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# JONES POLYNOMIALS OF PERIODIC LINKS

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**Let $L$ be a link in $S^3$ which has a prime period and $L_*$ be its factor link. Several relationships between the Jones polynomials of $L$ and $L_*$ are proved. As an application, it is shown that some knot cannot have a certain period.**

**1. Introduction.** Let $L$ be an oriented link that has period $r > 1$. That is, there exists an orientation preserving auto-homeomorphism $\phi: S^3 \to S^3$ of order $r$ with a set of fixed points $F \cong S^1$ disjoint from $L$ and which maps $L$ onto itself. By the positive solution of Smith Conjecture, $F$ is unknotted. Let $\Sigma^3 = S^3/\phi$ be the quotient space under $\phi$. Since $F$ is unknotted, $\Sigma^3$ is again a 3-sphere, and $S^3$ is the $r$-fold cyclic covering space of $\Sigma^3$ branched along $F$.

Let $\psi: S^3 \to \Sigma^3$ be the covering projection. Denote $\psi(L) = L_*$, which is called the *factor link*, and let $V_L(t)$ and $V_{L_*}(t)$ denote, respectively, the Jones polynomials of $L$ and $L_*$.

In this paper, we will prove some relationships between $V_L(t)$ and $V_{L_*}(t)$ which are analogous to those between their Alexander polynomials [**M2**]. In fact, we will prove

THEOREM 1. *Let $r$ be a prime and $L$ a link that has period $r^q$, $q \geq 1$. Then*

$$V_L(t) \equiv \left[V_{L_*}(t)\right]^{r^q} \mod(r, \xi_r(t)), \tag{1.1}$$

*where* $\xi_r(t) = \sum_{j=0}^{r-1}(-t)^j - t^{(r-1)/2}$.

If $L$ is not split, then we are able to prove a slightly more precise formula.

Let $\operatorname{lk}(X, Y)$ denote the linking number between two simple closed curves $X$ and $Y$ in $S^3$. Then we have

THEOREM 2. *Let $r$ be a prime and $L$ a non-split link that has period $r^q$, $q \geq 1$.*

(1) *If* $\mathrm{lk}(L, F) \equiv 1 \pmod 2$, *then*

$$V_L(t) \equiv \left[V_{L_*}(t)\right]^{r^q} \quad \mathrm{mod}(r, \eta_r(t)), \tag{1.2}$$

*where* $\eta_r(t) = [\sum_{j=0}^{r-2}(j+1)(-t)^j](1+t^r) - t^{r-1}$.

(2) *If* $\mathrm{lk}(L, F) \equiv 0 \pmod 2$, *then*

$$V_L(t) \equiv \left[V_{L_*}(t)\right]^{r^q} \quad \mathrm{mod}(r, \xi_r(t)). \tag{1.3}$$

Note that $\eta_r(t) \equiv 0 \ \mathrm{mod}(r, \xi_r(t))$. (See Lemma 6 in §3.) As a simple consequence, we obtain

COROLLARY 3. *Let* **b** *be an n-braid and let* $V_{\mathbf{b}}(t)$ *be the Jones polynomial of the closure* $\hat{\mathbf{b}}$ *of* **b**. *Let r be a prime and* $q \geq 1$. *Then*

$$V_{\mathbf{b}^{r^q}}(t) \equiv \left[V_{\mathbf{b}}(t)\right]^{r^q} \quad \mathrm{mod}(r, \xi_r(t)).$$

Formulas (1.1), (1.2), and (1.3) involve slightly larger ideals than those in the corresponding formulas about the Alexander polynomials [**M2**]. However, they are the best possible. To see this, consider an $n$-component trivial link $L$. $L$ has any period $r$ and a factor link $L_*$ is also an $n$-component trivial link. Since $V_L(t) = V_{L_*}(t) = (-1)^{n-1}(\sqrt{t} + 1/\sqrt{t})^{n-1}$, the formula $V_L(t) \equiv [V_{L_*}(t)]^r \pmod I$ holds only if the ideal $I$ contains $\xi_r(t)$. We should note that while the Alexander polynomial of a link may vanish, the Jones polynomial of a link never vanishes.

Corollary 3 is also verified for $n = 3$ by a direct computation using Theorem 21 [**J**] and Theorem [**M2**].

These formulas may have more theoretical values than practical values. (See Proposition 7 in §4.) Nevertheless, we can prove that $10_{105}$ cannot have period 7 (Proposition 10). This solves one of several undecided cases for knots with 10 crossings.

**2. Proof of Theorem 1.** Since it suffices to prove Theorem 1 for $q = 1$, we assume that $L$ has a prime period $r$. In this section, we prove that Theorem 2 implies Theorem 1.

Suppose that $L$ has period $r$ and let $\phi$ be an orientation preserving auto-homeomorphism of $S^3$ that maps $L$ onto itself. Suppose that $L$ splits into $k$ components $L_1, L_2, \ldots, L_k$. Then $\phi$ must map a split component not having period $r$ onto another split component not having period $r$. Therefore, split components of $L$ are divided into $h + 1$ sets $A_1 = \{L_1, \ldots, L_r\}$, $A_2 = \{L_{r+1}, \ldots, L_{2r}\}, \ldots, A_h = \{L_{(h-1)r+1}, \ldots, L_{hr}\}$ and $B = \{L_{hr+1}, \ldots, L_k\}$ such that any two links in $A_i$ $(i = 1, 2, \ldots, h)$ are ambient isotopic and a link in $B$ has period $r$. The factor link $L_*$, then, has $h + (k - hr)$ $(= k - h(r-1))$ split components. Noting that the factor link of the $r$-split component link $L_{sr+1} \cup \cdots \cup L_{(s+1)r}$ is

$L_{sr+1}$, $0 \le s \le h-1$, we have

$$(2.1)\ (1)\quad V_L(t) = \left[-\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)\right]^{k-1} \prod_{i=1}^{k} V_{L_i}(t), \quad \text{and}$$

$$(2)\quad V_{L_*}(t) = \left[-\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)\right]^{k-h(r-1)-1} \prod_{s=0}^{h-1} V_{L_{sr+1}}(t) \prod_{j=hr+1}^{k} V_{L_{j*}}(t).$$

Now Theorem 2 implies that for $j = hr+1, \dots, k$, $V_{L_j}(t) \equiv V_{L_{j*}}(t)^r \bmod(r, \xi_r(t))$ and hence

$$(2.2)\quad V_{L_*}(t)^r = \left[-\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)\right]^{r[k-h(r-1)-1]} \prod_{s=0}^{h-1} V_{L_{sr+1}}(t)^r \prod_{j=hr+1}^{k} V_{L_{j*}}(t)^r$$

$$\equiv \left[-\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)\right]^{k-h(r-1)-1} \prod_{i=1}^{k} V_{L_i}(t) \qquad \bmod(r, \xi_r(t)).$$

Comparing (2.2) with (2.1) (1), we see that Theorem 1 will follow from Lemma 4 below.

LEMMA 4. *For a prime $r$,*

$$(-1)^{k-h(r-1)-1}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^{k-h(r-1)-1} \equiv (-1)^{k-1}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^{k-1} \quad \bmod(r, \xi_r(t)).$$

*Proof.* Since $\xi_r(t) \equiv (1+t)^{r-1} - t^{(r-1)/2} \pmod r$ by Lemma 6 (proved in §3), it follows that $(\sqrt{t} + (1/\sqrt{t}))^{r-1} = ((t+1)/\sqrt{t})^{r-1} \equiv 1$ $(\bmod(r, \xi_r(t)))$. Since $r$ is a prime, $(-1)^{k-h(r-1)-1} \equiv (-1)^{k-1} \pmod r$. □

**3. Proof of Theorem 2.** We may assume that $q = 1$ and $L$ is not split.

Let $\zeta$ be the rotation of $R^2$ about the origin 0 through $2\pi/r$. Since $L$ is a link having period $r$, $L$ has a diagram $\tilde{L}$ ($\notin \{0\}$) on $R^2$ which is divided into $r$ pieces $\tilde{L}_0, \tilde{L}_1, \dots, \tilde{L}_{r-1}$ such that $\zeta(\tilde{L}_i) = \tilde{L}_{i+1}$, $i = 0, 1, \dots, r-1$, $\tilde{L}_r = \tilde{L}_0$. Let $R(0, 2\pi/r)$ be the closed domain bounded by two half lines $\theta = 0$ and $\theta = 2\pi/r$ in the polar coordinate system. We may assume that $\tilde{L}_0 = \tilde{L} \cap R(0, 2\pi/r)$. Let $A_1, A_2, \dots, A_l$ be the points of intersection of $\tilde{L}_0$ and the line $\theta = 0$ and let $\zeta(A_i) = B_i$, $i = 1, 2, \dots, l$, $A_i \neq B_i$. By joining $A_i$ and $B_i$ on $R^2$ by a circle $C_i$ centered 0, we obtain a diagram $\tilde{L}_*$ of the factor link $L_* = \psi(L)$. For simplicity, we write $\tilde{L}_* = \tilde{L}/\zeta$. $\tilde{L}_*$ divides $R^2$ into finitely many domains, which we classify as shaded or unshaded. Now unshading the domain containing 0, we have the graph $\Gamma_*$ of $\tilde{L}_*$. We may take 0 as one vertex of $\Gamma_*$. Furthermore, we can assign $+1$ or $-1$ to each edge of $\Gamma_*$ [**M4**]. Similarly, we have the graph $\Gamma$ of $\tilde{L}$ by unshading the domain containing 0. $\Gamma$ is also an oriented graph. Using $\Gamma$ and $\Gamma_*$, we can evaluate $V_L(t)$ and $V_{L_*}(t)$ as follows. (See [**M4**].)

Let $p'$ and $n'$ be, respectively, the number of positive and negative edges in $\Gamma$. Let $S(a, b)$, $0 \le a \le p'$ and $0 \le b \le n'$, be the collection of subgraphs obtained from $\Gamma$ by removing exactly $a$ positive edges and $b$ negative edges. $S(a, b)$ contains $\binom{p'}{a}\binom{n'}{b}$ subgraphs.

For $\gamma \in S(a, b)$, let $\mu(\gamma) = b_0(\gamma) + b_1(\gamma)$, where $b_i(\gamma)$, $i = 0, 1$, denotes the $i$th Betti number of $\gamma$ as a 1-complex. Then the bracket polynomial $P_{\tilde{L}}(A)$ defined in [**K**] associated with the link diagram $\tilde{L}$ is given by the following formula:

$$(3.1)\quad P_{\tilde{L}}(A) = \sum_{\substack{0\le a\le p'\\ 0\le b\le n'}} A^{p'-2a-n'+2b} \sum_{\gamma\in S(a,b)} \left[-(A^2 + A^{-2})\right]^{\mu(\gamma)-1}.$$

Note that (3.1) is equivalent to the formula (2.10) in [**M4**].

We will use (3.1) to evaluate $P_{\tilde{L}}(A)$ and $P_{\tilde{L}_*}(A)$.

Let $p$ and $n$ be, respectively, the number of positive and negative edges in $\Gamma_*$. Then $\Gamma$ has exactly $rp$ positive and $rn$ negative edges, i.e. $p' = rp$ and $n' = rn$. Let $S_*(a, b)$ be the collection of subgraphs of $\Gamma_*$ which is defined in a similar way to $S(a, b)$. Then we have

$$(3.2)\quad P_{\tilde{L}_*}(A) = \sum_{\substack{0\le a\le p\\ 0\le b\le n}} A^{p-2a-n+2b} \sum_{\gamma_*\in S_*(a,b)} \left[-(A^2 + A^{-2})\right]^{\mu(\gamma_*)-1}.$$

Since the rotation $\zeta\colon R^2 \to R^2$ maps $\tilde{L}$ onto itself, we may assume that $\zeta$ maps the graph $\Gamma$ onto itself, preserving the sign of each edge. In other words, $\zeta$ defines an automorphism of the oriented graph $\Gamma$.

If $\mathrm{lk}(L, F) \equiv 1 \pmod 2$, then the unbounded domain is shaded. Therefore, $\zeta$ fixes only the origin 0. If $\mathrm{lk}(L, F) \equiv 0 \pmod 2$, however, the unbounded domain is unshaded, and hence, $\zeta$ keeps exactly two vertices 0 and $\infty$ fixed, where $\infty$ is a point associated with the unbounded domain. Therefore, if $\mathrm{lk}(L, F) \equiv 0 \pmod 2$, $\zeta$ may be considered as an automorphism of the graph $\Gamma$ in $S^2$ which keeps the north and south poles fixed.

*Case* A. $\mathrm{lk}(L, F) \equiv 1 \pmod 2$.

In this case, $\Gamma$ is the $r$-fold cyclic covering of $\Gamma$ branched at 0. Take $\gamma \in S(a, b)$.

*Case* 1. $\gamma$ is not fixed under $\zeta$, i.e. $\zeta(\gamma) \ne \gamma$.

This is, of course, the case when $a \not\equiv 0 \pmod r$ or $b \not\equiv 0 \pmod r$. In this case, $\gamma$, $\zeta(\gamma)$, $\zeta^2(\gamma), \dots, \zeta^{r-1}(\gamma)$ are all distinct, but, since any two of these are isomorphic, we have exactly $r$ identical terms in $P_{\tilde{L}}(A)$, and they vanish by reducing modulo $r$.

*Case* 2. $\gamma$ is fixed under $\zeta$ setwise, i.e. $\zeta(\gamma) = \gamma$.

In this case, $a \equiv b \equiv 0 \pmod r$. Write $a = ra'$ and $b = rb'$. Then $\gamma$ defines a unique quotient subgraph $\gamma_*(= \gamma/\zeta) \in S_*(a', b')$.

Let $\alpha$ and $\alpha_*$, be, respectively, the terms in $P_{\tilde{L}}(A)$ and $P_{\tilde{L}_*}(A)$ which are associated with $\gamma$ and $\gamma_*$. Since $p' = rp$ and $n' = rn$, we have

$$\text{(3.3)}\quad (1)\quad \alpha = A^{r(p-2a'-n+2b')}\left[-(A^2 + A^{-2})\right]^{\mu(\gamma)-1}, \quad \text{and}$$

$$(2)\quad \alpha_* = A^{p-2a'-n+2b'}\left[-(A^2 + A^{-2})\right]^{\mu(\gamma_*)-1}.$$

We will compare $\mu(\gamma) - 1$ and $\mu(\gamma_*) - 1$.

If we use the fact that $\gamma$ is the $r$-fold cyclic cover of $\gamma_*$, it is not difficult to find some relationship between $b_1(\gamma)$ and $b_1(\gamma_*)$.

Consider connected components of $\gamma$. Let $D_0, D_1, \ldots, D_k, D_{1,1}, \ldots, D_{1,r}, D_{2,1}, \ldots, D_{2,r}, \ldots, D_{m,1}, \ldots, D_{m,r}$ be connected components of $\gamma$ such that

(3.4) (1) $D_0$ contains the origin $\{0\}$, and $\zeta(D_0) = D_0$,

(2) $D_i$ $(i = 1, 2, \ldots, k)$ is a component $(\notin \{0\})$ of $\gamma$ such that $\zeta(D_i) = D_i$,

(3) $\{D_{j,1}, \ldots, D_{j,r}\}$, $(j = 1, 2, \ldots, m)$ is a set of components of $\gamma$ which permutes by $\zeta$

Then connected components of $\gamma_*$ consist of the sets: $D_i' = D_i/\zeta$ $(i = 0, 1, 2, \ldots, k)$ and $D_{j,1}' = D_{j,1}$ $(j = 1, 2, \ldots, m)$.

We compare $b_1(D_i)$ and $b_1(D_{j,\lambda})$ with $b_1(D_i')$ and $b_1(D_{j,1}')$.

LEMMA 5.

(3.5) (1) $b_1(D_0) = rb_1(D_0')$,

(2) $b_1(D_i) - 1 = r\{b_1(D_i') - 1\}$ *for* $1 \le i \le k$,

(3) $b_1(D_{j,1}) = b_1(D_{j,\lambda}) = b_1(D_{j,1}')$ *for* $1 \le j \le m$ *and* $1 \le \lambda \le r$.

*Proof*. (1) Let $d_0'$ and $e_0'$, denote, respectively, the number of vertices and edges of $D_0'$. Then, since $D_0$ is the $r$-fold cyclic covering of $D_0'$ branched at 0, the number of vertices and edges of $D_0$ are given by

$r(d_0' - 1) + 1$ and $re_0'$ respectively. Therefore

$$1 - b_1(D_0) = r(d_0' - 1) + 1 - re_0' = r(d_0' - e_0') - r + 1$$
$$= r(1 - b_1(D_0')) - r + 1 = 1 - rb_1(D_0'),$$

and hence, $b_1(D_0) = rb_1(D_0')$.

(2) Since $D_i$ is the $r$-fold (unbranched) cyclic covering of $D_0'$, it follows that $\chi(D_i) = r\chi(D_i')$, where $\chi$ denotes the Euler characteristic. Since $\chi(D_i) = 1 - b_1(D_i)$, we have

$$1 - b_1(D_i) = r\chi(D_i') = r\{1 - b_1(D_i')\}$$

and hence, $b_1(D_i) - 1 = r\{b_1(D_i') - 1\}$.

(3) is obvious.

Now we compare $\mu(\gamma) - 1$ and $\mu(\gamma_*) - 1$. Using Lemma 5, we obtain

$$\begin{aligned}
\mu(\gamma) - 1 &= b_1(\gamma) + b_0(\gamma) - 1\\
&= b_1(D_0) + \sum_{i=1}^{k} b_1(D_i) + \sum_{j=1}^{m}\sum_{\lambda=1}^{r} b_1(D_{j,\lambda}) + k + 1 + rm - 1\\
&= rb_1(D_0') + \sum_{i=1}^{k}\{rb_1(D_i') - r + 1\} + \sum_{j=1}^{m} rb_1(D_{j,1}') + k + rm\\
&= r\left[b_1(D_0') + \sum_{i=1}^{k} b_1(D_i') + \sum_{i=1}^{m} b_1(D_{j,1}') + k + 1 + m - 1\right]\\
&\quad -rk - rm - rk + k + k + rm\\
&= r[b_1(\gamma_*) + b_0(\gamma_*) - 1] - 2k(r-1)\\
&= r\{\mu(\gamma_*) - 1\} - 2k(r-1).
\end{aligned}$$

Using this equality, we have

$$\alpha \equiv \alpha_*^r \mod\left[\{-(A^2 + A^{-2})\}^{2(r-1)} - 1\right]. \tag{3.6}$$

In fact, a simple computation shows that

$$\begin{aligned}
\alpha &= A^{r(p-2a'-n+2b')}\left[-(A^2 + A^{-2})\right]^{\mu(\gamma)-1}\\
&= A^{r(p-2a'-n+2b')}\left[-(A^2 + A^{-2})\right]^{r(\mu(\gamma_*)-1)}\left[-(A^2 + A^{-2})\right]^{-2k(r-1)}\\
&= \left\{A^{p-2a'-n+2b'}\left[-(A^2 + A^{-2})\right]^{\mu(\gamma_*)-1}\right\}^r\left[-(A^2 + A^{-2})\right]^{-2k(r-1)}\\
&= \alpha_*^r\left[-(A^2 + A^{-2})\right]^{-2k(r-1)}\\
&\equiv \alpha_*^r \mod\left(\left[-(A^2 + A^{-2})\right]^{2(r-1)} - 1\right).
\end{aligned}$$

*Case* B. $\mathrm{lk}(L, F) \equiv 0 \pmod 2$.

We consider connected components of $\gamma \in S(a, b)$. Let $D_0, D_1, \ldots, D_k$, $D_\infty, D_{1,1}, \ldots, D_{1,r}, D_{2,1}, \ldots, D_{2,r}, \ldots, D_{m,1}, \ldots, D_{m,r}$ be connected components of $\gamma$ which satisfy (3.4) (2) and (3). Furthermore, $D_0$ and $D_\infty$ are such that

(3.7) $D_0$ contains $\{0\}$ and $D_\infty$ contains $\{\infty\}$, and $\zeta(D_0) = D_0$ and $\zeta(D_\infty) = D_\infty$.

It may occur that $D_0 = D_\infty$. We should note that $\gamma$ is the $r$-fold cyclic covering of $\gamma_*$ branched at 0 and $\infty$.

Now (3.5) (2) and (3) are still valid under the present case. Only (3.5) (1) should be changed to the following.

(3.8) (i) If $D_0 \neq D_\infty$, then $b_1(D_0) = rb_1(D_0')$ and $b_1(D_\infty) = rb_1(D_\infty')$.

(ii) If $D_0 = D_\infty$, then $b_1(D_0) + 1 = r\{b_1(D_0') + 1\}$.

*Proof.* (i) follows from the fact that $D_0$ and $D_\infty$ are, respectively, the $r$-fold cyclic coverings of $D_0'$ and $D_\infty'$ branched at 0 and $\infty$.

(ii) $D_0(= D_\infty)$ is the $r$-fold cyclic covering of $D_0'$ branched at 0 and $\infty$. Let $d'$ and $e'$ denote the number of vertices and edges of $D_0'$. Then

$$1 - b_1(D_0) = 2 + r(d' - 2) - re' = r(d' - e') - 2r + 2$$
$$= r\big(1 - b_1(D_0')\big) - 2r + 2,$$

which yields $b_1(D_0) + 1 = r\{b_1(D_0') + 1\}$.

Using (3.8) (i) and (3.5) (1), (2), we obtain the following formulas.

(i) When $D_0 \neq D_\infty$,

$$\begin{aligned}\mu(\gamma) - 1 &= b_1(\gamma) + b_0(\gamma) - 1 \\ &= b_1(D_0) + \sum_{i=1}^{k} b_1(D_i) + b_1(D_\infty) \\ &\quad + \sum_{j=1}^{m}\sum_{\lambda=1}^{r} b_1(D_{j,\lambda}) + k + 2 + rm - 1 \\ &= rb_1(D_0') + \sum_{i=1}^{k}\{rb_1(D_i') - r + 1\} \\ &\quad + rb_1(D_\infty') + \sum_{j=1}^{m} rb_1(D_{j,1}') + k + 1 + rm\end{aligned}$$

(continues)

(continued)

$$= r\left\{b_1(D_0') + \sum_{i=1}^{k} b_1(D_i') + b_1(D_\infty') + \sum_{j=1}^{m} b_1(D_{j,1}') + k + 2 + m - 1\right\}$$

$$-kr + k - rk - r - rm + k + 1 + rm$$

$$= r\{\mu(\gamma_*) - 1\} - (2k+1)(r-1).$$

(ii) When $D_0 = D_\infty$,

$$\mu(\gamma) - 1 = b_1(D_0) + \sum_{i=1}^{k} b_i(D_i) + \sum_{j=1}^{m}\sum_{\lambda=1}^{r} b_1(D_{j,\lambda}) + k + 1 + rm - 1$$

$$= rb_1(D_0') + r - 1 + \sum_{i=1}^{k} rb_1(D_i') + \sum_{j=1}^{m} rb_1(D_{j,1}') + k + rm$$

$$= r\left\{b_1(D_0') + \sum_{i=1}^{k} b_1(D_i') + \sum_{j=1}^{m} b_1(D_{j,1}') + k + 1 + m - 1\right\}$$

$$-rk - rm + r - 1 + k + rm$$

$$= r[\mu(\gamma_*) - 1] - (k-1)(r-1).$$

Therefore, we have

$$\alpha \equiv \alpha_*^r \mod\left([-(A^2 + A^{-2})]^{r-1} - 1\right). \tag{3.9}$$

Now it only remains to show the following simple lemma.

LEMMA 6. *For any prime* $r$,

(1) $$(t+1)^{2(r-1)} - t^{r-1} \equiv \eta_r(t) \pmod r.$$

(2) $$(t+1)^{r-1} - t^{r-1/2} \equiv \xi_r(t) \pmod r.$$

*Proof*. If $r = 2$, the lemma is obvious. Therefore, we assume that $r$ is an odd prime. Then it suffices to prove the following.

(3.10) For $j = 0, 1, \ldots, r-1$,

(1) $$\binom{2r-2}{j} \equiv (-1)^j (j+1) \pmod r,$$

(2) $$\binom{2r-2}{r+j} \equiv (-1)^j (j+1) \pmod r,$$

(3) $$\binom{r-1}{j} \equiv (-1)^j \pmod r.$$

*Proof*. Firstly, (3) is obviously true for $j = 0$ and 1. Since $\binom{r}{j} = \binom{r-1}{j} + \binom{r-1}{j-1}$, it follows by the induction hypothesis that $0 \equiv \binom{r-1}{j} + (-1)^{j-1} \pmod r$ which yields $\binom{r-1}{j} \equiv (-1)^j \pmod r$. This proves (3). Secondly, (1) is trivially true for $j = 0$ and 1. Now for $1 \le j \le r-1$,

$$\binom{2r-2}{j} = \binom{2r-2}{j-1}\frac{2r-j-1}{j}.$$

Using the induction hypothesis, we can write

$$\binom{2r-2}{j-1} = (-1)^{j-1}j + rk$$

for some integer $k$. Then

$$\binom{2r-2}{j} = (-1)^j(j+1) + (-1)^{j-1}2r + \frac{rk}{j}(2r-j-1).$$

Since $\binom{2r-2}{j}$ is an integer and $r$ is a prime, $j \mid k(2r-j-1)$ and hence

$$\binom{2r-2}{j} \equiv (-1)^j(j+1) \pmod r.$$

This proves (1). Finally, since $r$ is odd and $r-j-2 \le r-1$ for $0 \le j \le r-1$, (3.10) (1) implies that

$$\binom{2r-2}{r+j} = \binom{2r-2}{r-j-2} \equiv (-1)^{r-j+2}(r-j-2+1)$$
$$\equiv (-1)^{r-j+1}(j+1) \equiv (-1)^j(j+1) \pmod r.$$

This proves (2). □

Let $I$ be the ideal in $Z[A, A^{-1}]$ generated by $r$ and

$$[-(A^2 + A^{-2})]^{2(r-1)} - 1 \quad \left(\text{or } [-(A^2+A^{-2})]^{r-1} - 1 \text{ in } Z[A, A^{-1}]\right)$$

when $\mathrm{lk}(L, F) \equiv 1 \pmod 2$ (or $\mathrm{lk}(L, F) \equiv 0 \pmod 2$). The Lemma 6 yields that $P_{\tilde{L}}(A) \equiv [P_{\tilde{L}_*}(A)]^r \pmod I$. Let $w(\tilde{L})$ be the twisting number (or the writhe) of $\tilde{L}$. Then, since $w(\tilde{L}) = rw(\tilde{L}_*)$, it follows that

$$f_L(A) = (-A)^{-3w(\tilde{L})}P_{\tilde{L}}(A) = (-A)^{-3rw(\tilde{L}_*)}P_{\tilde{L}}(A) \equiv \left[(-A)^{-3w(\tilde{L}_*)}P_{\tilde{L}_*}(A)\right]^r$$
$$= \left[f_{L_*}(A)\right]^r \pmod I.$$

Here $f_L(t^{-1/4}) = V_L(t)$ **[K]** and Theorem 1 follows from Lemma 6. A proof of Theorem 2 is now complete.

**4. Applications and remarks.** Formula (1.1) may not be used to determine whether a knot (but not a link) $K$ has small prime period $r \leq 5$. In fact, we have the following

PROPOSITION 7. *Let K be a knot. Then for* $r = 2, 3$ *or* $5$,

$$V_K(t) \equiv 1 \mod(r, \xi_r(t)). \tag{4.1}$$

*Proof*. First, note that $\xi_2(t) = 1 - t - \sqrt{t}$, $\xi_3(t) = 1 - 2t + t^2$ and $\xi_5(t) = 1 - t - t^3 + t^4$. Now, as is well known (Definition 17 [J]), $1 - V_k(t) \equiv 0 \mod \xi_5(t)$, and hence $V_K(t) \equiv 1 \mod \xi_5(t)$. Furthermore, congruences $1 - t + t^2 \equiv (1 - t - \sqrt{t})(1 - t - \sqrt{t}) \pmod 2$, $(1 - t)(1 - t^3) \equiv (1 - t + t^2)(1 + t^2) \pmod 2$ and $(1 - t)(1 - t^3) \equiv (1 - 2t + t^2)\cdot(1 + t + t^2) \pmod 3$ prove Proposition 7.

It is also easy to show that for any prime $r \geq 5$, $\xi_5(t) \mid \xi_r(t)$.

PROPOSITION 8. *Let r be an odd prime* $\geq 5$. *Let* $\omega$ *and* $\tau$ *denote, respectively, a primitive* $(r - 1)/2$*th-root and* $(r + 1)/2$*th-root of unity. If a link L has period r, then*

$$\text{(4.2)}\qquad \begin{aligned} &(1) \qquad V_L(\omega) \equiv V_{L_*}(\omega) \pmod r \\ &(2) \qquad V_L(\tau) \equiv V_{L_*}(\tau^{-1}) \pmod r. \end{aligned}$$

*Proof*. From Theorem 1, we see that $V_L(t) \equiv V_{L_*}(t)^r \equiv V_{L_*}(t^r)$ $\mod(r, \xi_r(t))$. Note that

$$\xi_r(t) = \frac{1 + t^r}{1 + t} - t^{(r-1)/2} = \frac{1}{1 + t}(1 - t^{(r-1)/2})(1 - t^{(r+1)/2}).$$

Since

$$\omega^r = \left(\omega^{(r-1)/2}\right)^2\omega = \omega \quad \text{and} \quad \tau^r = \left(\tau^{(r+1)/2}\right)^2\tau^{-1} = \tau^{-1},$$

a substitution $\omega$ or $\tau$ for $t$ in $V_L(t)$ and $V_{L_*}(t^r)$ proves (4.2).

COROLLARY 9. *Under the conditions of Proposition* 8, *if* $L_*$ *is unknotted, then*

$$V_L(\omega) \equiv V_L(\tau) \equiv 1 \pmod r. \tag{4.3}$$

Using Corollary 9, we can prove the following

PROPOSITION 10. *The knot* $10_{105}$ *in* [**R**] *has no period.*

*Proof*. According to [**B-Z**, p. 312], 7 is the only possible period of $10_{105}$. Suppose that $K$ has period 7. Since $K$ is alternating and fibred [**M1**], the factor knot $K_*$ is either unknotted or fibred [**M3**]. Since $\Delta_K(t) = 1 - 8t + 22t^2 - 29t^3 + 22t^4 - 8t^5 + t^6 \equiv (1+t)^6 \pmod 7$, it follows from [**M2**] that $K_*$ must be unknotted. Therefore, by Corollary 9, $V_K(\omega) \equiv 1$ and $V_K(\tau) \equiv 1 \pmod 7$, where $\omega = e^{2\pi i/3}$ and $\tau = e^{2\pi i/4} = \sqrt{-1}$. Since $V_K(t) = t^{-7} - 4t^{-6} + 8t^{-5} - 12t^{-4} + 15t^{-3} - 15t^{-2} + 14t^{-1} - 11 + 7t - 3t^2 + t^3$, we have $V_K(\sqrt{-1}) \equiv -1 \pmod 7$. Therefore, $K$ cannot have period 7.

REMARK. A similar argument reveals that if $K = 10_{101}$ in [**R**] has period 7, then the factor knot cannot be unknotted.

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