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POSITIVE ANALYTIC CAPACITY BUT ZERO BUFFON NEEDLE PROBABILITY

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There exists a compact set of positive analytic capacity but zero Buffon needle probability.

1. Introduction. For a compact set E in the complex plane $\mathbf{C}$, $H^\infty(E^c)$ denotes the Banach space of bounded analytic functions outside E with supremum norm $\|\cdot\|_{H^\infty(E^c)}$. The analytic capacity of E is defined by

$$\gamma(E) = \sup\{|f'(\infty)|; f \in H^\infty(E^c),\ \|f\|_{H^\infty(E^c)} \leq 1\},$$

where $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$ [**1**, p. 6]. Let $\mathscr{L}(r,\theta)$ $(r > 0,\ -\pi < \theta \leq \pi)$ denote the straight line defined by the equation $x\cos\theta + y\sin\theta = r$. The Buffon length of E is defined by

$$Bu(E) = \iint_{\{(r,\theta);\mathscr{L}(r,\theta)\cap E\neq\varnothing\}} dr\, d\theta.$$

Vitushkin [**7**] asked whether two classes of null-sets concerning $\gamma(\cdot)$ and $Bu(\cdot)$ are same or not (cf. [**2**], [**3**]). Mattila [**4**] showed that these two classes are different. (He showed that the class of null-sets concerning $Bu(\cdot)$ is not conformal invariant. Hence his method does not give the information about the implication between these two classes.) The second author [**5**] showed that, for any $0 < \varepsilon < 1$, there exists a compact set E_ε such that $\gamma(E_\varepsilon) = 1$, $Bu(E_\varepsilon) \leq \varepsilon$. The purpose of this note is to show

THEOREM. *There exists a compact set E_0 such that $\gamma(E_0) = 1$, $Bu(E_0) = 0$.*

2. Cranks. To construct E_0, we begin by defining cranks. The 1-dimension Lebesgue measure is denoted by $|\cdot|$. For a finite union E of segments in $\mathbf{C}$, its length is also denoted by $|E|$. For $\rho > 0, z \in \mathbf{C}$ and a set $E \subset \mathbf{C}$, we write $[\rho E + z] = \{\rho\zeta + z; \zeta \in E\}$. With $0 \leq \varphi < 1$ and a segment $J \subset \mathbf{C}$ parallel to the x-axis, we associate the closed segment $J(\varphi)$ of the same midpoint as J, parallel to the x-axis and of length $(1+\varphi)|J|$. With a positive integer $q, 0 \leq \varphi < 1$ and a segment J parallel to the x-axis, we associate

$$J(q,\varphi) = \bigcup_{k=1}^{2^{q-1}} [J_{2k-1}(\varphi) + i2^{-q}|J|] \cup \bigcup_{k=1}^{2^{q-1}} J_{2k}(\varphi),$$

where $\{J_k\}_{k=1}^{2^q}$ are mutually non-overlapping segments on J of length $2^{-q}|J|$; they are ordered from left to right. The set $J(q,\varphi)$ is a union of 2^q closed segments of length $2^{-q}(1+\varphi)|J|$. The segment $\Gamma_0 = \{x; 0 \leq x \leq 1\} \subset \mathbf{C}$ is called a crank of type 0. For a finite sequence $\{\varphi_j\}_{j=0}^n, \varphi_0 = 0$ $(n \geq 1)$ of non-negative numbers less than 1, a finite union Γ of closed segments is called a crank of type $\{\varphi_j\}_{j=0}^n$ if there exists a crank $\Gamma' = \bigcup_{k=1}^l J_k$ ($\{J_k\}_{k=1}^l$ are components of Γ') of type $\{\varphi_j\}_{j=0}^{n-1}$ such that

$$\Gamma = \bigcup_{k=1}^{l} J_k(q_k, \varphi_n)$$

for some l-tuple $(q_1, \ldots, q_l)$ of positive integers larger than or equal to $q_0 = 100$. We write $\Gamma' [_{\varphi_n} \Gamma$. For a sequence $\{\varphi_j\}_{j=0}^\infty, \varphi_0 = 0$ of non-negative numbers less than 1, a set Γ is called a crank of type $\{\varphi_j\}_{j=0}^\infty$, if there exists a sequence $\{\Gamma_n\}_{n=0}^\infty$ of cranks such that

(1) $$\Gamma_n \text{ is of type } \{\varphi_j\}_{j=0}^n,$$

(2) $$\Gamma_0 [_{\varphi_1} \Gamma_1 [_{\varphi_2} \cdots,$$

(3) $$\Gamma = \bigcap_{n=0}^\infty \overline{\bigcup_{j=n}^\infty \Gamma_j}.$$

We write by $\mathbf{O}_n$ the finite sequence of n zeros $(n \geq 1)$. For a finite union Γ of segments, $L^p(\Gamma)$ $(1 \leq p \leq \infty)$ denotes the L^p space on Γ with respect to the length element $|dz|$. We define an operator $\mathscr{H}_\Gamma$ on $L^p(\Gamma)$ by

$$\begin{aligned}\mathscr{H}_\Gamma f(z) &= \frac{1}{2\pi i} \text{p.v.} \int_\Gamma \frac{f(\zeta)}{\zeta - z}|d\zeta| \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{|\zeta - z| > \varepsilon, \zeta \in \Gamma} \frac{f(\zeta)}{\zeta - z}|d\zeta|.\end{aligned}$$

The following fact is already known.

LEMMA 1 ([5]). *For any positive integer m, there exist a crank Γ_m^* of type $\mathbf{O}_{m+1}$ and a non-negative function g_m^* on Γ_m^* such that g_m^* is a constant on each component of Γ_m^*,*

$$\|g_m^*\|_{L^1(\Gamma_m^*)} = 1, \quad \|g_m^*\|_{L^\infty(\Gamma_m^*)} \le C_1, \quad \|\mathrm{Re}\,\mathscr{H}_{\Gamma_m^*} g_m^*\|_{L^\infty(\Gamma_m^*)} \le C_1\sqrt{m},$$
$$Bu(\Gamma_m^*) \le C_1/m^{9/10},$$

where $\mathrm{Re}\,\zeta$ *is the real part of* ζ *and* C_1 *is an absolute constant.*

Our method is as follows. We define a sequence $\{n(k)\}_{k=0}^\infty$ of non-negative integers with large gaps. Choosing $\{\varphi_j\}_{j=0}^{10n(1)}$ suitably, we define a crank $\Gamma_{10n(1)}$ of type $\{\varphi_j\}_{j=0}^{10n(1)}$. Then $|\Gamma_{10n(1)}| = \prod_{\mu=1}^{10n(1)}(1+\varphi_\mu)$. Replacing each component of $\Gamma_{10n(1)}$ by a crank similar to $\Gamma_{n(2)-10n(1)}^*$ in Lemma 1, we construct a crank $\Gamma_{n(2)}$ of type $\{\varphi_j\}_{j=0}^{n(2)}$, where $\varphi_j = 0$ $(10n(1)+1 \le j \le n(2))$. Then we see that

$$1/\gamma(\Gamma_{n(2)}) \le 1/\gamma(\Gamma_{10n(1)}) + \mathrm{Const}(n(2) - 10n(1))^{1/2} / \prod_{j=1}^{10n(1)} (1+\varphi_j),$$

$$Bu(\Gamma_{n(2)}) \le C_1 \prod_{j=1}^{10n(1)} (1+\varphi_j)(n(2) - 10n(1))^{-9/10}.$$

Our sequence $\{\varphi_j\}_{j=0}^{10n(1)}$ is chosen so that

$$n(2) - 10n(1) = \left\{ \prod_{j=1}^{10n(1)} (1+\varphi_j) \right\}^{4/3}.$$

Hence

$$1/\gamma(\Gamma_{n(2)}) \le 1/\gamma(\Gamma_{10n(1)}) + \mathrm{Const}(n(2) - 10n(1))^{-1/4},$$
$$Bu(\Gamma_{n(2)}) \le C_1(n(2) - 10n(1))^{-3/20}.$$

Replacing each component of $\Gamma_{n(2)}$ by a suitable crank, we construct a crank $\Gamma_{10n(2)}$ of type $\{\varphi_j\}_{j=0}^{10n(2)}$. Replacing each component of $\Gamma_{10n(2)}$ by a crank similar to $\Gamma_{n(3)-10n(2)}^*$, we construct a crank $\Gamma_{n(3)}$ of type $\{\varphi_j\}_{j=0}^{n(3)}$, where $\varphi_j = 0$ $(10n(2)+1 \le j \le n(3))$. The sequence $\{\varphi_j\}_{j=n(2)+1}^{10n(2)}$ is chosen so that $|(n(3) - 10n(2)) - (\prod_{j=1}^{10n(2)}(1+\varphi_j))^{4/3}|$ is small. We see that

$$1/\gamma(\Gamma_{n(3)}) \le 1/\gamma(\Gamma_{10n(1)}) + \mathrm{Const}(n(2) - 10n(1))^{-1/4}$$
$$+ \mathrm{Const}(n(3) - 10n(2))^{-1/4} + (\text{negligible quantity}),$$
$$Bu(\Gamma_{n(3)}) \le C_1(n(3) - 10n(2))^{-3/20}.$$

Repeating this argument, we define a sequence $\{\Gamma_{n(k)}\}_{k=2}^{\infty}$ of cranks such that

$$\limsup_{k\to\infty} 1/\gamma(\Gamma_{n(k)}) < \infty, \qquad \lim_{k\to\infty} Bu(\Gamma_{n(k)}) = 0.$$

Then the analytic capacity of the limit crank is positive and its Buffon length is zero.

3. Lemmas.

LEMMA 2. *Let Γ_n be a crank of type $\{\varphi_j\}_{j=0}^{n}$, g_n be a non-negative function on Γ_n such that g_n is a constant on each component of Γ_n, and let $\{\varphi_j\}_{j=n+1}^{n+m}$ be non-negative numbers less than* 1. *Then there exist a crank Γ_{n+m} of type $\{\varphi_j\}_{j=0}^{n+m}$ and a non-negative function g_{n+m} on Γ_{n+m} such that*

(4) $\qquad g_{n+m}$ *is a constant on each component of* Γ_{n+m},

$$\|g_{n+m}\|_{L^1(\Gamma_{n+m})} = \|g_n\|_{L^1(\Gamma_n)}, \tag{5}$$

$$\|g_{n+m}\|_{L^\infty(\Gamma_{n+m})} \le \|g_n\|_{L^\infty(\Gamma_n)} \Big/ \prod_{\mu=n+1}^{n+m} (1+\varphi_\mu), \tag{6}$$

(7) $\|\operatorname{Re}\mathscr{H}_{\Gamma_{n+m}} g_{n+m}\|_{L^\infty(\Gamma_{n+m})}$

$$\le \|\operatorname{Re}\mathscr{H}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} + \|g_n\|_{L^\infty(\Gamma_n)} \sum_{j=n+1}^{n+m} \left\{ 1 \Big/ \prod_{\mu=n+1}^{j} (1+\varphi_\mu) \right\}.$$

We can write $\Gamma_n = \bigcup_{k=1}^{l_n} J_k^{(n)}$ with its components $\{J_k^{(n)}\}_{k=1}^{l_n}$. We put

$$\Gamma_{n+1} = \bigcup_{k=1}^{l_n} J_k^{(n)}(q_{n+1}, \varphi_{n+1}),$$

where q_{n+1} ($\ge q_0 = 100$) is determined later. Suppose that $\{\Gamma_\mu\}_{\mu=n+1}^{j}$ have been defined. We can write $\Gamma_j = \bigcup_{k=1}^{l_j} J_k^{(j)}$ with its components $\{J_k^{(j)}\}_{k=1}^{l_j}$. We put

$$\Gamma_{j+1} = \bigcup_{k=1}^{l_j} J_k^{(j)}(q_{j+1}, \varphi_{j+1}). \tag{8}$$

Thus $\{\Gamma_j\}_{j=n+1}^{n+m}$ are defined; $\{q_j\}_{j=n+1}^{n+m}$ are determined later. Let $n+1 \leq j \leq n+m$. We define a non-negative function g_j on Γ_j as follows. Each component $J_k^{(n)}$ of Γ_n generates $2^{q_{n+1}+\cdots+q_j}$ components of Γ_j. On these components, we put

$$g_j(z) = \left\{ \frac{1}{|J_k^{(n)}|} \int_{J_k^{(n)}} g_n(\zeta)|d\zeta| \right\} \Big/ \prod_{\mu=n+1}^{j} (1+\varphi_\mu).$$

Since the total length of these $2^{q_{n+1}+\cdots+q_j}$ components is

$$|J_k^{(n)}| \prod_{\mu=n+1}^{j} (1+\varphi_\mu),$$

the integration of g_j over these components is equal to $\int_{J_k^{(n)}} g_n(\zeta)|d\zeta|$. Hence $\|g_j\|_{L^1(\Gamma_j)} = \|g_n\|_{L^1(\Gamma_n)}$. Evidently, g_j is a constant on each component of Γ_j. We have

$$\|g_j\|_{L^\infty(\Gamma_j)} \leq \|g_n\|_{L^\infty(\Gamma_n)} \Big/ \prod_{\mu=n+1}^{j} (1+\varphi_\mu).$$

In particular, (4)–(6) hold. To prove (7), we estimate

$$\|\operatorname{Re} \mathscr{H}_{\Gamma_{j+1}} g_{j+1}\|_{L^\infty(\Gamma_{j+1})}.$$

Recall (8). We have

$$\begin{aligned} J_k^{(j)}(q_{j+1}, \varphi_{j+1}) = &\bigcup_{\mu=1}^{\sigma_{j+1}} [J_{k,2\mu}^{(j)}(\varphi_{j+1}) + i2^{-q_{j+1}}|J_k^{(j)}|] \\ &\cup \bigcup_{\mu=1}^{\sigma_{j+1}} J_{k,2\mu-1}^{(j)}(\varphi_{j+1}) \qquad (\sigma_{j+1} = 2^{q_{j+1}-1}, 1 \leq k \leq l_j), \end{aligned}$$

where $\{J_{k,\mu}^{(j)}\}_{\mu=1}^{2\sigma_{j+1}}$ are mutually non-overlapping segments on $J_k^{(j)}$ of length $2^{-q_{j+1}}|J_k^{(j)}|$; they are ordered from left to right. Let

$$z_0 \in \bigcup_{\mu=1}^{\sigma_{j+1}} [J_{k_0,2\mu}^{(j)}(\varphi_{j+1}) + i2^{-q_{j+1}}|J_{k_0}^{(j)}|]$$

and let z_0^* be the nearest point on $J_{k_0}^{(j)}$ to z_0. Then

$$\begin{aligned} L_1 &= \left| \operatorname{Re}\frac{1}{2\pi i}\text{p.v.} \int_{J_{k_0}^{(j)}(q_{j+1},\varphi_{j+1})} \frac{g_{j+1}(\zeta)}{\zeta - z_0}|d\zeta| \right. \\ &\qquad\qquad \left. - \operatorname{Re}\frac{1}{2\pi i}\text{p.v.} \int_{J_{k_0}^{(j)}} \frac{g_j(\zeta)}{\zeta - z_0^*}|d\zeta| \right| \\ &= \left| \operatorname{Re}\frac{1}{2\pi i}\sum_{\mu=1}^{\sigma_{j+1}}\text{p.v.} \int_{J_{k_0,2\mu-1}^{(j)}(\varphi_{j+1})} \frac{g_{j+1}(\zeta)}{\zeta - z_0}|d\zeta| \right| \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2^{-q_{j+1}}|J_{k_0}^{(j)}|}{(x-\operatorname{Re} z_0)^2 + (2^{-q_{j+1}}|J_{k_0}^{(j)}|)^2}\|g_{j+1}\|_{L^\infty(\Gamma_{j+1})}\,dx \\ &\le \|g_n\|_{L^\infty(\Gamma_n)} \Big/ \left\{ 2\prod_{\mu=n+1}^{j+1}(1+\varphi_\mu) \right\}. \end{aligned}$$

Let

$$\rho_j = \min_{1\le k\le l_j} \operatorname{dis}(J_k^{(j)}, \Gamma_j - J_k^{(j)}), \quad \tau(q_{j+1}) = 2^{-q_{j+1}} \max_{1\le k\le l_j} |J_k^{(j)}|,$$

where $\operatorname{dis}(\cdot,\cdot)$ is the distance. We choose, for a while, q_{j+1} ($\ge q_0$) so that $\tau(q_{j+1}) \le \rho_j/10$. Since

$$\begin{gathered} \int_{[J_{k,2\mu}^{(j)}(\varphi_{j+1})+i2^{-q_{j+1}}|J_k^{(j)}|]} g_{j+1}(\zeta)|d\zeta| = \int_{J_{k,2\mu}^{(j)}} g_j(\zeta)|d\zeta|, \\ \int_{J_{k,2\mu-1}^{(j)}(\varphi_{j+1})} g_{j+1}(\zeta)|d\zeta| = \int_{J_{k,2\mu-1}^{(j)}} g_j(\zeta)|d\zeta| \\ (1 \le k \le l_j, 1 \le \mu \le 2^{q_{j+1}-1} \; (= \sigma_{j+1})), \end{gathered}$$

we have

$$\begin{aligned}
L_2 = \Bigg| &\operatorname{Re}\frac{1}{2\pi i}\int_{\Gamma_{j+1}-J_{k_0}^{(j)}(q_{j+1},\varphi_{j+1})}\frac{g_{j+1}(\zeta)}{\zeta - z_0}|d\zeta| \\
&\qquad - \operatorname{Re}\frac{1}{2\pi i}\int_{\Gamma_j - J_{k_0}^{(j)}}\frac{g_j(\zeta)}{\zeta - z_0^*}|d\zeta|\Bigg| \\
\le \frac{1}{2\pi}\sum_{k\neq k_0}&\left\{\sum_{\mu=1}^{\sigma_{j+1}}\left|\int_{[J_{k,2\mu}^{(j)}(\varphi_{j+1})+i2^{-q_{j+1}}|J_k^{(j)}|]}\frac{g_{j+1}(\zeta)}{\zeta - z_0}|d\zeta|\right.\right. \\
&\qquad\left. - \int_{J_{k,2\mu}^{(j)}}\frac{g_j(\zeta)}{\zeta - z_0^*}|d\zeta|\right| \\
&\left. + \sum_{\mu=1}^{\sigma_{j+1}}\left|\int_{J_{k,2\mu-1}^{(j)}(\varphi_{j+1})}\frac{g_{j+1}(\zeta)}{\zeta - z_0}|d\zeta| - \int_{J_{k,2\mu-1}^{(j)}}\frac{g_j(\zeta)}{\zeta - z_0^*}|d\zeta|\right|\right\} \\
\le \operatorname{Const}\tau(q_{j+1})&\rho_j^{-2}\sum_{k\neq k_0}\sum_{\mu=1}^{2^{q_{j+1}}}\int_{J_{k,\mu}^{(j)}} g_j(\zeta)|d\zeta| \\
\le \operatorname{Const}\tau(q_{j+1})&\rho_j^{-2}\|g_j\|_{L^1(\Gamma_j)} \\
= \operatorname{Const}\tau(q_{j+1})&\rho_j^{-2}\|g_n\|_{L^1(\Gamma_n)}.
\end{aligned}$$

Thus

$$\begin{aligned}
(9)\quad |\operatorname{Re}\mathscr{H}_{\Gamma_{j+1}}g_{j+1}(z_0)| &\le |\operatorname{Re}\mathscr{H}_{\Gamma_j}g_j(z_0^*)| + L_1 + L_2 \\
&\le \|\operatorname{Re}\mathscr{H}_{\Gamma_j}g_j\|_{L^\infty(\Gamma_j)} + \|g_n\|_{L^\infty(\Gamma_n)}\Big/\left\{2\prod_{\mu=n+1}^{j+1}(1+\varphi_\mu)\right\} \\
&\quad + \operatorname{Const}\tau(q_{j+1})\rho_j^{-2}\|g_n\|_{L^1(\Gamma_n)}.
\end{aligned}$$

In the same manner, we have (9) for any point z_0 in

$$\bigcup_{\mu=1}^{\sigma_{j+1}} J_{k_0,2\mu-1}^{(j)}(\varphi_{j+1}).$$

Since k_0 $(1 \leq k_0 \leq l_j)$ is arbitrary, $\|\operatorname{Re}\mathscr{H}_{\Gamma_{j+1}} g_{j+1}\|_{L^\infty(\Gamma_{j+1})}$ is dominated by the summation of the last three quantities in (9). Consequently,

$$\begin{aligned}
(10)\ &\|\operatorname{Re}\mathscr{H}_{\Gamma_{n+m}} g_{n+m}\|_{L^\infty(\Gamma_{n+m})} \\
&\leq \|\operatorname{Re}\mathscr{H}_{\Gamma_{n+m-1}} g_{n+m-1}\|_{L^\infty(\Gamma_{n+m-1})} \\
&+ \|g_n\|_{L^\infty(\Gamma_n)} \Big/ \left\{ 2 \prod_{\mu=n+1}^{n+m} (1+\varphi_\mu) \right\} \\
&+ \operatorname{Const} \tau(q_{n+m}) \rho_{n+m-1}^{-2} \|g_n\|_{L^1(\Gamma_n)} \leq \cdots \leq \|\operatorname{Re}\mathscr{H}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} \\
&+ \|g_n\|_{L^\infty(\Gamma_n)} \sum_{j=n+1}^{n+m} 1 \Big/ \left\{ 2 \prod_{\mu=n+1}^{j} (1+\varphi_\mu) \right\} \\
&+ \operatorname{Const} \|g_n\|_{L^1(\Gamma_n)} \sum_{j=n+1}^{n+m} \tau(q_j) \rho_{j-1}^{-2}.
\end{aligned}$$

Since $\lim_{q\to\infty} \tau(q) = 0$, we can inductively define $\{q_j\}_{j=n+1}^{n+m}$ so that (7) holds. This completes the proof of Lemma 2.

LEMMA 3. *Let Γ_n be a crank of type $\{\varphi_j\}_{j=0}^n$, g_n be a non-negative function on Γ_n such that g_n is a constant on each component of Γ_n, and let m be a positive integer. Then there exist a crank Γ_{n+m} of type $\{\varphi_j\}_{j=0}^{n+m}$ with $\varphi_j = 0$ $(n+1 \leq j \leq n+m)$ and a non-negative function g_{n+m} on Γ_{n+m} such that*

(11) *g_{n+m} is a constant on each component of Γ_{n+m},*

$$(12)\qquad \|g_{n+m}\|_{L^1(\Gamma_{n+m})} = \|g_n\|_{L^1(\Gamma_n)},$$

$$(13)\qquad \|g_{n+m}\|_{L^\infty(\Gamma_{n+m})} \leq C_1 \|g_n\|_{L^\infty(\Gamma_n)},$$

$$\begin{aligned}(14)\qquad &\|\operatorname{Re}\mathscr{H}_{\Gamma_{n+m}} g_{n+m}\|_{L^\infty(\Gamma_{n+m})} \\ &\leq \|\operatorname{Re}\mathscr{H}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} + C_2\sqrt{m}\|g_n\|_{L^\infty(\Gamma_n)},\end{aligned}$$

$$(15)\qquad Bu(\Gamma_{n+m}) \leq C_1 |\Gamma_n| / m^{9/10},$$

where C_1 is the constant in Lemma 1 and C_2 is an absolute constant.

We can write $\Gamma_n = \bigcup_{k=1}^l J_k$ with its components $\{J_k\}_{k=1}^l$. Let z_k be the left endpoint of J_k $(1 \leq k \leq l)$. We put

$$\Gamma_{n+m} = \bigcup_{k=1}^{l} \Lambda_k, \quad \Lambda_k = [|J_k|\Gamma_m^* + z_k],$$

$$g_{n+m}(z) = g_m^*((z - z_k)/|J_k|)g_n(z_k) \qquad (z \in \Lambda_k,\ 1 \le k \le l),$$

where Γ_m^*, g_m^* are the crank and the function in Lemma 1, respectively. Then Γ_{n+m} is a crank of type $\{\varphi_j\}_{j=0}^{n+m}$. Evidently, (11) and (12) hold. Lemma 1 immediately yields (13) and (15). Let $z_0 \in \Lambda_{k_0}$ and let z_0^* be the projection of z_0 to J_{k_0}. Then Lemma 1 shows that

$$\begin{aligned}
&|\mathrm{Re}\,\mathscr{H}_{\Gamma_{n+m}} g_{n+m}(z_0) - \mathrm{Re}\,\mathscr{H}_{\Gamma_n} g_n(z_0^*)| \\
&\quad \le \left|\mathrm{Re}\frac{1}{2\pi i}\int_{\Lambda_{k_0}} \frac{g_{n+m}(\zeta)}{\zeta - z_0}|d\zeta| - \mathrm{Re}\frac{1}{2\pi i}\int_{J_{k_0}} \frac{g_n(\zeta)}{\zeta - z_0^*}|d\zeta|\right| + \frac{1}{2\pi}L^0 \\
&\quad = \left|\mathrm{Re}\frac{1}{2\pi i}\int_{\Lambda_{k_0}} \frac{g_{n+m}(\zeta)}{\zeta - z_0}|d\zeta|\right| + \frac{1}{2\pi}L^0 \\
&\quad = \left|\mathrm{Re}(\mathscr{H}_{\Gamma_m^*} g_m^*)\left(\frac{z_0 - z_k}{|J_k|}\right)\right| g_n(z_k) + \frac{1}{2\pi}L^0 \\
&\quad \le C_1\sqrt{m}\|g_n\|_{L^\infty(\Gamma_n)} + \frac{1}{2\pi}L^0,
\end{aligned}$$

where

$$L^0 = \sum_{k \ne k_0} \left|\int_{\Lambda_k} \frac{g_{n+m}(\zeta)}{\zeta - z_0}|d\zeta| - \int_{J_k} \frac{g_n(\zeta)}{\zeta - z_0^*}|d\zeta|\right|.$$

Let $\{\Gamma_j\}_{j=0}^n$ be cranks such that

$$\Gamma_0 [_{\varphi_1} \Gamma_1 [_{\varphi_2} \cdots [_{\varphi_n} \Gamma_n.$$

For $1 \le k \le l$, $0 \le j \le n$, $\gamma_k(j)$ denotes the component of Γ_j generating J_k. In particular, $\gamma_k(n) = J_k$ $(1 \le k \le l)$. We put

$$L_j^0 = \sum_{k \in \mathscr{F}_j} \left|\int_{\Lambda_k} \frac{g_{n+m}(\zeta)}{\zeta - z_0}|d\zeta| - \int_{J_k} \frac{g_n(\zeta)}{\zeta - z_0^*}|d\zeta|\right| \qquad (1 \le j \le n),$$

where

$$\mathscr{F}_j = \{1 \le k \le l; k \ne k_0, \gamma_k(j-1) = \gamma_{k_0}(j-1), \gamma_k(j) \ne \gamma_{k_0}(j)\}.$$

Then

$$L^0 = \sum_{j=1}^n L_j^0.$$

Since Γ_m^* is a crank of type $\mathbf{O}_{m+1}$, a geometric observation shows that, for any $z \in \Lambda_k$ $(1 \le k \le l)$,

$$\mathrm{dis}(z, J_k) \le 2|J_k|\{2^{-q_0} + 2^{-2q_0} + \cdots + 2^{-mq_0}\} \le \frac{1}{100}|J_k|.$$

Hence Λ_k is contained in the square $Q_k = \{z + is; z \in J_k, \ 0 \le s \le |J_k|/100\}$ $(1 \le k \le l)$. Since $|\gamma_k(n)| = |\gamma_{k_0}(n)|$ $(k \in \mathscr{F}_n)$, we have, for $k \in \mathscr{F}_n$,

$$\begin{aligned}\operatorname{dis}(Q_k, Q_{k_0}) &\ge \operatorname{dis}(\gamma_k(n), \gamma_{k_0}(n)) - \frac{1}{100}\{|\gamma_k(n)| + |\gamma_{k_0}(n)|\} \\ &= \operatorname{dis}(\gamma_k(n), \gamma_{k_0}(n)) - \frac{1}{50}|\gamma_{k_0}(n)|.\end{aligned}$$

For any $1 \le j \le n-1$, $z \in Q_k$,

$$\begin{aligned}\operatorname{dis}(z, \gamma_k(j)) &\le \sum_{\mu=j+1}^{n} \left\{\frac{|\gamma_k(\mu)|}{(1+\varphi_\mu)} + |\gamma_k(\mu)|\right\} + \frac{1}{100}|J_k| \\ &\le 2|\gamma_k(j)| \sum_{\mu=j+1}^{n} |\gamma_k(\mu)|/|\gamma_k(j)| + \frac{1}{100}|\gamma_k(j)| \\ &\le 2|\gamma_k(j)|\{2^{-q_0}(1+\varphi_{j+1}) + 2^{-2q_0}(1+\varphi_{j+1})(1+\varphi_{j+2}) \\ &\qquad + \cdots + 2^{-(n-j)q_0}(1+\varphi_{j+1})\cdots(1+\varphi_n)\} + \frac{1}{100}|\gamma_k(j)| \\ &\le 2|\gamma_k(j)|\{2^{-(q_0-1)} + 2^{-2(q_0-1)} + \cdots\} + \frac{1}{100}|\gamma_k(j)| \le \frac{1}{50}|\gamma_k(j)|.\end{aligned}$$

Since $|\gamma_k(j)| = |\gamma_{k_0}(j)|$ $(k \in \mathscr{F}_j)$, we have, for $k \in \mathscr{F}_j$, $1 \le j \le n-1$,

$$\begin{aligned}(16) \qquad \operatorname{dis}(Q_k, Q_{k_0}) &\ge \operatorname{dis}(\gamma_k(j), \gamma_{k_0}(j)) - \frac{1}{50}\{|\gamma_k(j)| + |\gamma_{k_0}(j)|\} \\ &= \operatorname{dis}(\gamma_k(j), \gamma_{k_0}(j)) - \frac{1}{25}|\gamma_{k_0}(j)|.\end{aligned}$$

Thus (16) holds for any $k \in \mathscr{F}_j$, $1 \le j \le n$. Let $1 \le j \le n$. Since

$$\int_{\Lambda_k} g_{n+m}(\zeta)|d\zeta| = \int_{J_k} g_n(\zeta)|d\zeta| \qquad (1 \le k \le l),$$

we have

$$\begin{aligned}(17) \qquad L_j^0 &= \sum_{k \in \mathscr{F}_j} \left| \int_{\Lambda_k} \left\{\frac{1}{\zeta - z_0} - \frac{1}{z_k - z_0^*}\right\} g_{n+m}(\zeta)|d\zeta| \right. \\ &\qquad \left. + \int_{J_k} \left\{\frac{1}{z_k - z_0^*} - \frac{1}{\zeta - z_0^*}\right\} g_n(\zeta)|d\zeta| \right| \\ &\le \mathrm{Const} \sum_{k \in \mathscr{F}_j} (|J_k| + |J_{k_0}|)\operatorname{dis}(Q_k, Q_{k_0})^{-2} \int_{J_k} g_n(\zeta)|d\zeta| \\ &\le \mathrm{Const}\|g_n\|_{L^\infty(\Gamma_n)} \sum_{k \in \mathscr{F}_j} (|J_k| + |J_{k_0}|)|J_k|\operatorname{dis}(Q_k, Q_{k_0})^{-2}.\end{aligned}$$

The segment $\gamma_{k_0}(j-1)$ generates 2^{q_j} components $\{\lambda_\nu\}_{\nu=1}^{2^{q_j}}$ of Γ_j of length $|\gamma_{k_0}(j)|$, where $q_j = \log\{(1+\varphi_j)|\gamma_{k_0}(j-1)|/|\gamma_{k_0}(j)|\}/\log 2$ ($\geq q_0$). We may assume that $\lambda_1 = \gamma_{k_0}(j)$. Let

$$\mathscr{F}_{j,\nu} = \{k \in \mathscr{F}_j; \lambda_\nu = \gamma_k(j)\} \qquad (2 \leq \nu \leq 2^{q_j}).$$

Then $\mathscr{F}_j = \bigcup_{\nu=2}^{2^{q_j}} \mathscr{F}_{j,\nu}$. We have, for $2 \leq \nu \leq 2^{q_j}$,

$$\begin{aligned}
&\sum_{k\in\mathscr{F}_{j,\nu}} (|J_k| + |J_{k_0}|)|J_k| \\
&\qquad \leq |\lambda_1| 2^{-q_0(n-j)} \prod_{j<\mu\leq n} (1+\varphi_\mu) \sum_{k\in\mathscr{F}_{j,\nu}} |J_k| \\
&\qquad = |\lambda_1|^2 2^{-q_0(n-j)} \left\{ \prod_{j<\mu\leq n} (1+\varphi_\mu) \right\}^2 \leq |\lambda_1|^2 2^{-(q_0-2)(n-j)},
\end{aligned}$$

where $\prod_{j<\mu\leq n}(1+\varphi_\mu)$ denotes 1 if $j = n$.

Hence a geometric observation and (16) show that the last quantity in (17) is dominated by

$$\begin{aligned}
&\mathrm{Const}\|g_n\|_{L^\infty(\Gamma_n)} \sum_{\nu=2}^{2^{q_j}} \sum_{k\in\mathscr{F}_{j,\nu}} (|J_k| + |J_{k_0}|)|J_k| \mathrm{dis}(Q_k, Q_{k_0})^{-2} \\
&\quad \leq \mathrm{Const}\|g_n\|_{L^\infty(\Gamma_n)} \sum_{\nu=2}^{2^{q_j}} \mathrm{dis}(\lambda_\nu, \lambda_1)^{-2} \sum_{k\in\mathscr{F}_{j,\nu}} (|J_k| + |J_{k_0}|)|J_k| \\
&\quad \leq \mathrm{Const}\|g_n\|_{L^\infty(\Gamma_n)} |\lambda_1|^2 2^{-(q_0-2)(n-j)} \sum_{\nu=2}^{2^{q_j}} \mathrm{dis}(\lambda_\nu, \lambda_1)^{-2} \\
&\quad \leq \mathrm{Const}\|g_n\|_{L^\infty(\Gamma_n)} |\lambda_1|^2 2^{-(q_0-2)(n-j)} \sum_{\mu=1}^{\infty} (|\lambda_1|\mu)^{-2} \\
&\quad \leq \mathrm{Const}\|g_n\|_{L^\infty(\Gamma_n)} 2^{-(q_0-2)(n-j)}.
\end{aligned}$$

Thus

$$\begin{aligned}
|\mathrm{Re}\,\mathscr{H}_{\Gamma_{n+m}} g_{n+m}(z_0)| &\leq |\mathrm{Re}\,\mathscr{H}_{\Gamma_n} g_n(z_0^*)| \\
&\qquad + C_1\sqrt{m}\|g_n\|_{L^\infty(\Gamma_n)} + \frac{1}{2\pi}\sum_{j=1}^{n} L_j^0 \\
&\leq \|\mathrm{Re}\,\mathscr{H}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} + C_1\sqrt{m}\|g_n\|_{L^\infty(\Gamma_n)} \\
&\qquad + \mathrm{Const}\|g_n\|_{L^\infty(\Gamma_n)} \sum_{j=1}^{n} 2^{-(q_0-2)(n-j)},
\end{aligned}$$

which shows that

$$|\mathrm{Re}\,\mathscr{H}_{\Gamma_{n+m}} g_{n+m}(z_0)| \leq \|\mathrm{Re}\,\mathscr{H}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} + C_2\sqrt{m}\|g_n\|_{L^\infty(\Gamma_n)}$$

for some absolute constant C_2. Since $z_0 \in \Gamma_{n+m}$ is arbitrary, this gives (14). This completes the proof of Lemma 3.

LEMMA 4. *Let Γ be a crank of type $\{\varphi_j\}_{j=0}^\infty$, and let $\{\Gamma_n\}_{n=0}^\infty$ be a sequence of cranks satisfying* (1)–(3). *If* $\liminf_{n\to\infty} Bu(\Gamma_n) = 0$, *then* $Bu(\Gamma) = 0$.

Let $\mathscr{P}^\theta$ $(-\pi/2 < \theta \leq \pi/2)$ denote the straight line defined by the equation $x\sin\theta - y\cos\theta = 0$. For a set $E \subset \mathbf{C}$, $\mathrm{proj}_\theta(E)$ denotes the projection of E to $\mathscr{P}^\theta$. We have

$$Bu(E) = \int_{-\pi/2}^{\pi/2} |\mathrm{proj}_\theta(E)|\, d\theta.$$

We can write $\Gamma_n = \bigcup_{k=1}^{l_n} J_k^{(n)}$ with its components $\{J_k^{(n)}\}_{k=1}^{l_n}$. In the same manner as in the proof of (14), we have

$$\Gamma \subset \bigcup_{k=1}^{l_n} \{z; \mathrm{dis}(z, J_k^{(n)}) \leq |J_k^{(n)}|\} \quad \left(= \bigcup_{k=1}^{l_n} R_k^{(n)}, \text{ say}\right).$$

Hence, for any $-\pi/2 < \theta \leq \pi/2$,

$$|\mathrm{proj}_\theta(\Gamma)| \leq \left|\mathrm{proj}_\theta\left(\bigcup_{k=1}^{l_n} R_k^{(n)}\right)\right|.$$

We can decompose $\{k; 1 \leq k \leq l_n\}$ into a finite number of mutually disjoint sets $\{\mathscr{G}_\mu^\theta\}_{\mu=1}^{\nu_\theta}$ so that $\mathrm{proj}_\theta(\bigcup_{k\in\mathscr{G}_\mu^\theta} J_k^{(n)})$ is connected. Then a geometric observation shows that

$$\begin{aligned}\left|\mathrm{proj}_\theta\left(\bigcup_{k\in\mathscr{G}_\mu^\theta} R_k^{(n)}\right)\right| &\leq \left|\mathrm{proj}_\theta\left(\bigcup_{k\in\mathscr{G}_\mu^\theta} J_k^{(n)}\right)\right| \\ &\qquad + \mathrm{Const}\left(\frac{\pi}{2} - |\theta|\right)^{-1} \max_{k\in\mathscr{G}_\mu^\theta} |\mathrm{proj}_\theta(J_k^{(n)})| \\ &\leq \mathrm{Const}\left(\frac{\pi}{2} - |\theta|\right)^{-1} \left|\mathrm{proj}_\theta\left(\bigcup_{k\in\mathscr{G}_\mu^\theta} J_k^{(n)}\right)\right| \qquad (1 \leq \mu \leq \nu_\theta),\end{aligned}$$

and hence

$$
\begin{aligned}
|\mathrm{proj}_\theta(\Gamma)| &\le \mathrm{Const}\left(\frac{\pi}{2}-|\theta|\right)^{-1}\sum_{\mu=1}^{\nu_\theta}\left|\mathrm{proj}_\theta\left(\bigcup_{k\in\mathscr{G}_\mu^\theta} J_k^{(n)}\right)\right| \\
&= \mathrm{Const}\left(\frac{\pi}{2}-|\theta|\right)^{-1}|\mathrm{proj}_\theta(\Gamma_n)|.
\end{aligned}
$$

We have, for any $0<\varepsilon<\pi/2$,

$$
\begin{aligned}
\int_{-(\pi/2)+\varepsilon}^{(\pi/2)-\varepsilon}|\mathrm{proj}_\theta(\Gamma)|\,d\theta &\le \mathrm{Const}\int_{-(\pi/2)+\varepsilon}^{(\pi/2)-\varepsilon}\left(\frac{\pi}{2}-|\theta|\right)^{-1}|\mathrm{proj}_\theta(\Gamma_n)|\,d\theta \\
&\le \mathrm{Const}\,\varepsilon^{-1}Bu(\Gamma_n).
\end{aligned}
$$

Since $\liminf_{n\to\infty} Bu(\Gamma_n)=0$, this shows that the first quantity equals zero. Since $0<\varepsilon<\pi/2$ is arbitrary, $Bu(\Gamma)=0$. This completes the proof of Lemma 4.

4. **Construction of E_0.** Let p_n be the integral part of $(3/2)^{4n/3}$ $(n\ge 1)$. We define a sequence $\{n(k)\}_{k=1}^\infty$ of positive integers by $n(1)=10$,

$$n(k+1)=10n(k)+p_{10n(k)} \qquad (k\ge 1).$$

We define a sequence $\{\varphi_j\}_{j=0}^\infty$ of non-negative numbers by $\varphi_0=0$,

$$
\begin{aligned}
\varphi_j &= \tfrac{1}{2} && (1\le j\le n(1)),\\
\varphi_j &= \tfrac{1}{2} && (n(k)<j\le 10n(k),\ k\ge 1),\\
\varphi_j &= 0 && (10n(k)<j\le n(k+1),\ k\ge 1).
\end{aligned}
$$

We use Lemma 2 with Γ_0, $g_0=1$ and $\{\varphi_j\}_{j=0}^{10n(1)}$. There exist a crank $\Gamma_{10n(1)}$ of type $\{\varphi_j\}_{j=0}^{10n(1)}$ and a non-negative function $g_{10n(1)}$ on $\Gamma_{10n(1)}$ such that $g_{10n(1)}$ is a constant on each component of $\Gamma_{10n(1)}$,

$$\|g_{10n(1)}\|_{L^1(\Gamma_{10n(1)})}=1, \qquad \|g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})}\le 1\Big/\prod_{\mu=1}^{10n(1)}(1+\varphi_\mu),$$

$$
\begin{aligned}
&\|\mathrm{Re}\,\mathscr{H}_{\Gamma_{10n(1)}}g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})}\\
&\quad\le \|\mathrm{Re}\,\mathscr{H}_{\Gamma_0}g_0\|_{L^\infty(\Gamma_0)}+\sum_{j=1}^{10n(1)}1\Big/\prod_{\mu=1}^{j}(1+\varphi_\mu)\\
&\quad=\left\{\sum_{j=1}^{n(1)}1\Big/\prod_{\mu=1}^{j}(1+\varphi_\mu)\right\}+\sum_{j=n(1)+1}^{10n(1)}1\Big/\prod_{\mu=1}^{j}(1+\varphi_\mu).
\end{aligned}
$$

Using Lemma 3 with $n = 10n(1), m = p_{10n(1)}$, we obtain a crank $\Gamma_{n(2)}$ of type $\{\varphi_j\}_{j=0}^{n(2)}$ and a non-negative function $g_{n(2)}$ on $\Gamma_{n(2)}$ such that $g_{n(2)}$ is a constant on each component of $\Gamma_{n(2)}$,

$$\|g_{n(2)}\|_{L^1(\Gamma_{n(2)})} = \|g_{10n(1)}\|_{L^1(\Gamma_{10n(1)})} = 1,$$

$$\|g_{n(2)}\|_{L^\infty(\Gamma_{n(2)})} \le C_0\|g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})} \le C_0 \Big/ \prod_{\mu=1}^{10n(1)} (1+\varphi_\mu),$$

$$\begin{aligned}
&\|\operatorname{Re}\mathscr{H}_{\Gamma_{n(2)}} g_{n(2)}\|_{L^\infty(\Gamma_{n(2)})} \\
&\quad \le \|\operatorname{Re}\mathscr{H}_{\Gamma_{10n(1)}} g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})} + C_0\sqrt{p_{10n(1)}}\|g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})} \\
&\quad \le \left\{\sum_{j=1}^{n(1)} 1 \Big/ \prod_{\mu=1}^{j}(1+\varphi_\mu)\right\} + \sum_{j=n(1)+1}^{10n(1)} 1 \Big/ \prod_{\mu=1}^{j}(1+\varphi_\mu) \\
&\qquad + C_0\sqrt{p_{10n(1)}} \Big/ \prod_{\mu=1}^{10n(1)} (1+\varphi_\mu),
\end{aligned}$$

$$Bu(\Gamma_{n(2)}) \le C_0|\Gamma_{10n(1)}|/p_{10n(1)}^{9/10} = C_0 \prod_{\mu=1}^{10n(1)} (1+\varphi_\mu)/p_{10n(1)}^{9/10},$$

where $C_0 = \max\{C_1, C_2\}$. Using Lemma 2 with $n = n(2), m = 9n(2)$, we obtain a crank $\Gamma_{10n(2)}$ and a non-negative function $g_{10n(2)}$. Using Lemma 3 with $n = 10n(1), m = p_{10n(2)}$, we obtain a crank $\Gamma_{n(3)}$ and a non-negative function $g_{n(3)}$. Repeating this argument, we obtain a crank $\Gamma_{n(k)}$ $(k \ge 2)$ of type $\{\varphi_j\}_{j=0}^{n(k)}$ and a non-negative function $g_{n(k)}$ on $\Gamma_{n(k)}$ such that $g_{n(k)}$ is a constant on each component of $\Gamma_{n(k)}$,

$$\|g_{n(k)}\|_{L^1(\Gamma_{n(k)})} = 1,$$

$$\|g_{n(k)}\|_{L^\infty(\Gamma_{n(k)})} \le C_0^{k-1} \Big/ \prod_{\mu=1}^{10n(k-1)} (1+\varphi_\mu),$$

$$\begin{aligned}
&\|\operatorname{Re}\mathscr{H}_{\Gamma_{n(k)}} g_{n(k)}\|_{L^\infty(\Gamma_{n(k)})} \\
&\quad \le \left\{\sum_{j=1}^{n(1)} 1 \Big/ \prod_{\mu=1}^{j}(1+\varphi_\mu)\right\} + \sum_{\nu=1}^{k-1} \sum_{j=n(\nu)+1}^{10n(\nu)} \left\{C_0^{\nu-1} \Big/ \prod_{\mu=1}^{j}(1+\varphi_\mu)\right\} \\
&\qquad + \sum_{\nu=1}^{k-1} \left\{C_0^{\nu}\sqrt{p_{10n(\nu)}} \Big/ \prod_{\mu=1}^{10n(\nu)} (1+\varphi_\mu)\right\},
\end{aligned}$$

$$Bu(\Gamma_{n(k)}) \le C_0 \prod_{\mu=1}^{10n(k-1)} (1+\varphi_\mu)/p_{10n(k-1)}^{9/10}.$$

Let $\Gamma = \bigcap_{j=1}^{\infty} \overline{\bigcup_{k=2}^{\infty} \Gamma_{n(k)}}$. Then Γ is a crank of type $\{\varphi_j\}_{j=0}^{\infty}$. We have

$$\begin{aligned} Bu(\Gamma_{n(k)}) &\le C_0 \prod_{\mu=1}^{10n(k-1)} (1+\varphi_\mu) p_{10n(k-1)}^{-9/10} \\ &\le \text{Const} \left(\frac{3}{2}\right)^{10n(k-1)} \left(\frac{3}{2}\right)^{-(4/3)(9/10)10n(k-1)} \\ &= \text{Const} \left(\frac{3}{2}\right)^{-2n(k-1)}, \end{aligned}$$

which shows that $\lim_{k\to\infty} Bu(\Gamma_{n(k)}) = 0$. Hence Lemma 4 gives that $Bu(\Gamma) = 0$.

We now show that $\gamma(\Gamma) > 0$. Let $k \ge 1$. Then

$$\int_{\Gamma_{n(k)}} g_{n(k)}(\zeta)|d\zeta| = 1.$$

Since $n(\nu) \ge 10n(\nu-1)$ $(\nu \ge 2), n(1) = 10$, we have $n(\nu) \ge 10^\nu$ $(\nu \ge 1)$, and hence

$$\|g_{n(k)}\|_{L^\infty(\Gamma_{n(k)})} \le C_0^{k-1} \left(\frac{3}{2}\right)^{-9n(k-1)} \le \text{Const}.$$

Since

$$\begin{aligned} \sqrt{p_{10n(\nu)}} \left\{ \prod_{\mu=1}^{10n(\nu)} (1+\varphi_\mu) \right\}^{-1} &\le \sqrt{p_{10n(\nu)}} \left(\frac{3}{2}\right)^{-9n(\nu)} \\ &\le \text{Const} \left(\frac{3}{2}\right)^{(4/3)(1/2)10n(\nu)} \left(\frac{3}{2}\right)^{-9n(\nu)} \\ &= \text{Const} \left(\frac{3}{2}\right)^{-(7/3)n(\nu)} \qquad (\nu \ge 1), \end{aligned}$$

we have

$$\|\operatorname{Re} \mathscr{H}_{\Gamma_{n(k)}} g_{n(k)}\|_{L^\infty(\Gamma_{n(k)})} \le \text{Const}.$$

Hence we can define a non-negative function h_k on $\Gamma_{n(k)}$ so that

$$\begin{aligned} &\int_{\Gamma_{n(k)}} h_k(\zeta)|d\zeta| = \eta_0, \quad \|h_k\|_{L^\infty(\Gamma_{n(k)})} \le 1/2, \\ &\|\operatorname{Re} \mathscr{H}_{\Gamma_{n(k)}} h_k\|_{L^\infty(\Gamma_{n(k)})} \le 1/2, \\ &h_k(\zeta) = 0 \quad \text{at endpoints of each component of } \Gamma_{n(k)}, \\ &h_k \text{ is differentiable along } \Gamma_{n(k)}, \end{aligned}$$

where η_0 is an absolute constant. Let

$$\hat{h}_k(z) = \frac{1}{2\pi i}\int_{\Gamma_{n(k)}} \frac{h_k(\zeta)}{\zeta - z}|d\zeta|,$$

$$u_k(z) = \operatorname{Re}\hat{h}_k(z), \quad v_k(z) = (\text{the imaginary part of } \hat{h}_k(z)),$$
$$f_k(z) = \{1 - \exp(i\hat{h}_k(z))\}/\{1 + \exp(i\hat{h}_k(z))\} \qquad (z \notin \Gamma_{n(k)})$$

(cf. [**1**, p. 30]). We see easily that f_k is analytic outside $\Gamma_{n(k)}$ and

$$f_k'(\infty) = \frac{1}{4\pi}\int_{\Gamma_{n(k)}} h_k(\zeta)|d\zeta| = \eta_0/4\pi.$$

The non-tangential limit of $|u_k(z)|$ to each point on $\Gamma_{n(k)}$ is dominated by

$$\|h_k\|_{L^\infty(\Gamma_{n(k)})} + \|\operatorname{Re}\mathscr{H}_{\Gamma_{n(k)}} h_k\|_{L^\infty(\Gamma_{n(k)})} \leq 1.$$

Since $|u_k|$ is sub-harmonic in $\Gamma^c_{n(k)}$ and continuous in $\mathbf{C} \cup \{\infty\}$, we have $\sup_{z \in \Gamma^c_{n(k)}} |u_k(z)| \leq 1$. Hence, for any $z \notin \Gamma_{n(k)}$,

$$|f_k(z)|^2 = \frac{1 + \exp(-2v_k(z)) - 2\exp(-v_k(z))\cos(u_k(z))}{1 + \exp(-2v_k(z)) + 2\exp(-v_k(z))\cos(u_k(z))} \leq 1,$$

which shows that $\|f_k\|_{H^\infty(\Gamma^c_{n(k)})} \leq 1$. Since $k \geq 1$ is arbitrary, using an argument of normal families, we obtain $f \in H^\infty(\Gamma^c)$ satisfying $f'(\infty) = \eta_0/4\pi$, $\|f\|_{H^\infty(\Gamma^c)} \leq 1$. This shows that $\gamma(\Gamma) \geq \eta_0/4\pi$. Normalizing Γ, we obtain the required set E_0.

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