**THE ADAMS SPECTRAL SEQUENCE OF THE REAL
PROJECTIVE SPACES**

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In this paper we study the mod 2 Adams spectral sequence for the infinite real projective space $P = RP^\infty$.

We recall ([1]) that the spectral sequence starts with

$$E_2^{s,t} = \text{Ext}_A^{s,t}(\tilde{H}^*(P), \mathbf{Z}/2)$$

and converges to the stable homotopy $_2\pi_*^s(P) = \pi_*^s(P)$ where A denotes the mod 2 Steenrod algebra and $\tilde{H}^*(P)$ is the reduced mod 2 cohomology of P. We simply write $\text{Ext}_A^{s,t}(P)$ for $\text{Ext}_A^{s,t}(\tilde{H}^*(P), \mathbf{Z}/2)$ and occasionally we abbreviate by ASS for "Adams spectral sequence".

Roughly, our main results consist of (1) a complete description of $\text{Ext}_A^{s,*}(P)$ for $0 \le s \le 2$ and also for $s = 3$ modulo indecomposable elements, (2) the determination of which classes in a substantial portion of $\text{Ext}_A^{1,*}(P)$ can detect homotopy elements in $\pi_*^s(P)$ (Adams's Hopf invariant Theorem solves the problem for $\text{Ext}_A^{0,*}(P)$) and (3) the construction of some infinite families in $\pi_*^s(P)$ at low Adams filtrations analogous to the ones in the 2-adic stable homotopy of spheres $_2\pi_*^s$ constructed in [9], [12] and [18], [22].

These Ext calculations were necessary in the work on the Kervaire invariant in [12]. The results are not surprising, but proving them is surprisingly difficult. In particular we make use of a calculational method that may be of independent interest.

To precisely state the results we first recall that the cohomology $\text{Ext}_A^{*,*} = \text{Ext}_A^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$ of the Steenrod algebra A is a commutative bigraded algebra over $\mathbf{Z}/2$ and that $\text{Ext}_A^{*,*}$ for $0 \le s \le 3$ is generated by $h_i \in \text{Ext}_A^{1,2^i}$ ($i \ge 0$) and $c_i \in \text{Ext}_A^{3,2^{i+3}+2^{i+1}+2^i}$ with relations $h_i h_{i+1} = 0, h_{i+1}^3 = h_i^2 h_{i+2}$ and $h_i h_{i+2}^2 = 0$ where h_i corresponds to the Steenrod square $\text{Sq}^{2^i} \in A$. The mod 2 cohomology $H^*(P)$ is a polynomial algebra $\mathbf{Z}/2[x]$ in one variable x with $\deg x = 1$ on which A acts by $\text{Sq}^k x^i = \binom{i}{k} x^{i+k}$. One easily proves that $\{x^{2^i-1} | i \ge 1\}$ is a minimal set of generators of $\tilde{H}^*(P)$ over A. The non-zero class in $\text{Ext}_A^{0,2^i-1}(P) = \mathbf{Z}/2$ corresponding to x^{2^i-1} is denoted by $\hat{h}_i$. The first part of the

following theorem is therefore clear but for completeness and for our computational purpose later we include it in the statement.

THEOREM 1.1. (1) $\{\hat{h}_i | i \geq 1\}$ *is a* $\mathbf{Z}/2$-*base for* $\mathrm{Ext}_A^{0,*}(P)$.
(2) $\mathrm{Ext}_A^{1,*}(P)$ *is generated by* $\hat{h}_i h_j$ *with relations* $\hat{h}_i h_{i-1} = 0$ $(i \geq 1)$.
(3) $\mathrm{Ext}_A^{2,*}(P)$ *is generated by* $\hat{h}_i h_j h_k$ *and* $\hat{c}_i \in \mathrm{Ext}_A^{2,2^{i+3}+2^{i+1}+2^i-1}(P)$ $(i \geq 0)$ *with the relations coming from* $\mathrm{Ext}_A^{*,*}$, *the relations implied by* (2) *and the relations*

$$\hat{h}_{i+2} h_i^2 = \hat{h}_{i+1} h_{i+1}^2 \quad (i \geq 0),$$
$$\hat{h}_{i+2} h_{i+2} h_i = 0 \quad (i \geq 0).$$

(4) *The products* $\hat{h}_i h_j h_k h_l$, $\hat{h}_i c_j$, $\hat{c}_i h_j$ *in* $\mathrm{Ext}_A^{3,*}(P)$ *are subject only to the relations from* $\mathrm{Ext}_A^{*,*}$, *the relations implied by* (2) *and* (3) *together with the following relations:*

$$\hat{h}_{i+3} h_i^2 h_{i+3} = 0, \qquad \hat{h}_{i+2} h_i h_{i+3}^2 = 0,$$
$$\hat{h}_i c_j = 0 \quad \text{for } i = j+2 \text{ and } i = j+3,$$
$$\hat{c}_i h_j = 0 \quad \text{for } i = j-3, j-2, j \text{ and } j+1,$$
$$\hat{c}_i h_{i+1} = \hat{h}_{i+1} c_i \neq 0.$$

THEOREM 1.2. *In the Adams spectral sequence for* P *there are the following differentials.*
(1) $d_2(\hat{h}_i h_j) = \hat{h}_i h_0 h_{j-1}^2 \neq 0$ *for* $i = 2, 3$ *and* $j \geq 5$.
(2) $d_2(\hat{h}_i h_j) = \hat{h}_{i-1} h_0 h_j h_{i-1} \neq 0$ *for* $i \geq 5, j = 0; i \geq 6, j = 2$ *and* $i \geq 7, j = 3$.
(3) $d_2(\hat{h}_i h_j) = \hat{h}_{i-1} h_0 h_{i-1} h_j + \hat{h}_i h_0 h_{j-1}^2 \neq 0$ *for* $i, j \geq 4$ *with* $i \neq j, j+1$.

THEOREM 1.3. *The following families in* $\mathrm{Ext}_A^{*,*}(P)$ *survive the Adams spectral sequence and so detect homotopy elements in* $\pi_*^s(P)$.
(1) $\hat{h}_1 h_i$ $(i \geq 1)$ (2) $\hat{h}_i h_1$ $(i \geq 3)$ (3) $\hat{h}_2 h_i^2$ $(i \neq 1)$ (4) $\hat{h}_i h_i h_2$ $(i \geq 5)$ (5) $\hat{h}_i c_1$ $(i \neq 3, 4)$.

We make some remarks on these results.

Theorem 1.2 is a consequence of Theorem 1.1 and Adams's Hopf invariant Theorem. Indeed, in the mod 2 ASS for $_2\pi_*^s$ in which $\mathrm{Ext}_A^{*,*}$ is the E_2 term there are differentials $d_2(h_i) = h_0 h_{i-1}^2 \neq 0$ for $i \geq 4$ ([2]) while h_i, for $0 \leq i \leq 3$, are well known to detect homotopy elements. Let $\varphi : P \to S^0$ be the standard transfer map from the

suspension spectrum of P to the sphere spectrum S^0 ([15], [19]). The Kahn-Priddy Theorem ([15]) says that the induced map $\varphi_{\sharp}: \pi_k^s(P) \to {}_2\pi_k^s(S^0) = {}_2\pi_k^s$ is epimorphism for $k > 0$. φ also induces a map

$$\varphi_*: \operatorname{Ext}_A^{s,t}(P) \to \operatorname{Ext}_A^{s+1,t+1} = \operatorname{Ext}_A^{s+1,t+1}(\tilde{H}^*(S^0) = \mathbf{Z}/2, \mathbf{Z}/2)$$

in E_2 terms of the ASS's such that $\varphi_*(\hat{h}_i) = h_i$ and $\varphi_*(\hat{c}_i) = c_i$ ([17], [19]). Since φ_* commutes with d_2 and $\operatorname{Ext}_A^{2,2^i}(P) = \mathbf{Z}/2$ is generated by $\hat{h}_{i-1}h_0h_{i-1}$ for $i \geq 4$ it follows that $d_2(\hat{h}_i) = \hat{h}_{i-1}h_0h_{i-1} \neq 0$ for $i \geq 4$ from which Theorem 1.2 follows.

The non-zero classes $\hat{h}_ih_j$ in $\operatorname{Ext}_A^{1,*}(P)$ not listed in (1.2) or (1.3) are classified in the following families:

(i) $\hat{h}_ih_j$, $i, j \leq 3$. (ii) $\hat{h}_2h_4$, $\hat{h}_3h_4$, $\hat{h}_4h_2$.

(iii) $\hat{h}_4h_4$, $\hat{h}_5h_5$. (iv) $\hat{h}_4h_0$, $\hat{h}_5h_2$, $\hat{h}_5h_3$.

(v) $\hat{h}_6h_3$. (vi) $\hat{h}_ih_i$, $i \geq 6$.

The first five of these are known to detect or not to detect homotopy elements in $\pi_*^s(P)$: (i), (ii) and (iii) do while (iv) and (v) do not. Indeed, (i) is trivial and is well known, (ii) is proved in [19] and (iii) is a consequence of Kahn-Priddy Theorem and the results in [20], [4] where h_4^2, h_5^2 are proved to survive the ASS for S^0. For (iv), it is proved in [19] that, in the ASS for P, $d_3(\hat{h}_4h_0) \neq 0$ and $d_4(\hat{h}_5h_2) \neq 0$ and $d_4(\hat{h}_5h_3) \neq 0$. The fact that (v) does not detect a homotopy class was proved by Mitchell (see [28]). For (vi) we refer to [3] for a discussion on this particular family.

One should compare Theorem 1.2 with the result in [21] where a similar situation in the spheres was analyzed.

Finally we remark on Theorem 1.3. In [9], [12], [18] and [22] the families $\{h_ih_1|i \geq 3\}$, $\{h_i^2h_2|i \geq 5\}$, $\{h_ic_1|i \geq 5\}$ in $\operatorname{Ext}_A^{*,*}$ are shown to detect homotopy elements in ${}_2\pi_*^s$. From Theorem 1.1 we see, under the map $\varphi_* : \operatorname{Ext}_A^{*,*}(P) \to \operatorname{Ext}_A^{*+1,*+1}$, there correspond two families in $\operatorname{Ext}_A^{*,*}(P)$ to each of the families $\{h_ih_1\}$, $\{h_i^2h_2\}$ and $\{h_ic_1\}$. Indeed, $\varphi_*(\hat{h}_ih_1) = \varphi_*(\hat{h}_1h_i) = h_ih_1$, $\varphi_*(\hat{h}_2h_i^2) = \varphi_*(\hat{h}_ih_2h_i) = h_i^2h_2$, $\varphi_*(\hat{c}_1h_i) = \varphi_*(\hat{h}_ic_1) = h_ic_1$. By Kahn-Priddy Theorem, homotopy elements in ${}_2\pi_*^s$ detected by h_ih_1 (resp. $h_i^2h_2, h_ic_1$) lift to elements in $\pi_*^s(P)$ which must be detected by $\hat{h}_ih_1$ or $\hat{h}_1h_i$ (resp. $\hat{h}_2h_i^2$ or $\hat{h}_ih_ih_2$, $\hat{h}_ic_1$ or $\hat{c}_1h_i$), and in general one does not know which of these two possibilities is the correct one. The result (1.3) shows that for the families $\{h_ih_1\}$ and $\{h_i^2h_2\}$ both choices are possible and for the family $\{h_ic_1\}$ one has a specific choice. If the strong Kervaire invariant conjecture (see

[3]) can be shown true then the family $\{\hat{c}_1 h_i\}$ can be easily shown to detect homotopy elements also. Otherwise we have no answer thus far about this family.

The family $\{\hat{h}_1 h_i\}$ in Theorem 1.3 was already proved in [12] to detect homotopy elements. It is included here for comparison with $\{\hat{h}_i h_1\}$ and also serves as a preparatory statement for going on proving (1.3)(3) as will be seen in §3.

This concludes our remarks.

Our work is therefore to prove Theorem 1.1 and Theorem 1.3. Theorem 1.1 is proved in §2 and Theorem 1.3 is proved in §3. The proof of Theorem 1.3 is shorter and one may proceed directly to §3 for the proof assuming the results in (1.1).

The proof of Theorem 1.1 is rather long. It is purely algebraic of course and is based on a lot of calculations. As mentioned earlier, a technical method, which later on is referred to as a "trick", is introduced to make these calculations tractable. In the beginning of the proof we give full details of our calculations. As the calculations progress we will begin omitting many of the details, as by then the reader should have no difficulty in using the "trick" to fill them in.

2. Proof of Theorem 1.1. Theorem 1.1 is proved by calculating with the lambda algebra Λ of Kan et al. ([6]).

We begin by recalling that Λ is a bigraded differential algebra over $\mathbb{Z}/2$ generated by $\lambda_i \in \Lambda^{1,i}$ ($i \geq 0$) with relations

$$\text{(a)} \quad \lambda_i \lambda_{2i+1+n} = \sum_{j \geq 0} \binom{n-j-1}{j} \lambda_{i+n-j} \lambda_{2i+1+j}.$$

The differential δ is given by

$$\text{(b)} \quad \delta(\lambda_k) = \sum_{j \geq 0} \binom{k-j-1}{j+1} \lambda_{k-j-1} \lambda_j$$

and $H^{s,t}(\Lambda, \delta) \simeq \operatorname{Ext}_A^{s,t+s}$. From (a) we see $\{\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_s} | 2i_j \geq i_{j+1}\}$ is a $\mathbb{Z}/2$-base for Λ. Such monomials are said to be admissible.

The class $h_i \in \operatorname{Ext}_A^{1,2^i}$ is represented by λ_{2^i-1} and $c_i \in \operatorname{Ext}_A^{3,2^{i+3}+2^{i+1}+2^i}$ is represented by $\lambda_{2^{i+1}+2^i-1}(\lambda_{2^{i+2}-1})^2$. Thus $h_i h_j$ for $0 \leq i \leq j$ (resp. $h_i h_j h_k, 0 \leq i \leq j \leq k$) is represented by $\lambda_{2^i-1}\lambda_{2^j-1}$ (resp. $\lambda_{2^i-1}\lambda_{2^j-1} \cdot \lambda_{2^k-1}$). Note that we have chosen these representations in the arithmetic linear ordering of the subscripts. Most of these representations are inadmissible. They are admissible only in the cases $i = j$ and $i = j = k$. One advantage of taking such representations is that

one recognizes immediately the relations $h_i h_{i+1} = 0$, $h_i^3 = h_{i-1}^2 h_{i+1}$ and $h_i h_{i+2}^2 = 0$ in the lambda algebra. Indeed, it is easy to see from (a) that the relations $\lambda_{2^i-1}\lambda_{2^{i+1}-1} = 0$, $\lambda_{2^i-1}^2\lambda_{2^{i+1}-1} = \lambda_{2^i-1}^3$ and $\lambda_{2^i-1}(\lambda_{2^{i+2}-1})^2 = 0$ hold in Λ. Finally note that the representations for c_i are admissible.

To compute $\mathrm{Ext}_A^{*,*}(P)$ we take $\tilde{H}_*(P)$, the mod 2 homology of P, and consider $\tilde{H}_*(P) \otimes \Lambda$ which is bigraded by $(\tilde{H}_*(P) \otimes \Lambda)^{s,t} = \sum_{k \geq 1} \tilde{H}_k(P) \otimes \Lambda^{s,t-k}$. $\tilde{H}_*(P) \otimes \Lambda$ is a differential right Λ-module with differential δ given by

$$\text{(c)} \quad \delta(e_k) = \sum_{j \geq 0} \binom{k-j-1}{j+1} e_{k-j-1} \otimes \lambda_j$$

where e_k is the generator of $\tilde{H}_k(P) = \mathbb{Z}/2$ $(k \geq 1)$. Then

$$H^{s,t}(\tilde{H}_*(P) \otimes \Lambda, \delta) \cong \mathrm{Ext}_A^{s,t}(P) \qquad ([13]).$$

Define a filtration $\{F(i)\}_{i \geq 1}$ of the differential module $\tilde{H}_*(P) \otimes \Lambda$ by $F(i) = \sum_{k \leq i} e_k \otimes \Lambda$. Clearly $F(i)/F(i-1) \cong \Sigma^i \Lambda$. This filtration gives rise to a spectral sequence $\{E_r^{i,s,t}\}_{r \geq 1}$ with $\sum_{i \geq 1} E_\infty^{i,s,t} \cong \mathrm{Ext}_A^{s,t}(P)$ as a vector space over $\mathbb{Z}/2$ and

$$\text{(d)} \quad E_1^{i,s,t} = H^{s,t}(F(i)/F(i-1)) \cong \mathrm{Ext}_A^{s,t+s-i}.$$

For each $r \geq 1$ the differential d_r of the spectral sequence goes from $E_r^{i,s,t}$ to $E_r^{i-r,s+1,t-1}$.

Our task is to make calculations with this spectral sequence. We need only compute the differentials $d_r : E_r^{i,s,t} \to E_r^{i-r,s+1,t-1}$ for $0 \leq s \leq 2$. By (d) we see

$\{e_i = e_i \otimes 1 | i \geq 1\}$ is a $\mathbb{Z}/2$-base for $E_1^{*,0,*}$,

$\{e_i h_j = e_i \otimes h_j | i \geq 1, j \geq 0\}$ is a $\mathbb{Z}/2$-base for $E_1^{*,1,*}$,

$\{e_i h_j h_k | i \geq 1, 0 \leq j \leq k, h_j h_k \neq 0\}$ is a $\mathbb{Z}/2$-base for $E_1^{*,2,*}$

and

$\{e_i h_j h_k h_l | i \geq 1, 0 \leq j \leq k \leq l, h_j h_k h_l \neq 0\} \cup \{e_i c_j | i \geq 1, j \geq 0\}$

is a $\mathbb{Z}/2$-base for $E_1^{*,3,*}$.

We begin with $d_r : E_r^{*,0,*} \to E_r^{*,1,*}$, the simplest differentials.

Note that each integer $i \geq 1$ can be expressed in the form $i = 2^l(2k+1) - 1$ for uniquely determined integers $l \geq 0$ and $k \geq 0$. Then we have the following differentials from $E_r^{*,0,*}$ to $E_r^{*,1,*}$.

(2.1.1) $\qquad\qquad d_r(e_{2^l-1}) = 0 \quad \text{for } r \geq 1, l \geq 1.$

(2.1.2) $\qquad e_{2^l(2k+1)-1} \to e_{2^{l+1}k-1} h_l \quad \text{for } k \geq 1 \text{ and } l \geq 0.$

Here "$e_i x \to e_{i-r} y$" means that $e_i x$ and $e_{i-r} y$ both survive to $E_r^{*,*,*}$ and $d_r(e_i x) = e_{i-r} y$; in the case of (2.1.2), $x = 1$ and $y = h_l$.

(2.1.1) is clear since e_{2^l-1} are cycles in $\tilde{H}_*(P) \otimes \Lambda$. It is easy to see, from (c), that in $\tilde{H}_*(P) \otimes \Lambda$

$$\delta(e_{2^l(2k+1)-1}) \equiv e_{2^{l+1}k-1}\lambda_{2^l-1} \quad \text{mod } F(2^{l+1}k - 2) \quad \text{for } k \geq 1$$

and this is (2.1.2).

Theorem 1.1 (1) follows from (2.1.1) and (2.1.2); the class $\hat{h}_i$ is represented by e_{2^i-1}.

Next we compute $d_r \colon E_r^{*,1,*} \to E_r^{*,2,*}$. By (2.1.2) we need only consider $d_r(e_{2^l(2k+1)-1} h_j)$ for $l \leq j$.

(2.2.1) $\qquad d_r(e_{2^l-1} h_j) = 0 \quad \text{for } r \geq 1,\ 1 \leq l \leq j.$

This is clear since $e_{2^l-1}\lambda_{2^j-1} \in F(2^l - 1)$ projects to $e_{2^l-1} h_j \in H^*(F(2^l - 1)/F(2^l - 2))$ and is a cycle.

(2.2.2) $e_{2^l(2k+1)-1} h_j \to e_{2^{l+1}k-1} h_l h_j$
$$\text{for } l < j - 1 \text{ or } l = j \text{ and } k \geq 1.$$

A direct calculation as that for (2.1.2) proves (2.2.2). Or, this follows directly from (2.1.2) by the fact that all our differentials are linear over $\text{Ext}_A^{*,*}$.

It remains to compute d_r on classes of the form $e_{2^l(2k+1)-1} h_{l+1}$, $k \geq 1$. Let $2k = 2^p(2k_1 + 1)$; so $p \geq 1$, $k_1 \geq 0$. Thus

$$e_{2^l(2k+1)-1} h_{l+1} = e_{2^{l+p+1}k_1 + 2^{l+p} + 2^l - 1} h_{l+1}.$$

(2.2.3) If $p \geq 2$, then

$$e_{2^{l+p+1}k_1 + 2^{l+p} + 2^l - 1} h_{l+1} \to e_{2^{l+p+1}k_1 + 2^{l+p} - 2^l - 1} h_{l+1}^2.$$

(2.2.4) If $p = 1$, $k_1 = 2^m - 1 \geq 0$ then

$$d_r(e_{2^{l+2}k_1 + 2^{l+1} + 2^l - 1} h_{l+1} = e_{2^{l+2+m} - 2^l - 1} h_{l+1}) = 0 \quad \text{for } r \geq 1.$$

(2.2.5) If $p = 1$, $k_1 = 2^{a+b} - 2\overline{k}_1 + 2^a - 1$ with $a \geq 0$, $b \geq 3$ and $\overline{k}_1 \geq 1$ then

$$e_{2^{l+2}k_1 + 2^{l+1} + 2^l - 1} h_{l+1} = e_{2^{l+a+b}\overline{k}_1 + 2^{l+a+2} - 2^l - 1} h_{l+1}$$
$$\to \begin{cases} e_{2^{l+a+b}\overline{k}_1 - 2^l - 1} h_{l+1} h_{l+a+2} & \text{for } a \geq 1 \\ e_{2^{l+b}\overline{k}_1 - 2^{l+1} - 2^l - 1} h_{l+2}^2 & \text{for } a = 0. \end{cases}$$

(2.2.3) again follows by a direct calculation as that for (2.1.2). However, in order to see what is really happening, we illustrate the calculation as follows. The class $e_{i=(2^{l+p+1}k_1 + 2^{l+p} + 2^l - 1)}\lambda_{2^{l+1}-1}$ in $F(i)$ projects

to $e_i h_{l+1} \in H^*(F(i)/F(i-1))$. We call the former a representation in $F(i)$ of the latter. From (c) we calculate

$$\delta(e_{2^{l+p+1}k_1+2^{l+p}+2^l-1}\lambda_{2^{l+1}-1})$$
$$\equiv e_{2^{l+p+1}k_1+2^{l+p}-1}\lambda_{2^l-1}\lambda_{2^{l+1}-1} + e_{2^{l+p+1}k_1+2^{l+p}-2^l-1}\lambda_{2^{l+1}-1}^2$$
$$\mathrm{mod}\ F(2^{l+p+1}k_1 + 2^{l+p} - 2^l - 2)$$
$$\equiv e_{2^{l+p+1}k_1+2^{l+p}-2^l-1}\lambda_{2^{l+1}-1}^2 \quad \mathrm{mod}\ F(2^{l+p+1}k_1 + 2^{l+p} - 2^l - 2)$$

since $\lambda_{2^l-1}\lambda_{2^{l+1}-1} = 0$ in Λ. This proves (2.2.3). Note that here the condition $p \geq 2$ is needed in order to have the term $e_{2^{l+p+1}k_1+2^{l+p}-2^l-1}\lambda_{2^{l+1}-1}$ present in the expansion of $\delta(e_{2^{p+l+1}k_1+2^{l+p}+2^l-1})$.

To show (2.2.4) we note the equation

$$(2.2.4)^* \quad \delta(e_{2^{l+2+m}+2^l-1}) \equiv e_{2^{l+2+m}-1}\lambda_{2^l-1} + e_{2^{l+2+m}-2^l-1}\lambda_{2^{l+1}-1}$$
$$\mathrm{mod}\ F(2^{l+2+m} - 2^l - 2).$$

Since the second term on the right side of the equation represents $e_{2^{l+2+m}-2^l-1}h_{l+1}$ and the first term is a cycle in $\tilde{H}_*(P) \otimes \Lambda$ it follows that $e_{2^{l+2+m}-2^l-1}h_{l+1}$ is an infinite cycle in the spectral sequence. This proves (2.2.4).

The proof of (2.2.5) is a little harder and is postponed.

Now we conclude Theorem 1.1 (2) from the above differentials. From (2.1.2) and (2.2.1) we see $\hat{h}_l h_j \neq 0$ in $\mathrm{Ext}_A^{1,*}(P)$ for $1 \leq l \leq j$. (2.1.2) shows that, for $l \geq j+1$, $e_{2^l-1}h_j$ is a boundary in the spectral sequence, but it does not necessarily follow that $\hat{h}_l h_j$ is zero in $\mathrm{Ext}_A^{1,*}(P)$. It only implies that $\hat{h}_l h_j$ has filtration less than $2^l - 1$. In fact, for $l > j+1$, the equation $(2.2.4)^*$ shows that $\hat{h}_l h_j$ has filtration $2^l - 2^j - 1$ and there it projects to

$$e_{2^l-2^j-1}h_{j+1} \in H^*(F(2^l - 2^j - 1)/F(2^P - 2^j - 2));$$

in particular $\hat{h}_l h_j$ is non-zero in $\mathrm{Ext}_A^{1,*}(P)$. This fails for $l = j+1$. Indeed, in $\tilde{H}_*(P) \otimes \Lambda$

$$\delta(e_{2^{j+1}+2^j-1}) = e_{2^{j+1}-1}\lambda_{2^j-1}.$$

Thus $\hat{h}_{j+1}h_j = 0$. By (2.2.2), (2.2.3) and (2.2.5) there are no other surviving elements in $E_\infty^{*,1,*}$. This proves Theorem 1.1(2).

It is not easy to see the differentials (2.2.5) by direct calculations as that for (2.2.3). To tackle (2.2.5) we will introduce a method which not only gives a quick proof for these differentials but is also efficient in recognizing more complicated differentials later.

Recall that the set of all the admissible monomials in the λ_i's is a $\mathbf{Z}/2$-base for Λ. Define an increasing filtration $\{\Lambda(i)\}_{i\geq 0}$ of the differential algebra Λ by $\Lambda(i) = \{\lambda_{i_1} \cdots \lambda_{i_s} | (i_1, \ldots, i_s)$ admissible and $i_1 \leq i\}$. Each $\Lambda(i)$ is indeed closed under δ ([6]). There is a short exact sequence

$$0 \to \Lambda(i-1) \xrightarrow{e} \Lambda(i) \xrightarrow{h} \Sigma^i \Lambda(2i) \to 0$$

where e is the inclusion map and h is given by

$$h(\lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_s}) = \begin{cases} \lambda_{i_2} \cdots \lambda_{i_s} & \text{if } i_1 = i, \\ 0 & \text{if } i_1 < i. \end{cases}$$

Thus $\Lambda(i)/\Lambda(i-1) \cong \Sigma^i \Lambda(2i)$ as a chain complex. This filtration gives rise to a spectral sequence $\{\overline{E}_r^{i,s,t}\}_{r\geq 1}$ with $\sum_{i\geq 1} \overline{E}_\infty^{i,s,t} \cong \mathrm{Ext}_A^{s,t}$ as a vector space over $\mathbf{Z}/2$ and

$$\overline{E}_1^{i,s,t} = H^{s,t}(\Lambda(i)/\Lambda(i-1)) \cong H^{s,t-i}(\Lambda(2i)).$$

For each $r \geq 1$ the differential $\overline{d}_r$ of the spectral sequence goes from $\overline{E}_r^{i,s,t}$ to $\overline{E}_r^{i-r,s+1,t-1}$.

The following is the basis of the to-be-introduced method.

THEOREM 2.3. *There is a chain map* $\phi: \Lambda \to \tilde{H}_*(P) \otimes \Lambda$ *(but not a differential Λ-map) such that*
(1) $\phi(\Lambda(i)) \subseteq F(i)$ *for all* $i \geq 0$.
(2) $\phi(\lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_s}) = e_{i_1} \otimes \lambda_{i_2} \cdots \lambda_{i_s}$ mod $F(i_1 - 1)$ *for* $(i_1, i_2, \ldots, i_s)$ *admissible and* $i_1 \geq 1$.

This result is actually proved in [17] except that it is not stated in this context there. We will recall the proof at the end of this section.

It follows from (2.3) that the chain map ϕ induces a map of the spectral sequences $\{\phi_r: \overline{E}_r^{i,s,t} \to E_r^{i,s,t}\}$ where

$$\phi_1: \overline{E}_1^{i,s,t} \cong H^{s,t-i}(\Lambda(2i)) \to E_1^{i,s,t} \cong H^{s,t-i}(\Lambda) \cong \mathrm{Ext}_A^{s,t+s-i}$$

is just the induced map of the inclusion $\Lambda(2i) \to \Lambda$. It is easy to see that $H^{s,t-i}(\Lambda(2i)) \cong H^{s,t-i}(\Lambda)$ if $t - i < 2i - 1$; so in this range ϕ_1 is an isomorphism. We call this range the "stable range".

Now we prove (2.2.5). Recall that we want to compute

$$d_r(e_{2^{l+a+b}\overline{k}_1 + 2^{l+a+2} - 2^l - 1} h_{l+1})$$

where $a \geq 0$, $b \geq 3$ and $\overline{k}_1 \geq 1$. Let $i = 2^{l+a+b}\overline{k}_1 + 2^{l+a+2} - 2^l - 1$. Then $2^{l+1} - 1 < 2i - 1$, and so we may consider

$$\lambda_i h_{l+1} \in H^*(\Lambda(i)/\Lambda(i-1)).$$

Our method of computing $d_r(e_i h_{l+1})$ is to compute $\overline{d}_r(\lambda_i h_{l+1})$ and then apply the equation

$$d_r(e_i h_{l+1}) = \phi_r \overline{d}_r(\lambda_i h_{l+1}).$$

Clearly $\lambda_i \lambda_{2^{l+1}-1} \in \Lambda(i)$ is a representation of $\lambda_i h_{l+1}$. The key point of the method is that the inadmissible monomial

$$\lambda_{2^l-1} \lambda_{2^{l+a+b}\overline{k}_1 + 2^{l+a+2}-1}$$

is also a representation in $\Lambda(i)$ of $\lambda_i h_{l+1}$ because

$$\lambda_{2^l-1} \lambda_{2^{l+a+b}\overline{k}_1 + 2^{l+a+2}-1} \equiv \lambda_i \lambda_{2^{l+1}-1} \qquad \mathrm{mod}\ \Lambda(i-1)$$

(recall $i = 2^{l+a+b}\overline{k}_1 + 2^{l+a+2} - 2^l - 1$), and it is easier to see the differential $\overline{d}_r(\lambda_i h_{l+1})$ from this representation. In fact, we have

$$\delta(\lambda_{2^l-1} \lambda_{2^{l+a+b}\overline{k}_1 + 2^{l+a+2}-1})$$
$$\equiv \lambda_{2^l-1} \lambda_{2^{l+a+b}\overline{k}_1 -1} \lambda_{2^{l+a+2}-1}$$
$$\equiv \begin{cases} \lambda_{2^{l+a+b}\overline{k}_1 - 2^l - 1} \lambda_{2^{l+1}-1} \lambda_{2^{l+a+2}-1} & \mathrm{mod}\ \Lambda(2^{l+a+b}\overline{k}_1 - 2^l - 2),\ a \geq 1 \\ \lambda_{2^{l+b}\overline{k}_1 - 2^{l+1} - 2^l - 1}(\lambda_{2^{l+2}-1})^2 & \mathrm{mod}\ \Lambda(2^{l+b}\overline{k}_1 - 2^{l+1} - 2^l - 2), \\ & \hspace{6cm} a = 0 \end{cases}$$

Thus

$$\lambda_i h_{l+1} \rightarrow \begin{cases} \lambda_{i-2^{l+a+2}} h_{l+1} h_{l+a+2}, & a \geq 1, \\ \lambda_{i-2^{l+2}-2^{l+1}} h_{l+2}^2, & a = 0. \end{cases}$$

So

$$e_i h_{l+1} \rightarrow \begin{cases} e_{i-2^{l+a+2}} h_{l+1} h_{l+a+2}, & a \geq 1, \\ e_{i-2^{l+2}-2^{l+1}} h_{l+2}^2, & a = 0. \end{cases}$$

This proves (2.2.5).

We proceed to compute $d_r : E_r^{*,2,*} \rightarrow E_r^{*,3,*}$, the last set of differentials. By (2.2.2) we need only consider $d_r(e_{2^l(2n+1)-1} h_j h_k)$ for $l \leq j$. By (2.2.3) and (2.2.5) we may impose some other restrictions on the subscripts of these elements when $l = j-1$ or $j-2$. These restrictions will be recalled at proper time.

To prove Theorem 1.1(3) it suffices to know which of these non-boundary elements are infinite cycles.

PROPOSITION 2.4. *The following elements in the first column are infinite cycles and form a* $\mathbb{Z}/2$-*base for* $E_\infty^{*,2,*}$; *they represent the elements*

in $\text{Ext}_A^{2,*}(P)$ *listed in the second column.*

(1) $e_{2^l-1}h_j h_k \longleftarrow\!\!\!\longrightarrow \hat{h}_l h_j h_k,$

$\qquad\qquad\qquad 1 \le l \le j \text{ and } j < k-1 \text{ or } j = k.$

(2) $e_{2^{j+1}+2^j-1}h_{j+2}^2 \longleftarrow\!\!\!\longrightarrow \hat{c}_j, \quad j \ge 0$

(3) $e_{2^m-2^j-1}h_j^2 \longleftarrow\!\!\!\longrightarrow \hat{h}_m h_{j-1}^2, \quad j \ge 1 \text{ and } j+2 \le m.$

(4) $e_{2^m-2^{j-1}-1}h_j h_k \longleftarrow\!\!\!\longrightarrow \hat{h}_m h_{j-1} h_k,$

$\qquad\qquad\qquad j \ge 1 \text{ and } j+1 \le m \le k, j < k-1.$

(5) $e_{2^{m+t}-2^m-2^{j-1}-1}h_j h_{m+1} \longleftarrow\!\!\!\longrightarrow \hat{h}_{m+t}h_{j-1}h_m,$

$\qquad\qquad\qquad j \ge 1, t \ge 2 \text{ and } j+1 \le m.$

We prove (2.4) excluding "they form a $\mathbf{Z}/2$-base for $E_\infty^{*,2,*}$" which will be clear after we compute non-trivial differentials on the remaining elements later.

First of all, a check against (2.2.2), (2.2.3) and (2.2.5) shows that the elements in the first column of (2.4) are non-boundary elements.

(2.4)(1) is clear. By a direct calculation we see $\delta(e_{2^{j+1}+2^j-1}\lambda_{2^{j+2}-1}^2) = 0$ in $\tilde{H}_*(P) \otimes \Lambda$. Let $\hat{c}_j = \{e_{2^{j+1}+2^j-1}\lambda_{2^{j+2}-1}^2\} \in \text{Ext}_A^{2,2^{j+3}+2^{j+1}+2^j-1}(P)$. Then $\hat{c}_j$ projects to $e_{2^{j+1}+2^j-1}h_{j+2}^2$ in the spectral sequence. This proves (2.4)(2). (2.4)(4) follows from the equation

$$\delta(e_{2^m+2^{j-1}-1}\lambda_{2^k-1})$$
$$\equiv e_{2^m-2^{j-1}-1}\lambda_{2^j-1}\lambda_{2^k-1} + e_{2^m-1}\lambda_{2^{j-1}-1}\lambda_{2^k-1} \quad \mod F(2^m-2^{j-1}-2).$$

To prove (2.4)(3) and (5) we use the "trick" that we tackled with (2.2.5). Note that in both cases we are in the stable range.

For (2.4)(3) consider the equation

$$\lambda_{2^m-2^j-1}\lambda_{2^j-1}^2 \equiv \lambda_{2^{j-1}-1}^2\lambda_{2^m-1} \quad \mod \Lambda(2^m-2^j-2)$$

in Λ. Since $Ext_A^{*,*}$ is commutative there exists $K \in \Lambda$ such that

$$\delta(K) = \lambda_{2^m-1}\lambda_{2^{j-1}-1}^2 + \lambda_{2^{j-1}-1}^2\lambda_{2^m-1}.$$

Thus $\lambda_{2^m-2^j-1}\lambda_{2^j-1}^2 + \delta(K) \equiv \lambda_{2^m-1}\lambda_{2^{j-1}-1}^2 \mod \Lambda(2^m-2^j-2)$. Note that both $(2^m-2^j-1, 2^j-1, 2^j-1)$ and $(2^m-1, 2^{j-1}-1, 2^{j-1}-1)$ are admissible. Applying the chain map ϕ in Theorem 2.3 to this equation we get

$$\cdot e_{2^m-2^j-1}\lambda_{2^j-1}^2 + \delta(\phi(K)) \equiv e_{2^m-1}\lambda_{2^{j-1}-1}^2 \quad \mod F(2^m-2^j-2).$$

This proves (2.4)(3).

For $(2.4)(5)$ consider the equations

$$\lambda_{2^{m+t}-2^m-2^{j-1}-1}\lambda_{2^j-1}\lambda_{2^{m+1}-1} \equiv \lambda_{2^{j-1}-1}\lambda_{2^{m+t}-2^m-1}\lambda_{2^{m+1}-1}.$$
$$\mathrm{mod}\ \Lambda(2^{m+t}-2^m-2^{j-1}-2)$$

and

$$\delta(\lambda_{2^{j-1}-1}\lambda_{2^{m+t}+2^m-1}) + \delta(K')$$
$$\equiv \lambda_{2^{j-1}-1}\lambda_{2^{m+t}-2^m-1}\lambda_{2^{m+1}-1} + \lambda_{2^{m+t}-1}\lambda_{2^{j-1}-1}\lambda_{2^m-1}$$
$$\mathrm{mod}\ \Lambda(2^{m+t}-2^m-2^{j-1}-2)$$

where

$$\delta(K') = \lambda_{2^{j-1}-1}\lambda_{2^{m+t}-1}\lambda_{2^m-1} + \lambda_{2^{m+t}-1}\lambda_{2^{j-1}-1}\lambda_{2^m-1}.$$

So

$$\delta(\lambda_{2^{j-1}-1}\lambda_{2^{m+t}+2^m-1}) + \delta(K')$$
$$\equiv \lambda_{2^{m+t}-2^m-2^{j-1}-1}\lambda_{2^j-1}\lambda_{2^{m+1}-1} + \lambda_{2^{m+t}-1}\lambda_{2^{j-1}-1}\lambda_{2^m-1}$$
$$\mathrm{mod}\ \Lambda(2^{m+t}-2^m-2^{j-1}-2).$$

Again this implies

$$\delta(\phi(\lambda_{2^{j-1}-1}\lambda_{2^{m+t}+2^m-1})) + \delta(\phi(K'))$$
$$\equiv e_{2^{m+t}-2^m-2^{j-1}-1}\lambda_{2^j-1}\lambda_{2^{m+1}-1} + e_{2^{m+t}-1}\lambda_{2^{j-1}-1}\lambda_{2^m-1}$$
$$\mathrm{mod}\ F(2^{m+t}-2^m-2^{j-1}-2).$$

This proves $(2.4)(5)$.

To complete the proof of Theorem $1.1(3)$ (assuming the whole (2.4)) it remains to show the relations

$$\hat{h}_{i+2}h_i^2 = \hat{h}_{i+1}h_{i+1}^2 \quad \text{and} \quad \hat{h}_{i+2}\hat{h}_{i+2}h_i = 0$$

in $\mathrm{Ext}_A^{2,*}(P)$. As suggested by $(2.2.2)$ we get these relations by the equations

$$\delta(e_{2^{i+2}+2^i-1}\lambda_{2^i-1}) + \delta(e_{2^{i+1}+2^i-1}\lambda_{2^{i+1}+2^i-1}) = e_{2^{i+2}-1}\lambda_{2^i-1}^2 + e_{2^{i+1}-1}\lambda_{2^{i+1}-1}^2$$

and

$$\delta(e_{2^{i+2}+2^i-1}\lambda_{2^{i+2}-1}) = e_{2^{i+2}-1}\lambda_{2^i-1}\lambda_{2^{i+2}-1}$$

in $\tilde{H}_*(P) \otimes \Lambda$. This proves Theorem $1.1(3)$.

The most difficult part of the work is to compute the differentials on the remaining non-boundary elements $e_{2^l(2n+1)-1}h_jh_k$ where $l \le j$. There are 16 sets of non-trivial differentials on these elements. We will first state them. Proofs will be given later. These differentials will be needed in the proof of Theorem $1.1(4)$.

First assume $j = k$. In this case there are 8 sets of non-trivial differentials. We state these differentials in the cases $l = j$, $l = j - 1$, $l = j - 2$ and $l \le j - 3$. Some of these cases will be divided into subcases.

(2.5.1) If $l = j = 0$ then $e_{2n}h_0^2 \to e_{2n-1}h_0^3$, $n \ge 1$.

(2.5.2) If $l = j \ge 1$ and $n + 1$ is odd, say $n = 2\bar{n}$ then

$$e_{2^j(2n+1)-1}h_j^2 = e_{2^{j+2}\bar{n}+2^j-1}h_j^2 \to e_{2^{j+2}\bar{n}-2^{j+1}-2^{j-1}-1}c_{j-1}, \qquad \bar{n} \ge 1.$$

(2.5.3) If $l = j \ge 1$, $n + 1 = 4\bar{n} + 2$ then

$$e_{2^j(2n+1)-1}h_j^2 = e_{2^{j+3}\bar{n}+2^{j+2}-2^j-1}h_j^2 \to e_{2^{j+3}\bar{n}-2^{j+1}-2^j-2^{j-1}-1}h_{j-1}h_{j+2}^2, \qquad \bar{n} \ge 1.$$

(2.5.4) If $l = j \ge 1$, $n + 1 = 2^{a+b}\bar{n} + 2^a$, $a \ge 2$ and $b \ge 1$, then

$$e_{2^j(2n+1)-1}h_j^2 = e_{2^{j+a+b+1}\bar{n}+2^{j+a+1}-2^j-1}h_j^2 \to e_{2^{j+a+b+1}\bar{n}-2^j-1}h_j^2h_{j+a+1}, \qquad \bar{n} \ge 1.$$

Suppose $l = j - 1 \ge 0$. Then by (2.2.3) we need only consider $n = 2\bar{n}$. In this case we have the following.

(2.5.5) $e_{2^{j-1}(2n+1)-1}h_j^2 = e_{2^{j+1}\bar{n}+2^{j-1}-1}h_j^2 \to e_{2^{j+1}\bar{n}-2^{j-1}-1}h_j^3$, $\qquad \bar{n} \ge 1$.

Suppose $l = j - 2 \ge 0$. Then by (2.2.5) we need only consider $n = 2\bar{n} + 1$ or $n = 4\bar{n}$.

(2.5.6) If $l = j - 2$, $n = 2\bar{n} + 1$ then

$$e_{2^{j-2}(2n+1)-1}h_j^2 = e_{2^j\bar{n}+2^{j-1}+2^{j-2}-1}h_j^2 \to e_{2^j\bar{n}-1}c_{j-2}, \qquad \bar{n} \ge 1.$$

(2.5.7) If $l = j - 2$, $n = 4\bar{n}$ then

$$e_{2^{j-2}(2n+1)-1}h_j^2 = e_{2^{j+1}\bar{n}+2^{j-2}-1}h_j^2 \to e_{2^{j+1}\bar{n}-2^j+2^{j-2}-1}h_j^3, \qquad \bar{n} \ge 1.$$

(2.5.8) If $l \le j - 3$ then

$$e_{2^l(2n+1)-1}h_j^2 \to e_{2^{l+1}n-1}h_lh_j^2, \qquad n \ge 1.$$

Next we state the remaining 8 sets of differentials for the case $j < k - 1$.

(2.5.9) If $l = j$ then

$$e_{2^l(2n+1)-1}h_jh_k \to e_{2^{l+1}n-1}h_j^2h_k, \qquad n \ge 1.$$

Suppose $l = j - 1 \ge 0$. Then by (2.2.5) we need only consider n such that

$$2^{j-1}(2n + 1) = 2^{m+1}\bar{n} + 2^m - 2^{j-1} \quad \text{where } j \le m \le k \text{ and } \bar{n} \ge 0.$$

The case $\bar{n} = 0$ is already covered by (2.4); so can assume $\bar{n} \geq 1$. Let $\bar{n} = 2^{p-1}(2q+1)$, $p \geq 1$, $q \geq 0$. Then

$$2^{j-1}(2n+1) - 1 = 2^{p+m}(2q+1) + 2^m - 2^{j-1} - 1.$$

The next 6 differentials are stated for the case $l = j - 1$ depending on various conditions on j, m, k, p and q.

(2.5.10) If $m = j$ then

$$e_{2^{j-1}(2n+1)-1}h_jh_k = e_{2^{p+j}(2q+1)+2^{j-1}-1}h_jh_k$$
$$\rightarrow e_{2^{p+j}(2q+1)-2^{j-1}-1}h_j^2h_k, \qquad p \geq 1, \ q \geq 0.$$

(2.5.11) If $j+1 \leq m < k - 1$ or if $k = m \geq j + 3$ then

$$e_{2^{j-1}(2n+1)-1}h_jh_k = e_{2^{p+m}(2q+1)+2^m-2^{j-1}-1}h_jh_k$$
$$\rightarrow \begin{cases} e_{2^{p+m}(2q+1)-2^{j-1}-1}h_jh_mh_k & \text{for } m \geq j+2, \qquad p \geq 1 \\ e_{2^{p+m}(2q+1)-2^{j+1}+2^{j-1}-1}h_{j+1}^2h_k & \text{for } m = j+1, \qquad q \geq 0. \end{cases}$$

(2.5.12) If $j+1 \leq m = k - 1$ and $p \geq 2$ then

$$e_{2^{j-1}(2n+1)-1}h_jh_k = e_{2^{p+m}(2q+1)+2^m-2^{j-1}-1}h_jh_{m+1}$$
$$\rightarrow \begin{cases} e_{2^{p+m}(2q+1)-2^m-2^{j-1}-1}h_jh_{m+1}^2 & \text{for } m \geq j+2, \\ e_{2^{p+j+1}(2q+1)-2^{j+2}-2^{j-1}-1}c_j & \text{for } m = j+1, \quad q \geq 0. \end{cases}$$

(2.5.13) If $j+1 \leq m = k - 1$, $p = 1$ and $2q + 1 = 2^t\bar{q} + 2^{t-1} - 1$, $t \geq 3, \bar{q} \geq 1$ then

$$e_{2^{j-1}(2n+1)-1}h_jh_k = e_{2^{m+t+1}\bar{q}+2^{m+t}-2^m-2^{j-1}-1}h_jh_{m+1}$$
$$\rightarrow e_{2^{m+t+1}\bar{q}-2^m-2^{j-1}-1}h_jh_{m+1}h_{m+t}.$$

(2.5.14) If $j+1 \leq m = k - 1$, $p = 1$, $2q + 1 = 4\bar{q} + 1$, $\bar{q} \geq 1$ then

$$e_{2^{j-1}(2n+1)-1}h_jh_k = e_{2^{m+3}\bar{q}+2^{m+1}+2^m-2^{j-1}-1}h_jh_{m+1}$$
$$\rightarrow e_{2^{m+3}\bar{q}-2^{m+1}-2^m-2^{j-1}-1}h_jh_{m+1}^2.$$

(2.5.15) If $m = k = j + 2$ then

$$e_{2^{j-1}(2n+1)-1}h_jh_k = e_{2^{p+j+2}(2q+1)+2^{j+2}-2^{j-1}-1}h_jh_{j+2}$$
$$\rightarrow e_{2^{p+j+2}(2q+1)-2^{j+1}-2^j-2^{j-1}-1}h_{j+2}^3, \qquad p \geq 1, \ q \geq 0.$$

Finally we have the following.

(2.5.16) If $l \leq j - 2$ then

$$e_{2^l(2n+1)-1}h_jh_k \rightarrow e_{2^{l+1}n-1}h_lh_jh_k, \qquad n \geq 1.$$

This completes the statements of the 16 sets of non-trivial differentials $d_r\colon E_r^{*,2,*} \to E_r^{*,3,*}$.

It is not difficult to check that the elements $e_{2^l(2n+1)-1}h_j h_k$ considered in (2.5.1) through (2.5.16) together with those listed in Proposition 2.4 form a complete set of non-boundary elements. Also, distinct elements in the domain of (2.5.1) through (2.5.16) have distinct values. This proves the second conclusion of Proposition 2.4.

Some of these differentials are straightforward as (2.2.2) or (2.2.3). They are (2.5.1), (2.5.5), (2.5.6), (2.5.7), (2.5.8), (2.5.9) and (2.5.16).

For the rest, we need the "trick" in the proof of (2.2.5) to handle them. We are not going to do all the calculations. Rather, we just do for two of these. One of them is the hardest of all, and the calculation for the other is very typical for doing the remaining 7 differentials which we leave to the reader.

First, the easy one which we choose to be (2.5.2). Consider $e_i\lambda_{2^{j}-1}^2$, where $i = 2^{j+2}\overline{n}+2^j - 1$, a representation in $F(i)$ of $e_i h_j^2$. Since $\overline{n} \geq 1$, we are in the stable range. The "trick" is to replace e by λ, that is, to consider $\lambda_i\lambda_{2^{j}-1}^2$, a representation in $\Lambda(i)$ of $\lambda_i h_j^2 \in H^*(\Lambda(i)/\Lambda(i-1))$ and then consider the equation

$$\lambda_i\lambda_{2^{j}-1}^2 \equiv \lambda_{2^{j}-1}^2\lambda_{i+2^j} \qquad \mathrm{mod}\ \Lambda(i-1)$$

which shows $\lambda_{2^{j}-1}^2\lambda_{i+2^j}$ is also a representation in $\Lambda(i)$ of $\lambda_i h_j^2$. We have

$$
\begin{aligned}
\delta(\lambda_{2^{j}-1}^2\lambda_{i+2^j}) &= \lambda_{2^{j}-1}^2\lambda_{2^{j+2}\overline{n}+2^{j+1}-1} \\
&\equiv \lambda_{2^{j}-1}^2\lambda_{2^{j+2}\overline{n}-1}\lambda_{2^{j+1}-1} \\
&\equiv \lambda_{2^{j}-1}\lambda_{2^{j+2}\overline{n}-2^{j+1}+2^{j}-1}\lambda_{2^{j+1}-1}^2 \quad (\text{since } \lambda_{2^{j}-1}\lambda_{2^{j+1}-1} = 0) \\
&\equiv \lambda_{2^{j+2}\overline{n}-2^{j+1}-2^{j}-1}\lambda_{2^{j}+2^{j}-1}\lambda_{2^{j+1}-1}^2 \quad (\text{since } \lambda_{2^{j}-1}\lambda_{2^{j+1}-1}^2 = 0) \\
&\qquad\qquad \mathrm{mod}\ \Lambda(2^{j+2}\overline{n} - 2^{j+1} - 2^{j-1} - 2).
\end{aligned}
$$

This implies

$$\lambda_i h_j^2 \to \lambda_{2^{j+2}\overline{n}-2^{j+1}-2^{j-1}-1}c_{j-1} \quad (\text{in the spectral sequence } \{\overline{E}_r^{*,*,*}\})$$

and therefore

$$e_i h_j^2 \to e_{2^{j+2}\overline{n}-2^{j+1}-2^{j-1}-1}c_{j-1} \quad (\text{by applying } \phi \text{ in Theorem 2.3}).$$

This proves (2.5.2).

Next we do the calculation for the hard one which is (2.5.3). We directly begin with $\lambda_i \lambda_{2^j-1}^2$ and the equation

$$\lambda_i \lambda_{2^j-1}^2 \equiv \lambda_{2^j-1-1}^2 \lambda_{2^j+i} \qquad \mod \Lambda(i-1)$$

where $i = 2^{j+3}\overline{n} + 2^{j+2} - 2^j - 1, \overline{n} \geq 1$. As above we compute

$$\delta(\lambda_{2^j-1-1}^2 \lambda_{i+2^j} = \lambda_{2^j-1-1}^2 \lambda_{2^{j+3}\overline{n}+2^{j+2}-1})$$
$$\equiv \lambda_{2^j-1-1}^2 \lambda_{2^{j+3}\overline{n}-1} \lambda_{2^{j+2}-1}$$
$$\equiv \lambda_{2^j-1-1} \lambda_{2^{j+3}\overline{n}-2^j-1-1} \lambda_{2^j-1} \lambda_{2^{j+2}-1}$$
$$\equiv \lambda_{2^{j+3}\overline{n}-2^j-1} \lambda_{2^j-1}^2 \lambda_{2^{j+2}-1} = \lambda_{2^{j+3}\overline{n}-2^j-1} \lambda_{2^{j+1}-1}^3$$

where the equivalence "$\equiv$" is yet to be determined. We pause at this point because the calculation seems to suggest

$$e_i h_j^2 \rightarrow e_{i-2^{j+2}} h_{j+1}^3.$$

However, by (2.5.5), $e_{i-2^{j+2}} h_{j+1}^3$ is already hit in the spectral sequence by an element of lower filtration, namely $e_{i-2^{j+1}} h_{j+1}^2$. So we have to take into account this differential also in order to get the correct answer. We need to pick up some other terms of lower filtration when we move λ_{2^j-1-1} across $\lambda_{2^{j+3}\overline{n}-2^j-1-1}$ in the calculation above. Note that in Λ, $\lambda_{2^j-1} \lambda_{2^{j+2}-1} = \lambda_{2^{j+1}+2^j-1} \lambda_{2^{j+1}-1}$. The calculation now continues as follows.

$$\delta(\lambda_{2^j-1-1}^2 \lambda_{2^{j+3}\overline{n}+2^{j+2}-1})$$
$$\equiv \lambda_{2^j-1-1} \lambda_{2^{j+3}\overline{n}-2^j-1-1} \lambda_{2^j-1} \lambda_{2^{j+2}-1}$$
$$\equiv \lambda_{2^{j+3}\overline{n}-2^j-1} \lambda_{2^{j+1}-1}^3$$
$$+ \lambda_{2^{j+3}\overline{n}-2^j-2^j-1-1} \lambda_{2^{j+2}-1-1} \lambda_{2^{j+1}+2^j-1} \lambda_{2^{j+1}-1}$$
$$+ \lambda_{2^{j+3}\overline{n}-2^{j+1}-1} \lambda_{2^{j+1}-1} \lambda_{2^j-1} \lambda_{2^{j+2}-1}$$
$$+ \lambda_{2^{j+3}\overline{n}-2^{j+1}-2^j-2^j-1-1} \lambda_{2^{j+1}+2^j+2^j-1-1} \lambda_{2^{j+1}+2^j-1} \lambda_{2^{j+1}-1}$$
$$\mod \Lambda(2^{j+3}\overline{n} - 2^{j+1} - 2^j - 2^{j-1} - 2).$$

We have the identities

$$\lambda_{2^j+2^j-1-1} \lambda_{2^{j+1}+2^j-1} = 0$$

and

$$\lambda_{2^{j+1}+2^j+2^j-1-1} \lambda_{2^{j+1}+2^j-1} \lambda_{2^{j+1}-1} = \lambda_{2^j-1-1} \lambda_{2^{j+2}-1}^2$$

in Λ. As suggested by (2.5.5) we find

$$\delta(\lambda_{2^{j+3}\overline{n}+2^j-1} \lambda_{2^{j+1}-1}^2) + \delta(\lambda_{2^{j+3}\overline{n}-2^{j+2}+2^j-1} \lambda_{2^{j+2}+2^{j+1}-1} \lambda_{2^{j+1}-1})$$
$$\equiv \lambda_{2^{j+3}\overline{n}-2^j-1} \lambda_{2^{j+1}-1}^3 \qquad \mod \Lambda(2^{j+3}\overline{n} - 2^{j+1} - 2^j - 2^{j-1} - 2).$$

It is easy to see that

$$\delta(\lambda_{2^{j+3}\overline{n}-2^{j+1}-1}\lambda_{2^{j+1}+2^{j}-1}\lambda_{2^{j+2}-1}) \equiv \lambda_{2^{j+3}\overline{n}-2^{j+1}-1}\lambda_{2^{j+1}-1}\lambda_{2^{j}-1}\lambda_{2^{j+2}-1}$$
$$\mod \Lambda(2^{j+3}\overline{n} - 2^{j+1} - 2^{j} - 2^{j-1} - 2).$$

From these equations and identities we get

$$(*) \quad \delta(L + \lambda^2_{2^{j-1}-1}\lambda_{2^{j+3}\overline{n}+2^{j+2}-1}) \equiv \lambda_{2^{j+3}\overline{n}-2^{j+1}-2^{j}-2^{j-1}-1}\lambda_{2^{j}-1}\lambda^2_{2^{j+2}-1}$$
$$\mod \Lambda(2^{j+3}\overline{n} - 2^{j+1} - 2^{j} - 2^{j-1} - 2)$$

where

$$L = \lambda_{2^{j+3}\overline{n}+2^{j}-1}\lambda^2_{2^{j+1}-1}$$
$$+ \lambda_{2^{j+3}\overline{n}-2^{j+2}+2^{j}-1}\lambda_{2^{j+2}+2^{j+1}-1}\lambda_{2^{j+1}-1}$$
$$+ \lambda_{2^{j+3}\overline{n}-2^{j+1}-1}\lambda_{2^{j+1}+2^{j}-1}\lambda_{2^{j+2}-1}.$$

Since $L \in \Lambda(i-1)$ (recall $i = 2^{j+3}\overline{n}+2^{j+2}-2^{j}-1$), $L+\lambda^2_{2^{j-1}-1}\lambda_{2^{j+3}\overline{n}+2^{j+2}-1}$ is also a representation in $\Lambda(i)$ of $\lambda_i h_j^2$. From (*) we thus see

$$\lambda_i h_j^2 \rightarrow \lambda_{2^{j+3}\overline{n}-2^{j+1}-2^{j}-2^{j-1}-1}h_{j-1}h_{j+2}^2.$$

So

$$e_i h_j^2 \rightarrow e_{2^{j+3}\overline{n}-2^{j+1}-2^{j}-2^{j-1}-1}h_{j-1}h_{j+2}^2.$$

This proves (2.5.3).

We proceed to the proof of Theorem 1.1(4).

PROPOSITION 2.6. *The following elements in the first column are infinite cycles and are linearly independent in $E_\infty^{*,3,*}$; they represent the elements of* $\text{Ext}_A^{3,*}(P)$ *listed in the second column.*

$$(1) \quad e_{2^i-1}c_j \longleftarrow\!\!\!\!\longrightarrow \hat{h}_i c_j, \qquad 0 \le i \le j+1.$$
$$(2) \quad e_{2^i-2^{j+3}-2^{j+1}-2^{j}-1}c_{j+1} \longleftrightarrow \hat{h}_i c_j, \qquad 0 \le j \le i-4.$$
$$(3) \quad e_{2^{i+1}+2^i-1}h_{i+2}^2 h_j \longleftarrow\!\!\!\!\longrightarrow \hat{c}_i h_j, \qquad 0 \le i \le j-4.$$
$$(4) \quad e_{2^{i+1}+2^i-2^j-1}h_{j+1}h_{i+2}^2 \longleftrightarrow \hat{c}_i h_j, \qquad 0 \le j \le i-2.$$

PROPOSITION 2.7. *The following elements in the first column are infinite cycles and are linearly independent in $E_\infty^{*,3,*}$; they represent the*

elements of $\text{Ext}_A^{3,*}(P)$ *listed in the second column.*

(1) $e_{2^i-1}h_j^3 \longleftarrow \hspace{1em} \hat{h}_i h_j^3,$ $\qquad 1 \le i \le j-1.$

(2) $e_{2^i-1}h_j^2 h_l \longleftarrow \hspace{1em} \hat{h}_i h_j^2 h_l,$ $\qquad 1 \le i \le j < l-2.$

(3) $e_{2^i-1}h_j h_k^2 \longleftarrow \hspace{1em} \hat{h}_i h_j h_k^2,$ $\qquad 1 \le i \le j < k-2.$

(4) $e_{2^i-1}h_j h_k h_l \longleftarrow \hspace{1em} \hat{h}_i h_j h_k h_l,$ $\qquad 1 \le i \le j < k-1$
$\hspace{13em} < l-2.$

(5) $e_{2^i-2^{j+1}-2^j-1}h_{j+1}^3 \longleftarrow \hspace{1em} \hat{h}_i h_j^3,$ $\qquad 0 \le j \le i-3.$

(6) $e_{2^{j+3}-2^{j+1}-1}h_{j+1}^2 h_l \longleftarrow \hspace{1em} \hat{h}_{j+3}h_j^2 h_l,$ $\qquad 0 \le j \le l-4.$

(7) $e_{2^i-2^{j+1}-1}h_{j+1}^2 h_l \longleftarrow \hspace{1em} \hat{h}_i h_j^2 h_l,$ $\qquad 0 \le j \le i-4 \le l-4.$

(8) $e_{2^l-2^j-2^{j+1}-1}h_{j+1}^2 h_{l+1} \longleftarrow \hspace{1em} \hat{h}_i h_j^2 h_l,$ $\qquad 0 \le j \le l-3 \le i-5.$

(9) $e_{2^i-2^j-1}h_{j+1}h_k^2 \longleftarrow \hspace{1em} \hat{h}_i h_j h_k^2,$ $\qquad 0 \le j \le i-2 \le k-2,$
$\hspace{13em} j \le k-4.$

(10) $e_{2^i-2^{k+1}-2^j-1}h_{j+1}h_{k+1}^2 \longleftarrow \hspace{1em} \hat{h}_i h_j h_k^2,$ $\qquad 0 \le j \le k-3 \le i-5.$

(11) $e_{2^i-2^j-1}h_{j+1}h_k h_l \longleftarrow \hspace{1em} \hat{h}_i h_j h_k h_l,$ $\qquad 0 \le j \le i-2 \le k-2,$
$\hspace{13em} j \le k-3 \le l-5.$

(12) $e_{2^i-2^k-2^j-1}h_{j+1}h_{k+1}h_l \longleftarrow \hspace{1em} \hat{h}_i h_j h_k h_l,$ $\qquad 0 \le j \le k-2 \le i-4$
$\hspace{13em} \le l-4,\ k < l-2.$

(13) $e_{2^i-2^l-2^k-2^j-1}h_{j+1}h_{k+1}h_{l+1} \longleftarrow \hspace{1em} \hat{h}_i h_j h_k h_l,$ $\qquad 0 \le j \le k-2$
$\hspace{13em} \le l-4 \le i-6.$

PROPOSITION 2.8. *In* $\text{Ext}_A^{3,*}(P)$ *there are the following relations.*

(1) $\hat{h}_{i+3}h_i^2 h_{i+3} = 0,\ i \ge 0.$

(2) $\hat{h}_{i+2}h_i h_{i+3}^2 = 0,\ i \ge 0.$

(3) $\hat{h}_i c_j = 0,\ 0 \le i = j+2,\ j+3.$

(4) $\hat{c}_i h_j = 0,\ 0 \le i = j-3,\ j-2,\ j\ \text{or}\ j+1.$

(5) $\hat{c}_i h_{i+1} = \hat{h}_{i+1}c_i \ne 0,\ i \ge 0.$

It is not difficult to check the following. (1) The elements listed in the first columns of (2.6) and (2.7) are not boundaries in the spectral sequence and are linearly independent. (2) Theorem 1.1(4) follows from Propositions 2.6, 2.7 and 2.8. The conclusion (1) is of course based on the above work on (2.5.1) through (2.5.16).

For the proofs of (2.6) and (2.7) we first note that 2.6(1) and 2.7(1) through 2.7(4) are obvious. Some of the rest follow directly from

Proposition 2.4 by the linearity of the spectral sequence over $\text{Ext}_A^{*,*}$: 2.6(3) follows from 2.4(2), 2.7(6) and 2.7(7) follow from 2.4(3), 2.7(9) and 2.7(11) follow from 2.4(4), and 2.7(12) follows from 2.4(5).

What remain are 2.6(2), 2.6(4), 2.7(5), 2.7(8), 2.7(10) and 2.7(13). Except 2.6(4), these have a pattern in common, namely the subscripts of the factors $h_j h_k h_l$ or c_j in the first columns of these rows each decreases by one when we go to the corresponding elements in the second columns. The proof for these cases is exactly the same as that for 2.4(3); one just twists one more factor in the lambda algebra representations. Note that the filtration subscripts in these cases are large enough compared to those of the attached factors $h_j h_k h_l$ or c_j so that we are in the stable range and so that we can use the "trick". We leave the proof to the reader.

There is only one case left—2.6(4). The calculation for this case is also straightforward. We have the equation

$$\delta(e_{2^{i+1}+2^i+2^j-1}\lambda_{2^{i+2}-1}^2 + e_{2^{i+1}+2^i-1}K)$$
$$\equiv e_{2^{i+1}+2^i-1}\lambda_{2^{i+2}-1}^2\lambda_{2^j-1} + e_{2^{i+1}+2^i-2^j-1}\lambda_{2^{j+1}-1}\lambda_{2^{i+2}-1}^2$$
$$\text{mod } F(2^{i+1} + 2^i - 2^j - 2)$$

where $K \in \Lambda$ is an element such that

$$\delta(K) = \lambda_{2^j-1}\lambda_{2^{i+2}-1}^2 + \lambda_{2^{i+2}-1}^2\lambda_{2^j-1}.$$

This proves 2.6(4) since $e_{2^{i+1}+2^i-1}\lambda_{2^{i+2}-1}^2$ represents $\hat{c}_i$.

Finally we prove Proposition 2.8. We have to show that appropriate cycle representations of the classes in (2.8) are boundaries in the differential module $\tilde{H}_*(P)\otimes\Lambda$. Most of the cases in (2.8) are not difficult to work out. Only two cases are rather tedious. Anyway, we will write down all the boundary homomorphism formulas, the details of which we leave to the reader to check. These formulas are derived from the differentials (2.5.1) through (2.5.16).

The case (2.8)(4) for $i = j - 3$ is the simplest one since

$$e_{2^{i+1}+2^i-1}\lambda_{2^{i+2}-1}^2\lambda_{2^{i+3}-1} = 0.$$

Next we have the following formulas according to the case.

(2.8.2)* $\delta(e_{2^{j+2}+2^j-1}\lambda_{2^{j+3}-1}^2) = e_{2^{j+2}-1}\lambda_{2^j-1}\lambda_{2^{j+3}-1}^2.$

(2.8.3)*$_{j+2}$ $\delta(e_{2^{j+2}+2^{j+1}+2^j-1}\lambda_{2^{j+2}-1}^2) = e_{2^{j+2}-1}\lambda_{2^{j+1}+2^j-1}\lambda_{2^{j+2}-1}^2.$

(2.8.4)*$_{j-2}$ $\delta(e_{2^j+2^{j-2}-1}\lambda_{2^{j-1}-1}\lambda_{2^{j+1}-1}) = e_{2^{j-1}+2^{j-2}-1}\lambda_{2^j-1}^2\lambda_{2^j-1}.$

$(2.8.4)^*_j$ $\qquad \delta(e_{2^{j+1}+2^j-1}K_0) = e_{2^{j+1}+2^j-1}\lambda^2_{2^{j+2}-1}\lambda_{2^j-1}$

where $K_0 = \lambda_{2^{j+2}-1}\lambda_{2^{j+2}+2^j-1} + \lambda_{2^{j+2}+2^j-1}\lambda_{2^{j+2}-1}$.

$(2.8.5)^*$ $\qquad \delta(e_{2^{i+1}+2^i-1}\lambda_{2^{i+2}-1}\lambda_{2^{i+2}+2^{i+1}-1})$
$$= e_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1}\lambda_{2^{i+1}-1} + e_{2^{i+1}-1}\lambda_{2^{i+1}+2^i-1}\lambda^2_{2^{i+2}-1}.$$

$(2.8.1)^*$ $\qquad \delta\left(\sum_{k=1}^5 A_k\right) = e_{2^{i+3}-1}\lambda^2_{2^i-1}\lambda_{2^{i+3}-1}$

where

$$A_1 = e_{2^{i+3}+2^i-1}\lambda_{2^i-1}\lambda_{2^{i+3}-1},$$
$$A_2 = e_{2^{i+3}-2^i-1}\lambda_{2^{i+1}+2^i-1}\lambda_{2^{i+3}-1},$$
$$A_3 = e_{2^{i+3}+2^{i+1}-1}\lambda^2_{2^{i+2}-1},$$
$$A_4 = e_{2^{i+2}+2^i-1}\lambda_{2^{i+2}+2^i-1}\lambda_{2^{i+3}-1},$$
$$A_5 = e_{2^{i+2}-1}\lambda_{2^{i+2}+2^{i+1}-1}\lambda_{2^{i+3}-1}.$$

For 2.8(4), $i = j+1$ and 2.8(3), $i = j+3$ we have only calculated the following.

$(2.8.4)^*_{j+1}$ $\qquad \delta(e_{2^{j+1}+2^{j+1}-1}K_1 + e_{2^{j+2}+2^{j+1}+2^j-1}\lambda_{2^{j+3}-1})$
$$= e_{2^{j+2}+2^{j+1}-1}\lambda^2_{2^{j+3}-1}\lambda_{2^j-1} + e_{2^{j+2}-1}L_1$$

where

$$K_1 = \lambda_{2^{j+3}+2^j-1}\lambda_{2^{j+3}-1} + \lambda_{2^{j+3}-1}\lambda_{2^{j+3}+2^j-1}$$

and

$$L_1 = \lambda_{2^{j+1}+2^j-1}\lambda^2_{2^{j+3}-1} + \lambda_{2^{j+1}-1}K_1.$$

$(2.8.3)^*_{j+3}$ $\qquad \delta\left(\sum_{k=1}^6 B_k\right) = e_{2^{j+3}-1}\lambda_{2^{j+1}+2^j-1} \qquad \lambda^2_{2^{j+2}-1} + e_{2^{j+2}-1}L_2$

where $L_2 \in \Lambda^{3,2^{j+4}-2^j-3}$ and

$$B_1 = e_{2^{j+3}+2^{j+1}+2^j-1}\lambda^2_{2^{j+2}-1},$$
$$B_2 = e_{2^{j+3}+2^j-1}\lambda_{2^{j+1}-1}\lambda_{2^{j+3}-1},$$
$$B_3 = e_{2^{j+2}+2^j-1}\lambda_{2^{j+2}+2^{j+1}-1}\lambda_{2^{j+3}-1},$$
$$B_4 = e_{2^{j+2}+2^{j+1}-1}\lambda_{2^{j+3}+2^j-1}\lambda_{2^{j+2}-1},$$
$$B_5 = e_{2^{j+2}+2^{j+1}-1}\lambda_{2^{j+3}-1}\lambda_{2^{j+2}+2^j-1},$$
$$B_6 = e_{2^{j+2}+2^{j+1}-1}\lambda_{2^{j+3}+2^{j+2}-1}\lambda_{2^j-1}.$$

It is clear from these two equations that $\delta(L_i) = 0$ in Λ, $i = 1, 2$. Since $\mathrm{Ext}_A^{3,2^{j+4}+2^{j+1}+2^j} = 0$ and $\mathrm{Ext}_A^{3,2^{j+4}-2^j} = 0$ it follows that L_i, and therefore $e_{2^{j+2}-1}L_i$, are boundaries and we are done.

This completes the proof of Proposition 2.8 and the whole of Theorem 1.1.

The last piece of work in this section is to prove Theorem 2.3. The chain map

$$\phi \colon \Lambda \to \tilde{H}_*(P) \otimes \Lambda$$

is constructed as follows.

$\phi(1) = 0$. For an admissible monomial $\lambda_I = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}$ define $\phi(\lambda_0) = 0$ and $\phi(\lambda_{i_1}) = e_{i_1}$ for $i_1 \geq 1$ if $s = 1$. If $s \geq 2$ then $\phi(\lambda_I) = 0$ for $i_1 = 0$ (which implies $i_\mu = 0$ for $\mu \geq 2$) and $\phi(\lambda_I)$ for $i_1 \geq 1$ is of the form

$$(*) \qquad\qquad e_{i_1}\lambda_{I'} + \sum_\nu e_{j_1(\nu)}\lambda_{J'(\nu)}$$

where $I' = (i_2, \ldots, i_s)$ and the second sum is described as follows. First we require each $J'(\nu) = (j_2(\nu), \ldots, j_s(\nu))$ be admissible and $J(\nu) = (j_1(\nu), j_2(\nu), \ldots, j_s(\nu))$ be inadmissible. Secondly, choose any large integer m (compared to i_j and s) and let

$$J(\nu, m) = (2^m + j_1(\nu), j_2(\nu), \ldots, j_s(\nu))$$

which is admissible. Then $e_{j_1(\nu)}\lambda_{J'(\nu)}$ appears in the second sum of $(*)$ if and only if, for some $q \geq 2$, $\lambda_{J(\nu,m)}$ appears in the admissible expansion of

$$\lambda_{i_1}\cdots\lambda_{i_{q-1}}\lambda_{i_q+2^m}\lambda_{i_{q+1}}\cdots\lambda_{i_s}.$$

It is proved in [17] that the second sum of $(*)$ is independent of the large integer m chosen and that the map ϕ defined above is a chain map. To complete the proof of Theorem 2.3 it suffices to show $j_1(\nu) < i_1$ for each ν. In the proof for Lemma 3.2 in [17] the following identity is obtained:

$$j_1(\nu) = i_q - i_{q-1} - \cdots - i_1 - \mu_{q-1} - \mu_{q-2} - \cdots - \mu_1 - q + 1$$

for some non-negative integers μ_t. Thus

$$j_1(\nu) \leq i_q - i_{q-1} - \cdots - i_1 - q + 1.$$

Since $(i_1, i_2, \ldots, i_q)$ is admissible, $i_1 \geq 2i_2$, $i_2 \geq 2i_3, \ldots, i_{q-1} \geq 2i_q$ which imply $i_1 \geq i_q/2^{q-1}$, $i_2 \geq i_q/2^{q-2}, \ldots, i_{q-1} \geq i_q/2$. Then

$$
\begin{aligned}
i_1 - j_1(\nu) &\geq i_1 - (i_q - i_{q-1} - \cdots - i_2 - i_1 - q + 1) \\
&= (2i_1 + i_2 + \cdots + i_{q-1}) - i_q + q - 1 \\
&\geq i_q \left(\frac{1}{2^{q-2}} + \frac{1}{2^{q-2}} + \frac{1}{2^{q-3}} + \cdots + \frac{1}{2} \right) - i_q + q - 1 \\
&= i_q - i_q + q - 1 = q - 1 \geq 1 > 0
\end{aligned}
$$

since $q \geq 2$. This proves Theorem 2.3.

3. Proof of Theorem 1.3. We begin by recalling the following result proved in [12].

(3.1)   For each $i \geq 0$ there is a stable map $f_i \colon S^{2^{i+1}} \to P$ such that in the mapping cone $X = P \cup_{f_i} e^{2^{i+1}+1}$ the Steenrod operation

$$
\mathrm{Sq}^{2^{i+1}} \colon H^1(X) = \mathbf{Z}/2 \to H^{2^{i+1}+1}(X) = \mathbf{Z}/2
$$

is non-zero.

This is Theorem 1.3(1). We will infer from this Theorem 1.3(3).

Let $\Phi_{i,i}$ be the secondary cohomology operation associated with the Adem relation

$$
\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i} + \sum_{k=2}^{i-1} \mathrm{Sq}^{2^{i+1}-2^k}\mathrm{Sq}^{2^k} + \mathrm{Sq}^4\mathrm{Sq}^{2^{i+1}-4} = 0, \qquad i \geq 2.
$$

$\Phi_{i,i}$ is defined on any cohomology class of a space on which the Steenrod operations Sq^l are zero for $l \geq 4$. From (3.1) and Adams's Hopf invariant Theorem ([2]) we have the following.

(3.2) $\Phi_{i,i} \colon H^2(X) = \mathbf{Z}/2 \to H^{2^{i+1}+1}(X) = \mathbf{Z}/2$ is non-zero for $i \geq 3$.

PROPOSITION 3.3. *There is a stable map $f \colon \Sigma P \to P$ such that $f^* \colon H^{2k+1}(P) = \mathbf{Z}/2 \to H^{2k+1}(\Sigma P) = H^{2k}(P) = \mathbf{Z}/2$ is an isomorphism, $k \geq 1$.*

There are several ways to see this. Perhaps, the simplest way now is the following. Let $\mathbf{C}P = \mathbf{C}P^\infty$ be the infinite complex projective space. It is proved in [23] that there is a stable map $g \colon \Sigma \mathbf{C}P \to P$ such that $g^* \colon H^*(P) \to H^*(\Sigma \mathbf{C}P)$ is onto. Let $\pi \colon P \to \mathbf{C}P$ be the standard map. Then $f = g \circ (\Sigma \pi) \colon \Sigma P \to P$ has the property in (3.3).

Taking the composite $\varphi_i = f \circ \Sigma f_i \colon S^{2^{i+1}} \to P$ of the maps in (3.1) and (3.3) we see from (3.2) the following.

(3.4)   In the mapping cone $Y = P \cup_{\varphi_i} e^{2^{i+1}+2}$ the secondary cohomology operation

$$\Phi_{i,i} \colon H^3(Y) = \mathbf{Z}/2 \to H^{2^{i+1}+2}(Y) = \mathbf{Z}/2$$

is non-zero for $i \geq 3$.

In the Adams spectral sequence setting this means that φ_i is detected by $\hat{h}_2 h_i^2 \in \mathrm{Ext}_A^{2,*}(P)$. This proves Theorem 1.3(3) for $i \geq 3$. The case $i = 2$ is trivial.

The following theorem is the main topological result of the paper from which the remaining cases of Theorem 1.3 are proved. To state it, let $B(k)$ be the kth 2-primary Brown-Gitler spectrum ([7]). Recall $H^*(B(k)) \cong A/A\{\chi(\mathrm{Sq}^i)|i > k\}$ which has $\{\chi(\mathrm{Sq}^I)|I = (i_1, \ldots, i_s)$ admissible, $i_1 \leq k\}$ a $\mathbf{Z}/2$-base where $\chi \colon A \to A$ is the canonical anti-automorphism of the Steenrod algebra A. In particular $H^l(B(2^i)) = 0$ for $l > 2^{i+1} - 1$ and

(a) $\mathrm{Sq}^1 \colon H^{2^{i+1}-2}(B(2^i)) = \mathbf{Z}/2 \to H^{2^{i+1}-1}(B(2^i)) = \mathbf{Z}/2$

is non-zero.

THEOREM 3.5. *For each $i \geq 1$ there is a stable map $f_i \colon \Sigma^{2^i} B(2^{i-1})$ $\to P$ such that $f_i^* \colon H^{2^{i+1}-1}(P) = \mathbf{Z}/2 \to H^{2^{i+1}-1}(\Sigma^{2^i} B(2^{i-1})) = \mathbf{Z}/2$ is non-zero and is zero in other dimensions.*

We prove this later. To prove (1.3)(2), (4) and (5) we derive from (3.5) a corollary and recall some results from [12], [18] and [22].

Let $X_i = \left[\Sigma^{2^i} B(2^{i-1})\right]^{2^{i+1}-2}$, $\bar{f}_i = f_i|X_i$ and $Y_i = P \cup_{\bar{f}_i} CX_i$. Then there is a short exact sequence

$$0 \leftarrow H^*(P) \xleftarrow{j^*} H^*(Y_i) \xleftarrow{q^*} H^*(\Sigma X_i) \leftarrow 0$$

where $j \colon P \to Y_i$ is the inclusion map and $q \colon Y_i \to \Sigma X_i$ is the pinching map. Recall $H^*(P) = \mathbf{Z}/2[x]$. Since j^* is isomorphism in dimensions $\leq 2^i$, $H^{2^i-1}(Y_i) \cong \mathbf{Z}/2$ is generated by x^{2^i-1}. Let $y_i \in H^{2^{i+1}-1}(Y_i)$ be the image of the non-zero class in $H^{2^{i+1}-1}(\Sigma X_i) = \mathbf{Z}/2$ under q^*.

COROLLARY 3.6. *In $H^*(Y_i)$, $\mathrm{Sq}^{2^i}(x^{2^i-1}) = y_i$ for $i \geq 3$.*

This follows from (a), Theorem 3.5 and Adams's differentials $d_2(\hat{h}_{i+1}) = \hat{h}_i h_i h_0$, $i \geq 3$ in the ASS for P.

We recall the following:

(3.7) ([22]) There is a map $g_1: S^{2^{i+1}} \to \Sigma^{2^i} B(2^{i-1})$ for each $i \geq 1$ such that

$$Sq^2: H^{2^{i+1}-1}\left(\Sigma^{2^i} B\left(2^{i-1}\right) \cup_{g_1} e^{2^{i+1}+1}\right)$$
$$= Z/2 \to H^{2^{i+1}+1}\left(\Sigma^{2^i} B\left(2^{i-1}\right) \cup_{g_1} e^{2^{i+1}+1}\right) = Z/2$$

is non-zero.

(3.8) ([12]) There is a map $g_2: S^{2^{i+1}+1} \to X_i$ for each $i \geq 1$ such that

$$Sq^4: H^{2^{i+1}-2}(X_i \cup_{g_2} e^{2^{i+1}+2})$$
$$= Z/2 \to H^{2^{i+1}+2}(X_i \cup_{g_2} e^{2^{i+1}+2}) = Z/2$$

is non-zero.

(3.9) ([18]) There is a map $g_3: S^{2^{i+1}+18} \to \Sigma^{2^i} B(2^{i-1})$ for each $i \geq 1$ such that the composite

$$S^{2^{i+1}+18} \xrightarrow{g_3} \Sigma^{2^i} B(2^{i-1}) \xrightarrow{p} \Sigma^{2^i} B(2^{i-1})/X_i \simeq S^{2^{i+1}-1}$$

is detected by $c_1 \in Ext_A^{3,*}$ in the Adams spectral sequence for $_2\pi_*^s$.

It follows from (3.5), (3.6), (3.7), (3.8) and (3.9) that the maps

$$f_i \circ g_1: S^{2^{i+1}} \to P,$$
$$\overline{f}_i \circ g_2: S^{2^{i+1}+1} \to P,$$

and

$$f_i \circ g_3: S^{2^{i+18}} \to P,$$

are detected respectively by $\hat{h}_i h_1$ ($i \geq 3$), $\hat{h}_i h_i h_2$ ($i \geq 5$) and $\hat{h}_i c_1$ ($i \neq 3, 4$). This proves Theorem 1.3.

To prove Theorem 3.5, consider the double loop space $\Omega^2 S^3$. There is a 2-primary stable decomposition

$$\Omega^2 S^3 \simeq \bigvee_{k=1}^{\infty} \Sigma^k B\left(\left[\frac{k}{2}\right]\right).$$

This is due to Brown-Peterson, Mahowald and Snaith ([8], [22], [26]) (odd primary analogues are proved in [11]). We will consider a composite

(b) $f: \Omega^2 S^3 \xrightarrow{\gamma} BO \xrightarrow{\sigma} BO(2) \xrightarrow{t} P \times P \xrightarrow{\pi} P$

of stable maps and show that for each $i \geq 1$, $f_i = f|\Sigma^{2^i} B(2^{i-1})$ is the map we want. Here $BO(n)$ is the classifying space for the orthogonal

group $O(n)$, $O = O(\infty)$. Note that $P = BO(1)$. The four maps γ, σ, t and π are described as follows.

γ and π are maps of spaces. π is the first factor projection map (any factor will do). To describe γ, let $\eta: S^1 \to BO$ be a map representing the generator of $\pi_1(BO) = \mathbf{Z}/2$. BO is an infinite loop space. In particular, $BO = \Omega^2 X$ for some space X. Let $\eta^*: S^3 \to X$ be the adjoint of η. Then $\gamma = \Omega^2 \eta^*: \Omega^2 S^3 \to \Omega^2 X = BO$.

V. P. Snaith shows in [27] that there is a stable decomposition

$$BO \simeq \bigvee_{n=1}^{\infty} \frac{BO(2n)}{BO(2n-2)}.$$

Then $\sigma: BO \to BO(2)$ is the projection to the first wedge summand $BO(2)$. $t: BO(2) \to BO(1) \times BO(1) = P \times P$ is the Becker-Gottlieb transfer map to the fibration $O(2)/O(1) \times O(1) \to BO(1) \times BO(1) \xrightarrow{p} BO(2)$ ([5]). Note that σ and t are defined only as stable maps.

To take the composite f in (b) we stabilize γ and π also.

Recall that $H_*(\Omega^2 S^3) \cong \mathbf{Z}/2[\xi_1, \xi_2, \ldots]$, $|\xi_k| = 2^k - 1$ ([10]). Assign to each monomial $\xi_1^{i_1} \cdots \xi_n^{i_n}$ the filtration $\Sigma_{k=1}^{n} i_k 2^{k-1}$. Then the wedge summand $\Sigma^{2^i} B(2^{i-1})$ of $\Omega^2 S^3$ has the set of all monomials of filtration 2^i as a $\mathbf{Z}/2$-base for its homology $H_*(\Sigma^{2^i} B(2^{i-1}))$ ([24]). In particular, $H_{2^{i+1}-2}(\Sigma^{2^i} B(2^{i-1})) = \mathbf{Z}/2$ is generated by ξ_i^2 and $H_{2^{i+1}-1}(\Sigma^{2^i} B(2^{i-1})) = \mathbf{Z}/2$ is generated by ξ_{i+1}. It suffices to show the following.

PROPOSITION 3.10. (1) $f_* = \pi_* t_* \sigma_* \gamma_*: H_*(\Omega^2 S^3) = \mathbf{Z}/2[\xi_1, \xi_2, \ldots] \to H_*(P)$ is zero on monomials of length ≥ 3.

(2) $f_*(\xi_i^2) = 0$.

(3) $f_*(\xi_{i+1}) = e_{2^{i+1}-1}$.

To prove this we recall some facts about the mod 2 cohomology and homology of BO and $BO(2)$.

First, $H^*(BO) = \mathbf{Z}/2[w_1, w_2, \ldots]$ and $H^*(BO(2)) = \mathbf{Z}/2[w_1, w_2]$ where w_i are the universal Stiefel-Whitney classes. If $j: BO(2) \to BO$ is the inclusion map then $j^*(w_i) = 0$ for $i \geq 3$ and $j^*(w_i) = w_i$ for $i = 1, 2$. In particular, j^* is onto. Induced by the loop multiplication $\mu: BO \times BO \to BO$ is the diagonal $\Delta = \mu^*$ in the Hopf algebra $H^*(BO)$

with $\Delta(w_i) = \sum_{k=0}^{i} w_k \otimes w_{i-k}$ $(w_0 = 1)$. Let $\bar{e}_i \in H_i(BO)$ be dual to w_1^i in the monomial basis of $H^*(BO)$. Then $H_*(BO) = \mathbb{Z}/2[\bar{e}_1, \bar{e}_2, \ldots]$. If $H_*(BO(2))$ is identified with $j_*H_*(BO(2)) \subset H_*(BO)$ then $H_*(BO(2)) = \mathbb{Z}/2$-submodule generated by monomials in the $\bar{e}_j$'s of length ≤ 2. The following is easy to prove; it will be needed in the proof of 3.10(2) and 3.10(3) later.

(3.11) $\langle \bar{e}_{2^i-1}^2, W \rangle = 0$ for any monomial $W \in H_*(BO(2))$ except $W = w_2^{2^i-1}$, in which case $\langle \bar{e}_{2^i-1}^2, w_2^{2^i-1} \rangle = 1$.
Snaith shows in [27] the following.

(3.12) The induced map $\sigma_*: H_*(BO) = \mathbb{Z}/2[\bar{e}_1, \bar{e}_2, \ldots] \rightarrow$ $H_*(BO(2))$ of the stable map $\sigma: BO \rightarrow BO(2)$ is the identity map on $H_*(BO(2))$ and is zero on any monomial of length ≥ 3.

This implies 3.10(1) since $\gamma_*: H_*(\Omega^2 S^3) \rightarrow H_*(BO)$ is a ring homomorphism.
In

$$H_*(\Omega^2 S^3) = \mathbb{Z}/2[\xi_1, \xi_2, \ldots], \quad \xi_i = Q_1^{i-1}\xi_1 = \underleftarrow{Q_1 \cdots}_{i-1} \underrightarrow{Q_1}\xi_1$$

for $i \geq 1$ where $Q_1: H_l(\Omega^2 S^3) \rightarrow H_{2l+1}(\Omega^2 S^3)$ is the Dyer-Lashof operation ([10]). By the construction of γ, $\gamma_*(\xi_1) = \bar{e}_1$. Since γ is a double loop map it follows that $\gamma_*(\xi_i) = \gamma_*(Q_1^{i-1}\xi_1) = Q_1^{i-1}\gamma_*(\xi_1) = Q_1^{i-1}\bar{e}_1$. Kochman proves in [16] the following.

(3.13) (1) $\gamma_*(\xi_i) = Q_1^{i-1}\bar{e}_1 \in H_{2^i-1}(BO)$ is the element dual to w_{2^i-1} in the monomial basis of $H^*(BO) = \mathbb{Z}/2[w_1, w_2, \ldots]$ for $i \geq 1$.
(2) $\gamma_*(\xi_i) = Q_1^{i-1}\bar{e}_1 = \bar{e}_{2^i-1} +$ decomposables, $i \geq 1$. (3.12) and (3.13)(2) imply the following.

(3.14) $\sigma_*\gamma_*(\xi_i^2) = \bar{e}_{2^i-1}^2, \quad i \geq 1$.

Recall that $H^*(P) = \mathbb{Z}/2[x]$ and $H^*(P \times P) = \mathbb{Z}/2[x_1, x_2]$. Then $\pi^*(x^{2^{i+1}-2}) = x_1^{2^{i+1}-2}$ and $\pi^*(x^{2^{i+1}-1}) = x_1^{2^{i+1}-1}$. Feshbach shows in [14] that the induced map $p^*t^*: H^*(P \times P) \rightarrow H^*(P \times P)$ of the composite

$$P \times P = BO(1) \times BO(1) \xrightarrow{p} BO(2) \xrightarrow{t} P \times P$$

maps $x_1^{2^{i+1}-2}$ to $x_1^{2^{i+1}-2} + x_2^{2^{i+1}-2}$ and $x_1^{2^{i+1}-1}$ to $x_1^{2^{i+1}-1} + x_2^{2^{i+1}-1}$. The ring homomorphism $p^*: H^*(BO(2)) = Z/2[w_1, w_2] \to H^*(P \times P) = Z/2[x_1, x_2]$ mapping w_1 to $x_1 + x_2$ and w_2 to $x_1 x_2$ is monomorphic with im $p^* = $ the subring consisting of all the symmetric polynomials in x_1 and x_2. One calculates then the following.

$$(3.15) \quad (1) \ t^* \pi^*(x^{2^{i+1}-2}) = \sum_{k=0}^{i-1} w_1^{2^{i+1}-2^{k+2}+2} w_2^{2^{k+1}-2}.$$

$$(2) \ t^* \pi^*(x^{2^{i+1}-1}) = w_1^{2^{i+1}-1} + w_1 w_2^{2^i-1} + \sum_{k=0}^{i-1} w_1^{2^{i+1}-2^{k+1}+1} w_2^{2^k-1}.$$

Note that the monomial $w_2^{2^i-1}$ does not appear in the expansion of $t^* \pi^*(x^{2^{i+1}-2})$ in (3.15)(1). Thus from (3.11) and (3.14) we see $f_*(\xi_i^2) = 0$. This proves 3.10(2).

To prove 3.10(3) we note that the result is trivially true for $i = 0$. So assume $i \geq 1$. We will show the following.

$$(3.16) \quad (1) \ \gamma^* \sigma^*(w_1^{2^{i+1}-1}) = 0.$$

$$(2) \ \gamma^* \sigma^*(w_1 w_2^{2^i-1}) \neq 0.$$

$$(3) \ \gamma^* \sigma^*(w_1^{2^{i+1}-2^{k+1}+1} w_2^{2^k-1}) = 0 \quad \text{for } 1 \leq k \leq i - 1.$$

This together with (3.15)(2) implies $f^*(x^{2^{i+1}-1}) \neq 0$ in

$$H^{2^{i+1}-1}\left(\Sigma^{2^i} B(2^{i-1})\right)$$

and this is 3.10(3).

To show (3.16)(2) and (3) we recall that in $H^*(BO(2))$, $\text{Sq}^1 w_2 = w_1 w_2$. Then

$$\text{Sq}^1(w_2^{2^i-1}) = w_1 w_2^{2^i-1}$$

and

$$\text{Sq}^1(w_1^{2^{i+1}-2^{k+1}} w_2^{2^k-1}) = w_1^{2^{i+1}-2^{k+1}+1} w_2^{2^k-1} \quad \text{for } 1 \leq k \leq i - 1.$$

By (3.11) and (3.14), $\gamma^* \sigma^*(w_2^{2^i-1}) \neq 0$ and $\gamma^* \sigma^*(w_1^{2^{i+1}-2^{k+1}} w_2^{2^k-1}) = 0$ for $1 \leq k \leq i - 1$ in $H^{2^{i+1}}(\Sigma^{2^i} B(2^{i-1})) = Z/2$. Since

$$\text{Sq}^1: H^{2^{i+1}-2}\left(\Sigma^{2^i} B\left(2^{i-1}\right)\right)$$

$$= Z/2 \to H^{2^{i+1}-1}\left(\Sigma^{2^i} B\left(2^{i-1}\right)\right) = Z/2$$

is non-zero it follows that $\gamma^*\sigma^*(w_1 w_2^{2^i-1}) \neq 0$ and

$$\gamma^*\sigma^*(w_2^{2^{i+1}-2^{k+1}+1} w_2^{2^k-1}) = 0 \quad \text{for } 1 \leq k \leq i-1.$$

This proves 3.16(2) and (3).

We prove 3.16(1) by contradiction. Suppose $\gamma^*\sigma^*(w_1^{2^{i+1}-1}) \neq 0$. By (3.12) $\sigma^*(w_1^{2^{i+1}-1}) = w_1^{2^{i+1}-1} + \sum_{l \geq 1} W_l$ where each W_l is a monomial in ker j^*, the ideal of $H^*(BO)$ generated by $w_3, w_4, \ldots$. Since $\gamma^*\sigma^*(w_1^{2^{i+1}-1}) \neq 0$ in $H^{2^{i+1}-1}(\Sigma^{2^i}B(2^{i-1})) = \mathbf{Z}/2$, by 3.13(1) we see one of the monomials W_l must be $w_{2^{i+1}-1}$ (since $2^{i+1} - 1 \geq 3$), say $W_1 = w_{2^{i+1}-1}$. Thus $\sigma^*(w_1^{2^{i+1}-1}) = w_1^{2^{i+1}-1} + w_{2^{i+1}-1} + \sum_{l \geq 2} W_l$ where, for $l \geq 2$, $W_l \neq w_{2^{i+1}-1}$.

Recall that in $H^*(BO)$, $\mathrm{Sq}^1 w_{2n} = w_{2n+1} + w_1 w_{2n}$ ($n \geq 1$) and $\mathrm{Sq}^1 w_{2n+1} = w_1 w_{2n+1}$ ($n \geq 0$). In particular, $\mathrm{Sq}^1 w_1^{2^{i+1}-1} = w_1^{2^{i+1}}$ and $\mathrm{Sq}^1 w_{2^{i+1}-1} = w_1 w_{2^{i+1}-1}$. Also, for any monomial $W \in H^{2^{i+1}-1}(BO)$ other than $w_{2^{i+1}-1}$, $w_1 w_{2^{i+1}-1}$ does not appear in the expansion of $\mathrm{Sq}^1 W$ as sum of monomials. So

$$\mathrm{Sq}^1\sigma^*(w_1^{2^{i+1}-1}) = w_1^{2^{i+1}} + w_1 w_{2^{i+1}-1} + \mathrm{Sq}^1\left(\sum_{l \geq 2} W_l\right) \neq w_1^{2^{i+1}}.$$

On the other hand, since σ^* commutes with the Steenrod operations, we see

$$\begin{aligned}
\mathrm{Sq}^1\sigma^*(w_1^{2^{i+1}-1}) &= \sigma^*(\mathrm{Sq}^1 w_1^{2^{i+1}-1}) = \sigma^*(w_1^{2^{i+1}}) \\
&= \sigma^*(\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1 w_1) \\
&= \mathrm{Sq}^{2^i}\mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1\sigma^*(w_1) \\
&= \mathrm{Sq}^{2^i}\mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1 w_1 = w_1^{2^{i+1}}.
\end{aligned}$$

This contradiction proves 3.16(1).

This completes the proof of Proposition 3.10 and therefore of Theorem 3.5.

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