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A holomorphic mapping $f: E \rightarrow F$ of complex Banach spaces is weakly compact if every $x \in E$ has a neighbourhood V_x such that $f(V_x)$ is a relatively weakly compact subset of F . Several characterizations of weakly holomorphic mappings are given which are analogous to classical characterizations of weakly compact linear mappings and the Davis-Figiel-Johnson-Pelczynski factorization theorem is extended to weakly compact holomorphic mappings. It is shown that the complex Banach space E has the property that every holomorphic mapping from E into an arbitrary Banach space is weakly compact if and only if the space $\mathcal{H}(E)$ of holomorphic complex-valued functions on E , endowed with the bornological topology τ_δ , is reflexive.

1. Introduction. Aron and Schottenloher [3], in a study of the approximation property for locally convex spaces of holomorphic functions on a complex Banach space, introduced the concept of a compact holomorphic mapping. If E and F are complex Banach spaces, a holomorphic mapping $f: E \rightarrow F$ is said to be compact if every $x \in E$ has a neighbourhood V_x such that $f(V_x)$ is a relatively compact subset of F . They obtained several characterizations of compact holomorphic mappings which were analogous to characterizations of compact linear mappings. For example, it is well-known that a continuous linear mapping $T: E \rightarrow F$ is compact if and only if its transpose $T': F' \rightarrow E'$ is compact. Now consider a holomorphic mapping $f: E \rightarrow F$. Denoting by $\mathcal{H}(E)$ the vector space of holomorphic functions on E (that is, holomorphic mappings from E into \mathbb{C}), the transpose of f may be defined as the linear mapping $f': F' \rightarrow \mathcal{H}(E)$ given by $f'(\psi) = \psi \circ f$ for $\psi \in F'$. Aron and Schottenloher show that when $\mathcal{H}(E)$ is given a suitable locally convex topology, the holomorphic mapping f is compact if and only if its transpose f' is a compact linear mapping of F' into $\mathcal{H}(E)$.

Motivated by this work, we carry out a similar study of weakly compact holomorphic mappings, which are defined in the same way: a holomorphic mapping $f: E \rightarrow F$ is weakly compact if every $x \in E$ has a neighbourhood V_x such that $f(V_x)$ is a relatively weakly compact subset of F . We extend some of the classical theory of weakly compact linear mappings to the holomorphic setting. Of course, some of our results are

analogous to those of Aron and Schottenloher, but, just as in the linear case, the proofs for the compact and weakly compact situations are often quite different.

We begin in §2 with weakly compact polynomials between Banach spaces. The properties of weakly compact polynomials which we require for the sequel follow easily from the fact that the continuous polynomials can be viewed as continuous linear mappings on a suitable tensor product.

In §3 we consider weakly compact holomorphic mappings. We show that weak compactness of a holomorphic mapping is equivalent to weak compactness of its derivatives. Next, we show that weak compactness of a holomorphic mapping $f: E \rightarrow F$ is equivalent to weak compactness of its transpose $f^t: F' \rightarrow (\mathcal{H}(E), \tau_\omega)$ where τ_ω is the Nachbin topology of $\mathcal{H}(E)$, and that this is in turn equivalent to $f''((\mathcal{H}(E), \tau_\omega)') \subset F$, generalising the fact that a continuous linear mapping $T: E \rightarrow F$ is weakly compact if and only if $T''(E'') \subset F$. This, and other results in this paper, reinforce the view expressed in [3] that $\mathcal{H}(E)$ can be thought of as a "holomorphic dual" of E . Next, we show that the Davis-Figiel-Johnson-Pełczyński theorem, which states that weakly compact linear mappings factor through reflexive spaces, extends to holomorphic mappings. Finally, we examine the relationship between weakly compact holomorphic mappings and reflexivity. We show that E has the property that for every compact Banach space F , every holomorphic mapping from E into F is weakly compact, if and only if the locally convex space $(\mathcal{H}(E), \tau_\delta)$ is reflexive, where τ_δ is the bornological topology associated with τ_ω .

We now outline our notation and terminology. E and F denote complex Banach spaces. E' , F' denote their strong duals, and F'_τ denotes the dual space of F endowed with the Mackey topology $\tau(F', F)$. $\bar{\Gamma}U$ denotes the closed convex balanced hull of the set U . If X is a locally convex space, X'_b denotes the strong dual of X , that is the dual space of X with the topology of uniform convergence on bounded subsets of X . For $n \in \mathbf{N}$, $L_s(^nE; F)$ denotes the Banach space of continuous symmetric n -linear mappings from E into F , where the norm is given by $\|A\| = \sup\{\|A(x_1, \dots, x_n)\| : \|x_1\|, \dots, \|x_n\| \leq 1\}$. A mapping $P: E \rightarrow F$ is a continuous n -homogeneous polynomial if there exists a necessarily unique element A of $L_s(^nE; F)$ such that $P(x) = A(x, \dots, x)$ for every $x \in E$. $P(^nE; F)$ denotes the Banach space of continuous n -homogeneous polynomials from E into F with the norm given by $\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\}$. The mapping $P \rightarrow A$ is an isomorphism from $P(^nE; F)$ onto $L_s(^nE; F)$.

A mapping $f: E \rightarrow F$ is holomorphic if for each $x \in E$ there exists a sequence of polynomials

$$\frac{1}{n!} \hat{d}^n f(x) \in P({}^n E; F)$$

and a neighbourhood V of x such that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(x)(y-x)$$

converges uniformly to $f(y)$ for $y \in V$. $\mathcal{H}(E; F)$ denotes the vector space of holomorphic mappings from E into F . When $F = \mathbf{C}$ the spaces of functions which we have introduced above are denoted $L_s({}^n E)$, $P({}^n E)$ and $\mathcal{H}(E)$. A seminorm p on $\mathcal{H}(E; F)$ is said to be ported by a compact subset K of E if for every neighbourhood V of K there exists $C_V > 0$ such that $p(f) \leq C_V \|f\|_V$ for every $f \in \mathcal{H}(E; F)$. The Nachbin topology, τ_ω is the locally convex topology on $\mathcal{H}(E; F)$ generated by the seminorms which are ported by compact subsets of E . τ_ω induces the norm topology on the subspaces $P({}^n E; F)$ of $\mathcal{H}(E; F)$. We refer to [6] and [11] for further details about polynomials and holomorphic mappings.

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2. Weakly compact polynomials. Let P be a continuous n -homogeneous polynomial from E into F and let A_p be the unique continuous symmetric n -linear mapping from E^n into F which generates P , so that $P(x) = A_p(x, \dots, x)$ for every $x \in E$. Let $L_p \in L(\hat{\otimes}^n E; F)$ be the continuous linear mapping associated with A_p where $\hat{\otimes}^n E = E \hat{\otimes} \cdots \hat{\otimes} E$ is the n -fold projective tensor product [8]. The polynomial P is said to be weakly compact if P maps the unit ball of E into a relatively weakly compact subset of F . It is shown in [12] that P is weakly compact if and only if the associated linear mapping L_p is weakly compact.

The transpose of $P \in P({}^n E; F)$ is the continuous linear mapping $P': F' \rightarrow P({}^n E)$ defined by $P'(\psi) = \psi \circ P$ for $\psi \in F'$. We wish to relate weak compactness of P to the behaviour of the linear mapping P' . P' can be identified with the transpose of the linear mapping L_p in the following way: L'_p maps F' into $L_s({}^n E)$, and $J \circ P' = L'_p$, where $J: P({}^n E) \rightarrow L_s({}^n E)$ is the isomorphism given by $J(Q) = A_Q$. We now apply the following well-known equivalent formulations of weak compactness of linear mappings (see, for example [7]). The following properties of a

continuous linear mapping $T: X \rightarrow Y$ are equivalent:

- (a) T is weakly compact;
- (b) $T': Y'_\tau \rightarrow X'$ is continuous;
- (c) $T': Y' \rightarrow X'$ is weakly compact.

Applying this to the linear mapping L_P , we obtain:

PROPOSITION 2.1. *Let $P \in P(^nE; F)$. The following are equivalent:*

- (i) P is weakly compact;
- (ii) $P': F'_\tau \rightarrow P(^nE)$ is continuous;
- (iii) $P': F' \rightarrow P(^nE)$ is weakly compact.

$P_{wK}(^nE; F)$ denotes the subspace of $P(^nE; F)$ of weakly compact n -homogeneous polynomials from E into F .

COROLLARY 2.2. $P_{wK}(^nE; F)$ is a closed subspace of $P(^nE; F)$.

Proof. Let $\{P_n\}$ be a sequence in $P_{wK}(^nE; F)$ which converges to $P \in P(^nE; F)$. Then P'_n converges to P' in $L(F'; P(^nE))$ and since $L_{wK}(F'; P(^nE))$ is closed in $L(F'; P(^nE))$, it follows that P' , and hence P , is weakly compact. \square

For further results concerning weakly compact polynomials on Banach spaces with the Dunford-Pettis property, we refer to [12, 13].

3. Weakly compact holomorphic mappings. We shall say that a holomorphic mapping $f: E \rightarrow F$ is weakly compact if every $x \in E$ has a neighbourhood V_x such that $f(V_x)$ is a relatively weakly compact subset of F . Our first objective is to establish a connection between weak compactness of a holomorphic mapping and weak compactness of its derivatives.

We call a subset U of E circled if for every $x \in U$ and every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have $\lambda x \in U$. The following lemma is an easy application of the Hahn-Banach Theorem and the Cauchy inequality (see the proof of [3, Proposition 3.4] for a similar result).

LEMMA 3.1. *Let $f \in \mathcal{H}(E; F)$. If $U \subset E$ is circled and $x \in E$, then*

$$\frac{1}{n!} \hat{d}^n f(x)(U) \subset \bar{\Gamma} f(x + U)$$

for every n .

We now show that weak compactness of a holomorphic mapping f is equivalent to weak compactness of its derivatives at every point, and also that weak compactness of f is determined by the behaviour of f at the

origin. We refer to [3, Proposition 3.4] for an analogous result for compact holomorphic mappings.

THEOREM 3.2. *Let $f \in \mathcal{H}(E; F)$. The following are equivalent:*

- (i) *f is weakly compact;*
- (ii) *f maps some neighbourhood of the origin in E into a relatively weakly compact subset of F ;*
- (iii) *$\hat{d}^n f(x)$ is a weakly compact polynomial for every $n \in \mathbf{N}$ and every $x \in E$;*
- (iv) *$\hat{d}^n f(0)$ is a weakly compact polynomial for every $n \in \mathbf{N}$.*

Proof. The implications (iii) \Rightarrow (iv) and (i) \Rightarrow (ii) are trivial, while (i) \Rightarrow (iii) and (ii) \Rightarrow (iv) follow immediately from Lemma 3.1. It only remains to show that (iv) implies (i).

Suppose, therefore, that $\hat{d}^n f(0)$ is a weakly compact polynomial for every $n \in \mathbf{N}$. Let $X = \{x \in E: f(V_x) \text{ is relatively weakly compact for some neighbourhood } V_x \text{ of } x\}$. We shall prove that X is a non-empty subset of E and is both open and closed, from which it follows that f is weakly compact.

First we show that X contains the origin. Choose $r > 0$ so that f is bounded on the ball $B(0, r)$. We claim that $f(B(0, r/2))$ is a relatively weakly compact subset of F . To see this, let $K_n = \overline{\Gamma}((1/n!) \hat{d}^n f(0)(B(0, r)))$. Since $(1/n!) \hat{d}^n f(0)$ is a weakly compact polynomial, K_n is weakly compact for every n . By Lemma 3.1 K_n is contained in the bounded set $\overline{\Gamma}f(B(0, r))$ for every n , and therefore if $z_n \in K_n$ then the series $\sum_{n=0}^{\infty} 2^{-n} z_n$ converges absolutely. Therefore $K = \{\sum_{n=0}^{\infty} 2^{-n} z_n: z_n \in K_n\}$ is a well-defined bounded subset of F . Now, for $x \in B(0, r/2)$, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(0)(x) = \sum_{n=0}^{\infty} 2^{-n} \left(\frac{1}{n!} \hat{d}^n f(0)(2x) \right),$$

and hence $f(B(0, r/2)) \subset K$. We show that K is weakly compact by proving that for every $\psi \in F'$ there exists $z \in K$ such that $\psi(z) = \|\psi\|_K$ [10].

Let $\psi \in F'$. Since K_n is weakly compact and balanced, there exist $w_n \in K_n$ such that $\psi(w_n) = \|\psi\|_{K_n}$. Then $w = \sum_{n=0}^{\infty} 2^{-n} w_n \in K$, and

$$\|\psi\|_K \leq \sum_{n=0}^{\infty} 2^{-n} \|\psi\|_{K_n} = \sum_{n=0}^{\infty} 2^{-n} \psi(w_n) = \psi(w) \leq \|\psi\|_K.$$

Therefore $\psi(w) = \|\psi\|_K$ and our claim is proved.

It is clear that X is an open subset of F . Let $\{x_k\}$ be a sequence in X , converging to $x \in E$. By Lemma 3.1 $\hat{d}^n f(x_k)$ is weakly compact for every n and every k . Since $P_{w_k}({}^n E; F)$ is a closed subspace of $P({}^n E; F)$, and

$\lim_{k \rightarrow \infty} \hat{d}^n f(x_k) = \hat{d}^n f(x)$ in $P(^n E; F)$, it follows that $\hat{d}^n f(x)$ is weakly compact for every n . Consider the holomorphic mapping $g: E \rightarrow F$ defined by $g(z) = f(x + z)$. We have $\hat{d}^n g(0) = \hat{d}^n f(x)$ for every n , and it follows from the first part of this proof that g maps some neighbourhood V of the origin into a relatively weakly compact subset of F . Therefore f maps the neighbourhood $x + V$ of x into a relatively weakly compact set, which shows that $x \in X$. Therefore X is closed. \square

We shall denote by $\mathcal{H}_{wK}(E; F)$ the vector space of weakly compact holomorphic mappings from E into F .

COROLLARY 3.3. *($\mathcal{H}_{wK}(E; F), \tau_\omega$) is a closed subspace of $(\mathcal{H}(E; F), \tau_\omega)$.*

Proof. Let $\{f_\lambda\}$ be a net in $\mathcal{H}_{wK}(E; F)$ which converges in the topology τ_ω to $f \in \mathcal{H}(E; F)$. Since τ_ω induces the norm topology on $P(^n E; F)$ it follows that $\hat{d}^n f_\lambda(0)$ converges to $\hat{d}^n f(0)$ in the norm of $P(^n E; F)$ for every n . But $\hat{d}^n f_\lambda(0) \in P_{wK}(^n E; F)$ for every λ , and $P_{wK}(^n E; F)$ is closed in $P(^n E; F)$. Therefore, by Theorem 3.2, f is weakly compact. \square

If f is a holomorphic mapping from E to F then, following Aron and Schottenloher, we define the transpose of f to the linear mapping $f': F' \rightarrow \mathcal{H}(E)$ given by $f'(\psi) = \psi \circ f$. This mapping is continuous for the norm topology on F' and the Nachbin topology τ_ω on $\mathcal{H}(E)$. To see this, let p be a τ_ω -continuous seminorm on $\mathcal{H}(E)$, so that p is ported by some compact subset K of E . Thus, for every open neighbourhood V of K there exists $C_V > 0$ such that $p(g) \leq C_V \|g\|_V$ for every $g \in \mathcal{H}(E)$. Choose an open neighbourhood V of K for which $f(V)$ is bounded. Then $p(f'(\psi)) \leq C_V \|f'(\psi)\|_V = C_V \|\psi\|_{f(V)}$, and hence $p \circ f'$ is a continuous seminorm on F' .

Since $f': F' \rightarrow (\mathcal{H}(E), \tau_\omega)$ is a continuous linear mapping, we may take its transpose in the usual linear sense to obtain a linear mapping $f'': (\mathcal{H}(E), \tau_\omega)' \rightarrow F''$. Our next result shows that the transpose and bitranspose of a holomorphic mapping can be used to characterize weak compactness. We recall that for a continuous linear mapping $T: E \rightarrow F$, weak compactness of T is equivalent to each of the following conditions:

- (a) $T': F' \rightarrow E'$ is weakly compact;
- (b) T'' maps E'' into the canonical image of F in F'' .

For holomorphic mappings we have: (cf. [3, Proposition 3.6])

THEOREM 3.4. *Let $f \in \mathcal{H}(E; F)$. The following are equivalent:*

- (i) f is weakly compact;
- (ii) $f^t: F'_\tau \rightarrow (\mathcal{H}(E), \tau_\omega)$ is continuous, where $\tau = \tau(F', F)$ is the Mackey topology of F' ;
- (iii) $f^t: F' \rightarrow (\mathcal{H}(E), \tau_\omega)$ is weakly compact;
- (iv) f'' maps $(\mathcal{H}(E), \tau_\omega)'$ into the canonical image of F in F'' .

Proof. (i) implies (ii): Let p be a continuous seminorm on $(\mathcal{H}(E), \tau_\omega)$. Then p is ported by some compact subset K of E , and so for every open set U containing K there exists $C_U > 0$ such that $p(g) \leq C_U \|g\|_U$ for every $g \in \mathcal{H}(E)$. Since f is weakly compact there exists an open neighbourhood V of K such that $f(V)$ is a relatively weakly compact subset of F . Therefore $\psi \rightarrow \|\psi\|_{f(V)}$ is a continuous seminorm on F'_τ , and since

$$p(f^t(\psi)) \leq C_V \|f^t(\psi)\|_V = C_V \|\psi\|_{f(V)},$$

it follows that $f^t: F'_\tau \rightarrow (\mathcal{H}(E), \tau_\omega)$ is continuous.

(ii) implies (iii): It follows from (ii) that f^t is continuous for the weak topologies $\sigma(F', F)$ and $\sigma(\mathcal{H}(E), (\mathcal{H}(E), \tau_\omega)')$. Since the closed unit ball of F' is $\sigma(F', F)$ -compact, it is mapped by f^t into a weakly compact subset of $(\mathcal{H}(E), \tau_\omega)$.

(iii) implies (i): The mapping $g \rightarrow \hat{d}^n g(0)$ is a continuous projection of $(\mathcal{H}(E), \tau_\omega)$ onto the subspace $P(^n E)$. If we compose f^t with this mapping we obtain the continuous mapping $(\hat{d}^n f(0))^t: F'_\tau \rightarrow P(^n E)$. It follows from Proposition 2.1 that $\hat{d}^n f(0)$ is a weakly compact polynomial for every n and hence, by Theorem 3.2, f is weakly compact.

(ii) implies (iv) is trivial, since $(F'_\tau)'$ is the canonical image of F in F'' . To see that (iv) implies (ii), suppose that f'' maps $(\mathcal{H}(E), \tau_\omega)'$ into the canonical image of F in F'' . Then f^t is continuous for the weak topologies $\sigma(F', F)$ and $\sigma(\mathcal{H}(E), (\mathcal{H}(E), \tau_\omega)')$, and it follows that f^t is continuous for the Mackey topologies associated with these weak topologies. But τ_ω is weaker than the Mackey topology associated with it, and hence f^t is a continuous mapping of F'_τ into $(\mathcal{H}(E), \tau_\omega)$. \square

Now by Theorem 3.4 if $f \in \mathcal{H}_{wK}(E; F)$ then the transpose f^t belongs to the space $L(F'_\tau; (\mathcal{H}(E), \tau_\omega))$. We shall endow this space of continuous linear mappings with the topology of uniform convergence on the equicontinuous subset of F' , and we denote the resulting locally convex space by $L_e(F'_\tau; (\mathcal{H}(E), \tau_\omega))$. We wish to show that the mapping $f \rightarrow f^t$ establishes an isomorphism of $(\mathcal{H}_{wK}(E; F), \tau_\omega)$ with $L_e(F'_\tau; (\mathcal{H}(E), \tau_\omega))$.

For each $x \in E$ let $\delta(x)$ be the continuous linear functional on $(\mathcal{H}(E), \tau_\omega)$ given by $f \rightarrow f(x)$.

LEMMA 3.5. δ is a holomorphic mapping from E into $(\mathcal{H}(E), \tau_\omega)'_\beta$.

Proof. We show that δ has a Fréchet derivative at every point of E . For each $x \in E$ let π_x denote the continuous projection of $(\mathcal{H}(E), \tau_\omega)$ onto the subspace E' which associates with $f \in \mathcal{H}(E)$ its derivative at x , $df(x)$. Let J denote the canonical embedding of E into E'' . We claim that the continuous linear mapping $\pi'_x \circ J$ is the Fréchet derivative of δ at x . Thus, we must show that

$$\lim_{\|y\| \rightarrow 0} \frac{p[\delta(x + y) - \delta(x) - \pi'_x \circ J(y)]}{\|y\|} = 0$$

for every continuous seminorm p on $(\mathcal{H}(E), \tau_\omega)'_\beta$. By the definition of the strong topology, we may assume without loss of generality that there exists a bounded subset A of $(\mathcal{H}(E), \tau_\omega)$ such that $p(\phi) = \sup\{|\phi(f)| : f \in A\}$ for every $\phi \in (\mathcal{H}(E), \tau_\omega)'$. By [11, §12, Proposition 1] there exist positive real numbers C, γ such that $\|(1/n!) \hat{d}^n f(x)\| \leq C\gamma^n$ for every $f \in A$. Therefore, if $\|y\| < \gamma^{-1}$ we have

$$\begin{aligned} & \frac{p[\delta(x + y) - \delta(x) - \pi'_x \circ J(y)]}{\|y\|} \\ &= \|y\|^{-1} \sup_{f \in A} |f(x + y) - f(x) - df(x)(y)| \\ &= \|y\|^{-1} \sup_{f \in A} \left| \sum_{n=2}^{\infty} \frac{1}{n!} \hat{d}^n f(x)(y) \right| \leq \|y\|^{-1} \sum_{n=2}^{\infty} C\gamma^n \|y\|^n = \frac{C\gamma^2 \|y\|}{1 - \gamma \|y\|}, \end{aligned}$$

and since this tends to zero with $\|y\|$, our claim is proved. □

Aron and Schottenloher show that the mapping $f \rightarrow f^t$ establishes an isomorphism between the space $(\mathcal{H}_K(E; F), \tau_\omega)$ of compact holomorphic mappings from E into F endowed with the Nachbin topology, and the space $L_\varepsilon(F'_c; (\mathcal{H}(E), \tau_\omega))$, where F'_c is the dual space of F with the topology of uniform convergence on compact subset of F [3, Theorem 4.1] and ε denotes the topology of uniform convergence on equicontinuous sets. In the weakly compact case we have an analogous result:

PROPOSITION 3.6. *The mapping $f \rightarrow f^t$ is an isomorphism from $(\mathcal{H}_{wK}(E; F), \tau_\omega)$ onto $L_\varepsilon(F'_t; (\mathcal{H}(E), \tau_\omega))$.*

Proof. It is easy to see that the mapping $f \rightarrow f'$ is linear and injective. To see that this mapping is surjective, let $T \in L(F'_\tau; (\mathcal{H}(E), \tau_\omega))$. Then $T' \in L((\mathcal{H}(E), \tau_\omega)'_\tau; F)$ and therefore by Lemma 3.5 $f = T' \circ \delta$ is a holomorphic mapping from E into F . If $\psi \in F'$ and $x \in E$ then

$$(f'\psi)(x) = \langle \psi, f(x) \rangle = \langle \psi, T'\delta(x) \rangle = \langle T\psi, \delta(x) \rangle = (T\psi)(x).$$

Therefore $f' = T$ and it follows that the mapping $f \rightarrow f'$ is a bijection from $\mathcal{H}_{wK}(E; F)$ onto $L(F'_\tau; (\mathcal{H}(E), \tau_\omega))$ with inverse given by $T \rightarrow T' \circ \delta$.

The proof that the mapping $f \rightarrow f'$ and its inverse are continuous is the same as the proof of the analogous assertion for the compact case [3, Theorem 4.1]. \square

Davis, Figiel, Johnson and Pełczyński [4] have shown that weakly compact linear mappings factor through reflexive spaces in the following sense: if T is a weakly compact linear mapping of E into F , there exists a reflexive Banach space G and continuous linear mappings $S: E \rightarrow G$, $R: G \rightarrow F$ such that $T = R \circ S$. We show next that weakly compact holomorphic mappings also factor through reflexive spaces.

If A is a closed, convex, balanced and bounded subset of a Banach space F , then F_A will denote the Banach space obtained by taking the linear span of A with norm given by the Minkowski functional of A , so that A is the closed unit ball of F_A .

Aron and Schottenloher show that compact holomorphic mappings have “thin” images, in the following sense: $f \in \mathcal{H}(E; F)$ is compact if and only if there exists a compact, convex, balanced subset K of F such that $f(x) \in F_K$ for every $x \in E$, and f is holomorphic as a mapping from E into F_K [3, Proposition 3.5]. We prove a similar result for weakly compact holomorphic mappings, and the Davis-Figiel-Johnson-Pełczyński theorem then enables us to factor these mappings through reflexive spaces.

THEOREM 3.7. *Let $f \in \mathcal{H}(E; F)$. The following are equivalent:*

- (i) *f is weakly compact;*
- (ii) *There exists a weakly compact convex balanced subset K of F such that f is a holomorphic mapping from E into F_K ;*
- (iii) *There exists a reflexive Banach space G , a continuous linear mapping $T: G \rightarrow F$ and a holomorphic mapping $g: E \rightarrow F$ such that $f = T \circ g$.*

Proof. (i) implies (ii): As in the proof of [3, Proposition 3.5], there exists a sequence $\{U_m\}$ of open circled subsets of E such that $f(U_m)$ is a bounded subset of F for every m , and $E = \bigcup_{m=1}^\infty U_m$. Let W_m be the balanced hull of U_m . Since U_m is circled, it follows from the maximum modulus theorem that $f(W_m)$ is bounded for every m . Let $V_m = \frac{1}{2}W_m$. Then $E = \bigcup_{m=1}^\infty V_m$ and hence $f(E) = \bigcup_{m=1}^\infty f(V_m)$. Let

$$K_{mn} = \bar{\Gamma} \left(\frac{1}{n!} \hat{d}^n f(0)(W_m) \right).$$

Then K_{mn} is a weakly compact convex balanced subset of F and by Lemma 3.1 $K_{mn} \subset \bar{\Gamma}f(W_m)$ for every m, n . Therefore, arguing in the same way as in the proof of Theorem 3.2, we find that the set

$$K_m = \left\{ \sum_{n=0}^\infty 2^{-n} z_n : z_n \in K_{mn} \right\}$$

is weakly compact, and that $f(V_m) \subset K_m$ for every m . Hence $\bar{\Gamma}f(V_m)$ is a weakly compact convex balanced subset of F for every m , and

$$(1) \quad f(E) \subset \bigcup_{m=1}^\infty \bar{\Gamma}f(V_m).$$

Since $\bar{\Gamma}f(V_m)$ is a weakly compact convex balanced set for every m , there exists a sequence $\{\beta_m\}$ of positive real numbers such that $K = \bigcup_{m=1}^\infty \beta_m \bar{\Gamma}f(V_m)$ is weakly compact [9, page 184]. It follows from (1) that f maps E into F_K . That f is a holomorphic mapping from E into F_K is proved in exactly the same way as the analogous result for compact holomorphic mappings [3, Proposition 3.5].

(ii) implies (iii): Suppose there exists a weakly compact convex balanced subset K of F such that f is a holomorphic mapping from E into F_K . Let $f_K: E \rightarrow F_K$ be the mapping $x \rightarrow f(x)$ and let J denote the injection of F_K into F , so that $f = J \circ f_K$. Applying the Davis-Figiel-Johnson-Pelczynski theorem to J , there exists a reflexive Banach space G , and continuous linear mappings $T: G \rightarrow F$ and $S: F_K \rightarrow G$ such that $J = T \circ S$. Let $g = S \circ f_K$. Then $f = J \circ f_K = T \circ S \circ f_K = T \circ g$.

(iii) implies (i) is trivial. □

We now look at the connection between reflexivity and weakly compact holomorphic mappings. It is easy to see that the Banach space E is reflexive if and only if for every Banach space F , every continuous linear mapping from E into F is weakly compact. We wish to characterize those Banach spaces E with the property that every holomorphic mapping with domain E is weakly compact.

The bornological topology on $\mathcal{H}(E)$ associated with the Nachbin topology τ_ω is denoted by τ_δ . This topology is also the barreled topology associated with τ_ω [6]. Dineen has shown that $(\mathcal{H}(E), \tau_\delta)$ is reflexive if and only if $P(^nE)$ is a reflexive Banach space for every n [5, Proposition 2.5].

THEOREM 3.8. *Let E be a complex Banach space. The following are equivalent:*

- (i) $(\mathcal{H}(E), \tau_\delta)$ is reflexive;
- (ii) for every complex Banach space F every holomorphic mapping from E into F is weakly compact.

Proof. (i) implies (ii): Let $f \in \mathcal{H}(E; F)$. Consider the continuous linear mapping $f^t: F' \rightarrow (\mathcal{H}(E), \tau_\omega)$. Since τ_δ and τ_ω have the same bounded sets, f^t is a continuous linear mapping of F' into $(\mathcal{H}(E), \tau_\delta)$. Since $(\mathcal{H}(E), \tau_\delta)$ is reflexive, f^t maps the unit ball of F' into a relatively weakly compact subset of $(\mathcal{H}(E), \tau_\delta)$. But τ_δ is finer than τ_ω and so f^t is a weakly compact linear mapping from F' into $(\mathcal{H}(E), \tau_\omega)$. Hence by Theorem 3.4 f is weakly compact.

(ii) implies (i): For each n let $\delta_n: E \rightarrow P(^nE)'$ be the continuous n -homogeneous polynomial defined by $\langle \delta_n(x), P \rangle = P(x)$ for $P \in P(^nE)$. By (ii) δ_n is weakly compact and therefore by Proposition 2.1 δ_n^t is a weakly compact linear mapping from $P(^nE)''$ into $P(^nE)$. But the restriction of δ_n^t to $P(^nE)$ is the identity mapping on $P(^nE)$. Therefore $P(^nE)$ is reflexive for every n . \square

Now E' is a subspace of $(\mathcal{H}(E), \tau_\delta)$ and so if $(\mathcal{H}(E), \tau_\delta)$ is reflexive then in particular E' is reflexive. The converse, however, is false. For example, $(\mathcal{H}(l_p), \tau_\delta)$ is not reflexive for any p [1]. Examples of spaces E for which $(\mathcal{H}(E), \tau_\delta)$ is reflexive are Tsirelson's space T' [1], and all quotient spaces of T' [2].

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