that, if $t = \exp(X)$, tZ is a generator of T/Z (i.e. the powers of tZ form a dense subgroup of T/Z). Let H be the group generated by t and let H' be the one-parameter group generated by X . Now $X = X_1 + X_2$, $X_1 \in \mathfrak{z}_{\mathfrak{f}}$, $X_2 \in \mathfrak{k}'$ and $X_1 \neq 0$, since tZ is a generator. This implies that both H and H' are closed subgroups.

On the other hand G acts effectively on G/H since $\bigcap_{x \in G} x^{-1}Tx = Z$ and by the choice of t it is clear that $H \cap Z = \{e\}$. Similarly, if $H' \cap Z = \{e\}$, the action on G/H' is effective. Otherwise $p(H')$ is a circle and then $p(H') = T$. Therefore G is isomorphic to the universal covering group of $\mathrm{SL}(2, \mathbf{R})$, against our assumption. Thus, by (2.4) $\dim \bar{G}_L$ is as asserted, since, if $S = H$ or H' , by Proposition 1.3 $\bar{G}_L = G_L \cdot (\bar{ZS})_R = G_L \cdot T_R$. Finally, by a result of C. Gordon ([G], Theorem 4.1)

$$I(M)_0 = G_L \cdot (U_0)_R$$

where, as usual, $U = \{x \in N(S): R_x \text{ is an isometry}\}$. Now

$$T \subseteq U_0 \subset N(S)_0 \subset N(T)_0 = T.$$

Thus the last assertion is clear. This concludes the proof. \square

2.3. REMARK. It is easy to modify the example in (3.10) [DMW] to show that any connected semisimple Lie group with infinite center G admits a closed, non-compact, discrete subgroup H satisfying (2) and G acting effectively on G/H .

APPENDIX. In this appendix we list some classical facts on the subgroups of \mathbf{R}^n . We refer to [B], Ch. VII for a detailed treatment. If $x, y \in \mathbf{R}^n$ denote by $x \cdot y$ the canonical inner product. Given $G \subset \mathbf{R}^n$, a subgroup, set $G^* = \{g' \in \mathbf{R}^n \mid g' \cdot g \in \mathbf{Z}, \text{ for all } g \in G\}$.

A.1. THEOREM (see [B], p. 74). Let $G \subset \mathbf{R}^n$ be a subgroup. Then $\overline{G} = G^{**}$. In particular G is dense if and only if $G^* = \{0\}$.

A.2. COROLLARY. Fix $v = (a_1, \dots, a_n) \in \mathbf{R}^n$ and let $\{e_j\}_1^n$ be the canonical basis. Let $G_1 = \mathbf{Z}v + \sum_1^n \mathbf{Z}e_j$ (resp. $G_2 = \mathbf{R}v + \sum_1^n \mathbf{Z}e_j$). Then G_1 (resp. G_2) is dense in \mathbf{R}^n if and only if $1, a_1, \dots, a_n$ are \mathbf{Q} -linearly independent (resp. a_1, \dots, a_n are \mathbf{Q} -linearly independent).

A.3. LEMMA. If $1 \leq k \leq n$, let $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^k \times T^{n-k}$ be the canonical map, $\phi(t_1, \dots, t_k, \frac{t_{k+1}}{2\pi i}, \dots, \frac{t_n}{2\pi i}) = (t_1, \dots, t_k, e^{2\pi i t_{k+1}}, \dots, e^{2\pi i t_n})$. If $G \subset \mathbf{R}^n$ is a subgroup, then $\phi(\overline{G}) = \overline{\phi(G + \sum_{k+1}^n \mathbf{Z}e_j)}$.

A.4. LEMMA. There exists an n -dimensional subspace \mathbf{V} of \mathbf{R}^{n+1} and $v \in \mathbf{V}$ such that $\mathbf{V} \cap \sum_1^{n+1} \mathbf{Z}e_j = \{0\}$ and $\mathbf{Z}v + \sum_1^{n+1} \mathbf{Z}e_j$ is dense in \mathbf{R}^{n+1} .

Proof. The lemma follows in a standard fashion from Theorem A.1. The proof will thus be omitted.

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