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In this paper we determine the set of all second homology classes in $C\mathbf{P}^2 \# \overline{C\mathbf{P}}^2$ which can be represented by smoothly embedded two-spheres in $C\mathbf{P}^2 \# \overline{C\mathbf{P}}^2$.

We say a class $u \in H_2(M^4, \mathbf{Z})$ can be represented by S^2 if it can be represented by a smoothly embedded 2-sphere in M^4. The purpose of this note is to prove the following.

THEOREM. *Let η, ξ be canonical generators of $H_2(C\mathbf{P}^2 \# \overline{C\mathbf{P}}^2, \mathbf{Z})$. Then $\gamma = a\eta + b\zeta$, $a, b \in \mathbf{Z}$, can be represented by S^2 if and only if a, b satisfy one of the following conditions.*
(i) $||a| - |b|| \leq 1$, *or*
(ii) $(a, b) = (\pm 2, 0)$ *or* $(0, \pm 2)$.

REMARK 1. The "if" part of the theorem is known (see Wall [7], Mandelbaum [5, the proof of Theorem 6.6]).

REMARK 2. If $p \in \mathbf{Z}$, then $p\eta$ (or $p\xi$) is represented by S^2 if and only if $|p| \leq 2$ (see Rohlin [6]).

REMARK 3. If a, b are relatively prime integers, then $\gamma = a\eta + b\xi$ is realized by a topologically embedded locally flat 2-sphere by Freedman [2]. Hence smoothness condition in the theorem is essential.

By Remarks 1 and 2, the Theorem follows from the following.

PROPOSITION. *Let a and b be two integers satisfying*

$$(*) \qquad \begin{cases} \text{(i)} & ab \neq 0, \text{ and} \\ \text{(ii)} & ||a| - |b|| \geq 2. \end{cases}$$

Then $a\eta + b\xi$ is not represented by S^2.

Proof. Suppose conversely that $a\eta + b\xi$ is represented by S^2. By reversing orientation if necessary, we may assume $n = b^2 - a^2 > 0$. Let $M^4 = C\mathbf{P}^2 \# \overline{C\mathbf{P}}^2 \# (n-1)C\mathbf{P}^2$ with ξ_i's the generators of

$H_2(M^4, \mathbf{Z})$ with respect to the additional $C\mathbf{P}^2$'s. Then the homology class $\gamma = a\eta + b\xi + \sum_{i=1}^{n-1} \xi_i$ can be represented by a smoothly embedded 2-sphere S in M^4. The self-intersection number of S is $S \cdot S = a^2 - b^2 + n - 1 = -1$. Hence the tubular neighborhood N of S in M^4 is the (-1)-Hopf bundle over S and ∂N is diffeomorphic to S^3. Set $W^4 = (M^4 - \mathring{N}) U_\partial D^4$. It is known that W^4 is a closed, simply connected smooth 4-manifold with a positive definite intersection form (see Kuga [**4**, claim 1]). By Donaldson's result (see Donaldson [**1**]), the intersection form of W^4 is standard. On the other hand, $M^4 = W^4 \# \hat{N}^4$ where $\hat{N}^4 = N^4 U_\partial D^4$. So, $(H_2(W^4, \mathbf{Z}), \langle\ ,\ \rangle_{W^4})$ is isomorphic to $(\gamma^\perp, \langle\ ,\ \rangle_{M^4})$. Hence there exist exactly $2n$ $\alpha \in H_2(M^4, \mathbf{Z})$ such that $\alpha \cdot \gamma = 0$ and $\alpha \cdot \alpha = 1$. Writing out the conditions in terms of the base $(\eta, \xi, \xi_1, \xi_2, \ldots, \xi_{n-1})$ by letting $\alpha = x\eta + y\xi + \sum_{i=1}^{n-1} z_i \xi_i$, we obtain $2n$ (≥ 16) solutions of the system of Diophantine equations

$$\begin{cases} ax - by + \sum_{i=1}^{n-1} z_i = 0, & (1) \\ x^2 - y^2 + \sum_{i=1}^{n-1} z_i^2 = 1. & (2) \end{cases}$$

Claim. If a, b satisfy $(*)$, the above equations have at most four solutions.

Proof. We have $y^2 - x^2 = \sum_{i=1}^{n-1} z_i^2 - 1 \geq -1$. If $y^2 - x^2 = -1$, then $y = 0$, $x = \pm 1$, and $z_i = 0$ for all i. By (1), this implies $a = 0$; if $y^2 - x^2 = 0$, then only one of z_i's is ± 1, all others are zero. By (1), this implies that $||a| - |b|| \leq 1$; If $y^2 - x^2 = 1$, then $y = \pm 1$, $x = 0$, and only two of z_i's are ± 1, all others are zero. So (1) implies $|b| \leq 2$, but $|a| \leq |b|$ by assumption. Therefore, in all cases, a, b fail to satisfy $(*)$. Hence we have $y^2 - x^2 \geq 3$.

Assume n' of the z_i's are nonzero, say z_{i_j}, $j = 1, 2, \ldots, n'$. Then we have

$$\begin{aligned} (3)\quad (ax - by)^2 &= \left(\sum_{j=1}^{n'} z_{i_j}\right)^2 \leq n' \cdot \left(\sum_{j=1}^{n'} z_{i_j}^2\right) \\ &= n'(1 + y^2 - x^2) = n' + n'(y^2 - x^2) \\ (4)\quad &\leq n' + (n-1)(y^2 - x^2) = n' + (b^2 - a^2 - 1)(y^2 - x^2) \\ &= n' + b^2y^2 - b^2x^2 + a^2x^2 - a^2y^2 - (y^2 - x^2) \\ &= n' + a^2x^2 + b^2y^2 - b^2x^2 - a^2y^2 - \sum_{j=1}^{n'} z_{i_j}^2 + 1, \end{aligned}$$

where (3) follows from Cauchy-Schwarz inequality.

Expanding and re-arranging this implies

$$(5)\qquad (bx - ay)^2 \leq \left(n' - \sum_{j=1}^{n'} z_{i_j}^2 \right) + 1.$$

Since each $z_{i_j} \neq 0$, (5) implies all these z_{i_j}'s are ± 1, and $(bx-ay)^2 \leq 1$.

There are now only two cases that might happen.

Case 1. $bx - ay = \pm 1$.

Then equalities in (3) and (4) hold. So $z_1 = \cdots = z_{n-1} = \pm 1$, and (1), (2) reduce to

$$(6)\qquad \begin{aligned} ax - by &= \pm(n-1), \\ x^2 - y^2 + (n-1) &= 1. \end{aligned}$$

The equation (6) and $bx - ay = \pm 1$ give at most four solutions to the Diophantine equations (1), (2) according to the choice of plus or minus signs.

Case 2. $bx - ay = 0$.

Then the equality in (3) must hold because if inequality holds, the left hand side of (3) will reduce at least -4 which contradicts (5) where the right hand side exceeds the left hand side by $+1$. By the same argument, the equality in (4) must hold since we have shown that $y^2 - x^2 \geq 3$. Therefore, the equality in (5) holds which is again a contradiction. Hence this case gives no solution.

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After submitting the note, the author learned that similar results were also obtained by T. Lawson.

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