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The ring R is right FPF if each faithful, finitely generated right R-module is a generator of MOD-R. C. Faith has conjectured that a two sided FPF ring has a self-injective classical ring of quotients. We provide a partial answer to Faith's conjecture by studying Ore localizations and cogeneration properties of right FPF rings R of finite right dimension. Many results from the literature on quotient rings of FPF rings are then reproven.

Introduction. An associative ring with identity R is right FPF (finitely pseudo-Frobenius) if each faithful, finitely generated right R-module is a generator of the category MOD-R of right R-modules. Examples of FPF rings include Dedekind prime rings and PF (pseudo-Frobenius) rings. C. Faith in [**Fa1**, Theorem 5.1] has shown that the classical ring of quotients of a commutative FPF ring is self-injective. He conjectures that each (two sided) FPF ring has a self-injective quotient ring [**Fa1**]. Such a quotient ring exists for FPF rings R which are either Noetherian or semi-perfect. (See [**Fa4**], [**FP**], [**Ft1**], [**Ft2**], [**Pa2**], and [**Pa3**].) However, the techniques employed in these papers are ad hoc. The results of the present paper provide a unified approach to Ore localization in right FPF rings of finite right dimension.

A description of the results follows.

Let R denote a right FPF ring of finite right dimension. We begin by studying those generators which are of minimal dimension in MOD-R, showing that any two such generators are similar. Consequently, each finitely generated, Lambek torsion-free, right R-module is torsionless (Corollary 2.9).

An unexpected consequence of our techniques is Corollary 4.2: If the (right and left) FPF ring R has finite right and finite left dimension, then the maximal right ring of quotients of R equals the maximal left ring of quotients of R. This provides a partial answer to a question raised by Faith. Other results from §4 bring out the connection between the set of right regular elements of R and the various types of localizations of R. Thus the right FPF ring R possesses a right self-injective, semi-perfect, maximal right ring of quotients iff R has finite right dimension and the right regular elements of R are regular, (Theorem 4.3). We then classify those right FPF rings possessing a right PF or right QF ring of quotients. Finally, some results from **[FP]**, **[Ft1]**, **[Ft2]**, **[Pa2]**, **[Pa3]** are revisited in light of the above results.

1. Preliminaries. Unexplained notations and terminologies can be found in the standard reference **[St]**.

We fix the following notation and convention for the remainder of the paper. Until stated otherwise, R denotes a ring of finite right (Goldie) dimension n, E denotes the injective hull of R in the category of right R-modules MOD-R, and the term module means right R-module. Since E has finite right dimension, there are integers $m, n_1, \ldots, n_m > 0$ and direct sum decompositions of nonzero modules

$$(1.1)\qquad \begin{aligned} E &= K_1 \oplus \cdots \oplus K_m, \quad \text{and} \\ K_i &= E_{i1} \oplus \cdots \oplus E_{in_i} \quad \text{for } 1 \leq i \leq m, \end{aligned}$$

such that

$$(1.2)\qquad \text{for all } 1 \leq i, k \leq m,\ 1 \leq j \leq n_i, \text{and } 1 \leq l \leq n_k, \\ E_{ij} \cong E_{kl} \quad \text{iff } i = k.$$

For $1 \leq i \leq m$, we set $E_i = E_{i1}$ and note that E_i is not isomorphic to E_k for $1 \leq i \neq k \leq m$. Let

$$(1.3)\qquad \{\pi_{ij} \colon E \to E_{ij} | 1 \leq i \leq m,\ 1 \leq j \leq n_i\}$$

be the set of projections corresponding to the direct sum decomposition in (1.1), i.e. $\bigoplus_{ij} \pi_{ij} = 1_E$ and for pairs (i, j), $E = E_{ij} \oplus \ker \pi_{ij}$, where $\ker \pi_{ij} = \bigoplus \{E_{kl} | (k, l) \neq (i, j)\}$.

More common convention follows. The singular submodule of the module M is $Z(M) = \{x \in M | r_R(x)$ is an essential right ideal of $R\}$. Then $Z = Z(R)$ is the right singular ideal of R. The injective hull of M is denoted $E(M)$, (or $E(M_R)$ if we wish to emphasise side). Given an ideal I of R, $\mathscr{C}^R(I)$, ($\mathscr{C}_r^R(I)$), is the set of (right) regular elements modulo I. The classical right (left) ring of quotients of R is denoted $Q_c^r(R)$, ($Q_c^l(R)$), while $Q_c(R)$ denotes the classical (left and right) ring of quotients of R. Similarly, we let $Q_m^r(R)$ and $Q_m^l(R)$ denote the maximal right and maximal left ring of quotients of R, respectively.

The ideal I of R is called *classically left localizable* if (i) I is a semi-prime left Goldie ideal, and (ii) $\mathscr{C}^R(I)$ is a left denominator set in R. (See **[St**, Chapter II].) In this case, (i) the localization $R_I = [\mathscr{C}^R(I)^{-1}]\, R$ of R at I is a semi-local ring with Jacobson radical I_I, and (ii) R_I/I_I is canonically the semi-simple classical left ring of quotients of the semi-prime left Goldie ring R/I. Our use of the term "classically localizable ideal" differs from some of the literature on Ore localization in that we do not assume an Artin-Rees property of I. Finally, a ring or ideal property not modified by "right" or "left" is meant to hold on both sides. Thus FPF means right and left FPF, ideal means right and left ideal, etc.

2. Mixed modules and cogeneration. Throughout this section and in addition to the conventions established in §1, *R denotes a right FPF ring* of finite right dimension.

The right R-module M is a *generator of minimal dimension in* MOD-R if M is a generator of MOD-R and if given a generator N of MOD-R, then $\dim(M) \leq \dim(N)$. The right R-module M is called *mixed* if M is not a singular module. i.e. There is an element $x \in M$ such that $r_R(x)$ is not an essential right ideal of R.

In our setting, MOD-R has a generator of minimal dimension since R has finite right dimension. By [**FP**, Theorem 1.2B], the basic module of a semi-perfect ring is a generator of minimal dimension. Examples of mixed modules include non-torsion abelian groups, nonsingular modules, and projective modules. If M is a mixed module and if $M \subset N$ then N is a mixed module. If $M \to N$ is a surjection and if N is a mixed module then M is a mixed module. Thus the class of mixed modules is closed under injective hulls and direct sums.

The following technical lemma illustrates how mixed modules are used in the sequel.

LEMMA 2.1. *Let M be a generator of* MOD-R. *Let $\varphi: N \to L$ be a map in* MOD-R *such that $\varphi(N)$ is a mixed submodule of L.*

(a) *There is a map $\gamma: M \to N$ such that $\varphi\gamma(M)$ is a mixed submodule of L.*

(b) *Assume $M = M_1 \oplus \cdots \oplus M_t$ where each M_i is a uniform module. There is a map $\gamma: M \to N$ and an index i such that the restriction map $\gamma|M_i: M_i \to L$ is an injection.*

Proof. (a) There is no loss of generality in assuming $\varphi(N) = L$. Then, because M is a generator, L is generated by the images of composite maps $M \to N \xrightarrow{\varphi} L$. By hypothesis L is a mixed module, so there is a map $\gamma: M \to N$ such that $\varphi\gamma(M)$ is a mixed submodule of L. (Otherwise L is the sum of singular modules, contrary to hypothesis.)

(b) By part (a) there is a map $\gamma: M \to N$ such that $\varphi\gamma(M) = \sum_{i=1}^{t} \varphi\gamma(M_i)$ is a mixed module. Thus for some index i, $\varphi\gamma(M_i)$

is a mixed module, so that M_i is a mixed uniform module. Since each proper homomorphic image of a uniform module is singular, $\varphi\gamma|M_i\colon M_i \to L$ is an injection. □

In case $\varphi = 1_N\colon N \to N$ then in the notation of (2.1), N contains a copy of one of the M_i.

Before showing that two generators of minimal dimension are similar, we construct a prototype generator.

LEMMA 2.2. *Let E_i be as in* (1.2), $1 \leq i \leq m$. *Each E_i contains a finitely generated, mixed, uniform module G_i. Moreover, $G = G_1 \oplus \cdots \oplus G_m$ is a generator of* MOD-R.

Proof. For each $1 \leq i \leq m$ and $1 \leq j \leq n_i$ let $\varphi_{ij}\colon E_{ij} \to E_i$ be an isomorphism, where the E_{ij} are defined in (1.1), and let $\pi_{ij}\colon E \to E_{ij}$ be the map defined in (1.3). Because $\ker \pi_{ij}$ is not essential in E, $(E_{ij} \neq 0)$, $\pi_{ij}(1)$ is not a singular element of $\pi_{ij}(R)$. Thus $\pi_{ij}(R) \cong \varphi_{ij}\pi_{ij}(R)$ is a mixed submodule of E_i. For fixed i let $G_i = \sum_j \varphi_{ij}\pi_{ij}(R)$. Then G_i is clearly a finitely generated, mixed, uniform submodule of E_i.

We prove that $G = G_1 \oplus \cdots \oplus G_m$ is a generator of MOD-R. Because $\bigoplus_{ij} \pi_{ij} = 1_E$, $R \subset \bigoplus_{ij} \pi_{ij}(R)$, so that

$$\bigcap_{ij} r_R\left(\pi_{ij}\left(R\right)\right) = r_R\left(\bigoplus_{ij} \pi_{ij}\left(R\right)\right) = 0.$$

Thus

$$r_R(G) = \bigcap_{i=1}^{m} r_R(G_i) = \bigcap_{i=1}^{m} \left[\bigcap_{j=1}^{n_i} r_R(\varphi_{ij}\pi_{ij}(R))\right] = \bigcap_{ij} r_R(\pi_{ij}(R)) = 0$$

since the φ_{ij} are isomorphisms. As R is right FPF, G is a generator of MOD-R. □

LEMMA 2.3. *Let G be as constructed in* (2.2).

(a) *If M is a generator of* MOD-R *then G embeds in M. Thus, G is a generator of minimal dimension in* MOD-R.

(b) *Let M be a generator of minimal dimension in* MOD-R. *Then M is similar to G.*

Proof. (a) Let E_i be as in (1.2). By (2.1a), there is a map $\lambda_i\colon M \to E_i$ such that $\lambda_i(M)$ is a mixed module. Note $E_i = E(\lambda_i(M))$. Then by

(2.1b) and (2.2), there is an index k and a map $\gamma_k\colon G \to M$ such that $\lambda_i \gamma_k | G_k$ is an injection. This injection lifts to an isomorphism of the indecomposable injective modules $E_k \cong E_i$. By (1.2), $k = i$. Thus, for each $1 \le i \le m$ there is an injection $\gamma_i\colon G_i \to M$.

The proof of (a) is complete once we have shown that $\sum_i \gamma_i(G_i)$ is direct. The above paragraph forms the basis for an induction. Assume for some value $1 \le k < m$ that we have shown the sum $\sum_{i=1}^k \gamma_i(G_i)$ is direct. Let $K = \gamma_{k+1}(G_{k+1}) \cap \left[\sum_{i=1}^k \gamma_i(G_i)\right]$. Observe that $E(K)$ is a summand of $E(\sum_{i=1}^k \gamma_i(G_i)) = \bigoplus_{1=1}^k E_i$ and the indecomposable injective module $E_{k+1} = E(G_{k+1})$. By the Azumaya-Krull-Schmidt Theorem [St, Corollary V.5.5] either $K = 0$ or $E(K) \cong E_{k+1}$ is isomorphic to E_i for some $1 \le i \le k$. Since (1.2) shows $E_{k+1} \not\cong E_i$ for each $1 \le i \le k$, $K = 0$. It follows that the sum $\sum_{i=1}^{k+1} \gamma_i(G_i)$ is direct, completing the induction. Hence $G \cong \sum_{i=1}^m \gamma_i(G_i)$ is a submodule of M.

(b) By part (a) and the minimality of $\dim(M)$, we may assume G is an essential submodule of M. By (2.1a), there are maps $\gamma_i\colon M \to G_i$ such that $\gamma_i(M)$ is a mixed module. Since $\ker\gamma_i$ is not essential in M and since G is essential in M, $\gamma_i(G)$ is mixed. As in the proof of part (a), the restriction $\gamma_i|G_i\colon G_i \to G_i$ is an injection. Hence $G_i \cap \ker\gamma_i = 0$. Now γ_i lifts to a map $\gamma_i\colon E(G) \to E_i$, and because $G_i \cap \ker\gamma_i = 0$, $E_i \cap E(\ker\gamma_i) = 0$. It is readily seen that $E(G) = E_i \oplus E(\ker\gamma_i)$, so by the Azumaya-Krull-Schmidt Theorem and (1.2) E_i is not isomorphic to a summand of $E(\ker\gamma_i)$. Therefore no E_i appears as a summand of $E(\bigcap_{i=1}^m \ker\gamma_i) \subset \bigcap_{i=1}^m E(\ker\gamma_i)$. Another application of the Azumaya-Krull-Schmidt Theorem shows that $E(\bigcap_{i=1}^m \ker\gamma_i) = 0$. But then $\gamma\colon M \to G$ defined by $\gamma(x) = \gamma_1(x) \oplus \cdots \oplus \gamma_m(x)$ is an injection since $\ker\gamma = \bigcap_{i=1}^m \ker\gamma_i = 0$. Thus M is similar to G. □

Notice (2.3) shows that any generator of minimal dimension in MOD-R embeds in a finitely generated module. The next lemma collects some properties of generators of minimal dimension in MOD-R.

LEMMA 2.4. (a) *Let F be a mixed, indecomposable, injective right R-module. Then $F \cong E_i$ for some $1 \le i \le m$.*

(b) *Let M and N be finitely generated, mixed, uniform right R-modules. Then M is similar to N iff $E(M) \cong E(N)$.*

(c) *For $1 \le i \le m$, let M_i be a finitely generated mixed submodule of E_i. Then $M = \bigoplus_{i=1}^m M_i$ is a generator of minimal dimension in* MOD-R.

(d) *The finitely generated faithful module M is a generator of minimal dimension iff $E(M) \cong \bigoplus_{i=1}^{m} E_i$.*

Proof. (a) By (2.1) and (2.2) there is a map $\gamma\colon G \to F$ and an index i such that $\gamma|G_i$ is an injection. Then $\gamma|G_i$ lifts to an isomorphism of the indecomposable injective modules $E_i = E(G_i) \cong F$.

(b) The only if part is clear. So assume $E(M) \cong E(N)$ and by part (a) there is no loss of generality in assuming $E(M) = E_1$. Then $G \subset G + M$ and M is finitely generated, so $G + M$ is a generator of MOD-R. Note that $G + M = (G_1 + M) \oplus G_2 \oplus \cdots \oplus G_m$. Now by (2.1b), there is an index k and a map $\gamma\colon G + M \to N$ such that $\gamma|G_k$ is an injection. As in the proof of (2.3a), $k = 1$, so that $M \subset G_1 + M$ embeds in N. Similarly, N embeds in M, which proves that M is similar to N.

(c) By part (b), G_i and M_i are similar for each $1 \leq i \leq m$. Thus M is a finitely generated faithful module satisfying $\dim(M) = \dim(G)$. Finally, recall that R is right FPF.

(d) Use (2.3b). □

REMARK. Compare (2.3) and (2.4) to the facts that (i) a basic module M of a semi-perfect ring S is unique up to isomorphism, (ii) M is a summand of each generator of MOD-S, and (iii) M is a generator of minimal dimension in MOD-S. To further illustrate the parallel between basic modules and generators of minimal dimension, we offer the following extensions of [**FP**, Theorems 2.1A, 2.1B].

PROPOSITION 2.5. (a) *Let M be a finitely generated faithful module. Then M is a generator of minimal dimension in* MOD-R *iff $r_R(M/K) \neq 0$ for each nonzero submodule K of M.*

(b) *If R is a generator of minimal dimension in* MOD-R *then R is right strongly bounded. i.e. Each nonzero right ideal of R contains a nonzero ideal of R.*

Proof. (a) Let M be a generator of minimal dimension in MOD-R and let $0 \neq K \subset M$. There is no loss of generality in assuming K is uniform. Then by (2.3b) and the Azumaya-et.al. Theorem [**St**, Corollary V.5.5], $E(K) \cong E_i$ for some $1 \leq i \leq m$. Without loss of generality, we assume $K \subset E_1 \subset E(M) = E_1 \oplus \cdots \oplus E_m$. Let $\pi\colon E(M) \to E_1$ be the projection with kernel $\bigoplus_{j \neq 1} E_j$, and let $\rho = 1_{E(M)} - \pi$. Then $\pi + \rho = 1$ so that $M \subset \pi(M) \oplus \rho(M)$. Hence

$\pi(M) \oplus \rho(M)$ is a generator of minimal dimension in MOD-R. Consider $(\pi(M) \oplus \rho(M))/K \cong \pi(M)/K \oplus \rho(M)$. Since $\pi(M)$ is uniform, $\pi(M)/K$ is singular. Thus $\pi(M)/K$ does not generate the mixed module E_1. Further, $\rho(E(M)) = \bigoplus_{j\neq 1} E_j$ and E_j does not embed in the indecomposable injective module E_1, (1.2). Then each map $E_j \to E_1$ has nonzero kernel and singular image. That is $\rho(E(M))$ does not generate E_1. Because E_1 is injective, $\rho(M)$ does not generate E_1. Hence $\pi(M)/K \oplus \rho(M)$ is not a generator of MOD-R. Since R is right FPF, $\pi(M)/K \oplus \rho(M)$ is not a faithful module. But $\pi(M)/K \oplus \rho(M)$ contains a copy of M/K, so $r_R(M/K) \neq 0$, as required.

Conversely, assume $r_R(M/K) \neq 0$ for each $0 \neq K \subset M$. Let G be the module constructed in (2.2). By (2.3b) we may assume G is a submodule of M. Let $K \subset M$ be such that $G \oplus K$ is an essential submodule of M. Then $M/K \supset (G \oplus K)/K \cong G$ is a faithful module. By hypothesis $K = 0$ and M has minimal dimension m.

(b) Follows from (a). □

We do not know if a generator of minimal dimension in MOD-R need be a projective right R-module. If so, then a right FPF ring of finite right dimension is Morita equivalent to a right strongly bounded ring. This would answer a question posed by Faith in [**FP**]. Also, compare with [**FP**, Theorem 2.1A].

From these techniques, we show that R possesses a cogeneration property.

THEOREM 2.6. *Let R be a right FPF ring of finite right dimension and let E be the injective hull of R in* MOD-R. *Then the finitely generated R-submodules of E embed in R.*

Proof. Let M be a finitely generated submodule of E. There is no loss of generality in assuming $M = M + R$. Let $\pi_{ij}\colon E \to E_{ij}$ be the projection map of (1.3), and for $i = 1, \dots, m$ let $M_i = \pi_{i1}(M)$. The proof is broken up into three steps.

Step 1. $N = \bigoplus_{i=1}^{m} M_i^{n_i}$ embeds in R.

First observe that $M_1 \oplus \cdots \oplus M_m$ is a generator of minimal dimension in MOD-R, (2.4c). We construct an embedding of $N \to R$ by induction on the lexicographically ordered set $\beta = \{(0,0), (i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n_i\}$.

The basis of the induction is to set $M_{00} = F_{00} = 0$ and $0 = \gamma_{00}\colon M_{00} \to R$. Assume for some pair $(0,0) \neq (k,p) \in \beta$ that we have produced a direct sum decomposition

$$(2.7) \qquad E = \left[\bigoplus_{(i,j)<(k,p)} F_{ij} \right] \oplus \left[\bigoplus_{(k,p)\le(i,j)} E_{ij} \right]$$

for some indecomposable injective modules F_{ij}. Further assume we have constructed injections

$$(2.8) \qquad \gamma_{ij}\colon M_i \to R \cap F_{ij} \quad \text{for } (i, j) < (k, p) \in \beta.$$

Now, let $\rho\colon E \to E_{kp}$ be the projection map such that

$$\ker \rho = \left[\bigoplus_{(i,j)<(k,p)} F_{ij} \right] \oplus \left[\bigoplus_{(k,p)<(i,j)} E_{ij} \right].$$

As in the proof of (2.2a), $\rho(R)$ is a finitely generated, mixed, uniform submodule of E_{kp}. By (2.1b), there is a map $\gamma_{kp}\colon M_1 \oplus \cdots \oplus M_m \to R$ and an index l such that $\rho\gamma_{kp}|M_l\colon M_l \to \rho(R)$ is an injection. As in (2.3a), $k = l$ so $\gamma_{kp}|M_k\colon M_k \to R$ is an injection.

Let $F_{kp} = E(\gamma_{kp}(M_k))$ and note that $F_{kp} \cap \ker \rho = 0$. We then have the direct sum decomposition of injective modules

$$E = F_{kp} \oplus \ker \rho = \left[\bigoplus_{(i,j)\le(k,p)} F_{ij} \right] \oplus \left[\bigoplus_{(k,p)<(i,j)} E_{ij} \right]$$

and injections

$$\gamma_{ij}\colon M_i \to R \cap F_{ij} \quad \text{for } (i, j) \le (k, p) \in \beta,$$

as required by (2.7) and (2.8). This completes the induction on β.

Therefore $E = \bigoplus_\beta F_{ij}$ and $\bigoplus_\beta \gamma_{ij}(M_i) \subset R$. Now for fixed i we have n_i pairs (i, j) in β. Thus $\bigoplus_\beta \gamma_{ij}(M_i) \cong \bigoplus_{i=1}^m M_i^{n_i} = N$ embeds in R, which completes Step 1.

Step 2. For fixed $i, M_i^{n_i}$ is similar to $\bigoplus_{j=1}^{n_i} \pi_{ij}(M)$.

Recall that $R \subset M$, so for $1 \le j \le n_i$ $\pi_{ij}(M)$ is a finitely generated, mixed, uniform module. Since $\pi_{ij}(M) \subset E_{ij} \cong E_i$ and since $M_i = \pi_{i1}(M)$, (2.4b) shows M_i is similar to $\pi_{ij}(M)$ for $1 \le j \le n_i$. Thus, $M_i^{n_i}$ is similar to $\bigoplus_{j=1}^{n_i} \pi_{ij}(M)$, completing Step 2.

Step 3. M embeds in R.

Since $\bigoplus_\beta \pi_{ij} = 1_E$ (1.3) and by Step 2 there are embeddings

$$M \to \bigoplus_{i=1}^{m} \left[\bigoplus_{j=1}^{n_i} \pi_{ij}(M) \right] \to \bigoplus_{i=1}^{m} M_i^{n_i} = N.$$

By Step 1, N embeds in R, so M embeds in R. This completes the proof of the theorem. □

This section closes with some consequences of (2.6). For the purposes of the next several results, call a module M *Lambek torsion-free* if M is a submodule of a product of copies of E. (See [**St**, page 149].)

COROLLARY 2.9. *Each finitely generated Lambek torsion-free module is torsionless.*

Proof. Say $M \subset \prod_I E = P$ for some index set I. By (2.6), to each $i \in I$ and projection map $\pi_i\colon P \to E$, there is an embedding $\varphi_i\colon \pi_i(M) \to R$. Hence M embeds in $\prod_I \varphi_i \pi_i(M) \subset \prod_I R$. That is, M is torsionless. □

We remark that over a right PF (injective cogenerator) ring, each right module is Lambek torsion-free and torsionless, [**FP**, Theorem 1.7A].

COROLLARY 2.10. *Let L be a submodule of E. Assume that for each of the projection maps π_{ij} in* (1.3), *the module $\pi_{ij}(L)$ is mixed. Then each finitely generated submodule of E embeds in L.*

Proof. Use the proof of (2.6) verbatim. □

COROLLARY 2.11. *Let I be a right ideal of R.*

(a) *Assume R/I is a Lambek torsion-free module.* (*i.e. Assume I is closed in the topology of dense right ideals on R.*) *Then $I = r_R(X)$ for some set $X \subset R$.*

(b) *I is dense in R iff $l_R(I) = 0$.*

Proof. (a) R/I is Lambek torsion-free iff R/I embeds in a product of copies of E. By (2.10) R/I is torsionless. Now argue as in [**Fa3**, Lemma 20.26].

(b) I is dense in R iff $\mathrm{Hom}_R(R/I, E) = 0$ iff $\mathrm{Hom}_R(R/I, R) = 0$ (2.6) iff $\mathrm{trace}_R(R/I) = Rl_R(I) = 0$ iff $l_R(I) = 0$. □

Note (2.11b) extends [**Fa4**, page 162, both corollaries].

PROPOSITION 2.12. (a) *E is a flat right R-module and each finitely generated submodule of E is contained in a free submodule of E.* (*M. Finkel-Jones* [**FJ**] *refers to such modules as f-projective.*)

(b) *There are finitely generated, mixed, uniform right ideals $I_1, \ldots, I_n$ and a right regular element c of R such that $cR \subset \bigoplus_{i=1}^n I_i$.*

Proof. (a) By (2.6), each finitely generated submodule $M \subset E$ embeds in R. Assume without loss of generality that $R \subset M$. Then an embedding $\varphi\colon M \to R$ lifts to an embedding $\varphi\colon E \to E$. Since E has finite dimension, $E = \varphi(E)$, which shows that φ is an automorphism. Hence $M \subset \varphi^{-1}(R) \cong R$ and since E is then a union of free modules, E is flat.

(b) Recall the projections π_{ij} from (1.3). Because $\bigoplus_{ij} \pi_{ij} = 1_E$, $R \subset \bigoplus_{ij} \pi_{ij}(R)$, and by part (a), $\bigoplus_{ij} \pi_{ij}(R) \subset xR \cong R$ for some element $x \in E$. Given an isomorphism $\varphi\colon xR \to R$, relabel the set $\{\varphi\pi_{ij}(R) | i, j\}$ as $\{I_1, \ldots, I_n\}$ and let $c = \varphi(1)$. It is readily verified that these choices satisfy the proposition. $\square$

Observe that the argument in (2.12a) can be used to rephrase (2.6) as follows. To each finitely generated submodule M of E, there is an automorphism φ of E such that $\varphi(M) \subset R$. We use this restatement of (2.6) without fanfare in §§3 and 4.

3. Preliminaries for localization. In this section, we investigate the set $\mathscr{C}_r^R(0)$ of right regular elements of R. These results are prerequisite to our discussion of classical localization in right FPF rings contained in §4.

In addition to the convention and notation established in previous sections, R is a right FPF ring of finite right dimension. Recall that Z denotes the right singular ideal of R. We fix an embedding $R \to E$ and denote the image of 1 by 1. Then let

$$\begin{aligned}
\Lambda &= \operatorname{End}_R(E), \\
L_1 &= l_\Lambda(1), \\
R_1 &= \{\lambda \in \Lambda | \lambda(R) \subset R\} = \{\lambda \in \Lambda | \lambda(1) \in R\}, \\
Z_1 &= \{\lambda \in \Lambda | \lambda(1) \in Z\} = R_1 \cap J(\Lambda),
\end{aligned}$$

where the last equality holds since $J(\Lambda)$ is the set of maps with essential kernel, [St, Proposition XIV.1.1]. Further, we set

$$\mathscr{C} = \{\varphi \in \Lambda | \varphi(1) \in \mathscr{C}_r^R(0)\} = \{\varphi \in R_1 | \ker \varphi = 0\}.$$

The following facts about the above sets are well known or easily proven. Λ is a semi-perfect ring, [St, Proposition XIV.1.7]. R_1 is a subring of Λ, and Z_1 is an ideal of R_1. Further, $L_1 = l_{R_1}(1) \subset Z_1$, and L_1 is an ideal of R_1. We consider E as a Λ-R-bimodule, and so as an R_1-R-bimodule. When possible, we avoid the use of parentheses in the action of Λ on E. As left Λ-modules, $E = \Lambda 1 \cong \Lambda/L_1$. $\mathscr{C}$ is a set of units of Λ, so that $\mathscr{C} \subset \mathscr{C}^{R_1}(0)$.

LEMMA 3.1. (a) *The evaluation map* $\xi\colon \Lambda \to E$ *with* $\xi(\lambda) = \lambda 1$ *induces a ring surjection* $\xi\colon R_1 \to R$ *and an isomorphism* $R_1/Z_1 \cong R/Z$.
(b) $\mathscr{C} = \mathscr{C}^{R_1}(Z_1) \subset \mathscr{C}^{R_1}(0)$.
(c) $\mathscr{C}$ *is a left Ore set in* R_1.
(d) Z_1 *is a classically left localizable ideal of* R_1 *and* $(R_1)_{Z_1} = \Lambda$.

Proof. (a) The definitions of ξ and R_1 show that $\xi(R_1) = R$, $\xi(Z_1) = Z$, and $\ker\xi = L_1 \subset Z_1$. Because L_1 is an ideal of R_1, ξ is a ring surjection which induces the isomorphism $R_1/Z_1 \cong R/Z$.

(b) As noted above, $\mathscr{C} \subset \mathscr{C}^{R_1}(0)$ since $\mathscr{C}$ is a set of units of Λ. Thus $\mathscr{C}$ maps onto a set of units of $\Lambda/J(\Lambda)$. Because

$$R_1/Z_1 = R_1/(R_1 \cap J(\Lambda)) \cong (R_1 + J(\Lambda))/J(\Lambda) \subset \Lambda/J(\Lambda)$$

we have $\mathscr{C} \subset \mathscr{C}^{R_1}(Z_1)$. On the other hand, given $\varphi \in \mathscr{C}^{R_1}(Z_1)$ we claim $\varphi + Z_1$ is a unit of $\Lambda/J(\Lambda)$.

(3.2) Suppose $\lambda \in \Lambda$ is such that $\lambda\varphi \in J(\Lambda)$. By (2.6) there is an automorphism $\gamma \in \Lambda$ of E such that $\gamma(R + \lambda R) \subset R$. Then $\gamma 1 \in \mathscr{C}_r^R(0)$ and $\gamma\lambda 1 \in R$ so that $\gamma \in \mathscr{C}$ and $\gamma\lambda \in R_1$. But then $(\gamma\lambda)\varphi \in R_1 \cap J(\Lambda) = Z_1$. Since $\varphi \in \mathscr{C}^{R_1}(Z_1)$, $\gamma\lambda \in Z_1 \subset J(\Lambda)$. Because γ is a unit of Λ, $\lambda \in J(\Lambda)$, which proves that $\varphi + Z_1$ is left regular in the semi-simple ring $\Lambda/J(\Lambda)$. Then as claimed, $\varphi + Z_1$ is a unit of $\Lambda/J(\Lambda)$.

Since units lift modulo $J(\Lambda)$, φ is a unit of Λ and hence $\varphi \in \mathscr{C}$. This proves $\mathscr{C} = \mathscr{C}^{R_1}(Z_1)$.

(c) Note (3.2) proves that to each $\lambda \in \Lambda$ there is a $\gamma \in \mathscr{C}$ such that $\gamma\lambda \in R_1$. Now for $\varphi \in \mathscr{C}$, $\Lambda/R_1 \cong \Lambda\varphi/R_1\varphi = \Lambda/R_1\varphi$, so that for each $\varphi \in \mathscr{C}$ and $\lambda \in R_1$ there is a $\gamma \in \mathscr{C}$ such that $\gamma\lambda \in R_1\varphi$. i.e. $\mathscr{C}$ is a left Ore set in R_1.

(d) By parts (b) and (c) $\mathscr{C} = \mathscr{C}^{R_1}(Z_1)$ is a left Ore set of regular elements of R_1. Thus the left ring of quotients $[\mathscr{C}^{-1}]\, R_1$ exists. Further, the proof of part (c) shows that Λ/R_1 is a $\mathscr{C}$-torsion left R_1-module. Since $\mathscr{C}$ is a set of units of Λ, the universal property of localization proves that $\Lambda \cong [\mathscr{C}^{-1}]\, R_1$. Finally, because Λ is a semi-perfect ring, Z_1 is a classically left localizable ideal of R_1. □

COROLLARY 3.3. (a) *R/Z is a semi-prime left Goldie ring.*
(b) $\mathscr{C}_r^R(0) = \mathscr{C}^R(Z)$.
(c) *$\mathscr{C}_r^R(0)$ is a left Ore set in R.*

Proof. Use (3.1a) and (3.1d) to prove (a). By (3.1a), $\xi(\mathscr{C}) = \mathscr{C}_r^R(0)$ and $\xi(Z_1) = Z$ so that (b) and (c) follow from (3.1b) and (3.1c). □

We have not claimed that Z is a classically left localizable ideal of R since this requires that $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$. This line of discussion is developed in §4. For the present, we content ourselves with a non-Ore localization. Consider the filter of left ideals

$$\text{(3.4)} \qquad \mathscr{F} = \{\text{left ideals } I \subset R \mid I \cap \mathscr{C}_r^R(0) \neq \varnothing\}.$$

Because $\mathscr{C}_r^R(0)$ is a left Ore set in R, the linear topology on R generated by $\mathscr{F}$ is a 1-topology. (See [St, section XI.6].) Further, R is an $\mathscr{F}$-torsion-free left R-module since each $I \in \mathscr{F}$ contains a right regular element. Therefore, the localization $R_{\mathscr{F}}$ exists and by [St, Proposition IX.2.4],

$$\text{(3.5)} \qquad \begin{aligned} R_{\mathscr{F}} &= \{x \in E({}_RR) \mid Ix \subset R \text{ for some } I \in \mathscr{F}\} \\ &= \{x \in E({}_RR) \mid cx \in R \text{ for some } c \in \mathscr{C}_r^R(0)\}, \end{aligned}$$

where $E({}_RR)$ is the injective hull of the left R-module R.

LEMMA 3.6. *Let $\mathscr{F}$ be the filter of left ideals defined in* (3.4).
(a) $R_{\mathscr{F}} = E \cap E({}_RR) = \{x \in E \mid L_1 \subset l_{R_1}(x)\}$.
(b) *$\mathscr{F}$ generates a perfect left topology on R iff $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$.*
(c) *$\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$ iff $\Lambda = E$. In this case, $R_1 = R$, $Z_1 = Z$, and $\mathscr{C} = \mathscr{C}^R(0) = \mathscr{C}^R(Z)$.*

Proof. (a) The largest R_1-submodule of $E({}_{R_1}R)$ possessing a natural action by R is

$$E({}_RR) = \{x \in E({}_{R_1}R) \mid L_1 \subset l_{R_1}(x)\},$$

[Fa3, Proposition 19.12]. Now by (3.1d) the left R_1-module R possesses a left module of quotients $[\mathscr{C}^{-1}]\, R \subset E({}_{R_1}R)$. Indeed, because $\xi(\mathscr{C}) = \mathscr{C}_r^R(0)$,

$$R_{\mathscr{F}} \subset \left[\mathscr{C}^{-1}\right] R \cong \Lambda \otimes_{R_1} R \cong \Lambda \otimes_{R_1} (R_1/L_1) \cong \Lambda/L_1 = E,$$

so by (3.5) $R_{\mathscr{F}} \subset E \cap E(_R R)$. But then

$$R_{\mathscr{F}} \subset E \cap E(_R R) \subset \{x \in E | L_1 \subset l_{R_1}(x)\}$$
$$\subset \{x \in E(_R R) | cx \in R \text{ for some } c \in \mathscr{C}_r^R(0)\} = R_{\mathscr{F}}.$$

(b) If $\mathscr{F}$ generates a perfect left topology on R, then $R_{\mathscr{F}} = R_{\mathscr{F}} Rc = R_{\mathscr{F}} c$ for each $c \in \mathscr{C}_r^R(0)$. Then each right regular element of R is left invertible in $R_{\mathscr{F}}$. By part (a), $R_{\mathscr{F}} \subset E$ has finite right dimension, so $\mathscr{C}_r^R(0)$ is a set of units of $R_{\mathscr{F}}$, **[St**, Lemma XV.5.4]. Thus $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$.

Conversely, if $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$ then $\mathscr{C}^R(0)$ is a left denominator set in R, (3.3c). By **[St**, Proposition XI.6.4], $\mathscr{F}$ generates a perfect left topology on R.

(c) The "if" part is clear from (3.1). Assume $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$ and claim L_1 is an ideal of Λ. By (2.6)

$$\{x \in E | xc = 0 \text{ for some } c \in \mathscr{C}^R(0)\} = 0.$$

(xR embeds in R and R is $\mathscr{C}^R(0)$-torsion-free.) Now L_1 is an ideal of R_1, so for $\varphi \in \mathscr{C}$, $L_1 = L_1\varphi\varphi^{-1} \subset L_1\varphi^{-1} = l_\Lambda(\varphi 1)$. But then $l_\Lambda(\varphi 1)/L_1$ is a subset of $E = \Lambda/L_1$ annihilated by $\varphi 1 \in \xi(\mathscr{C}) = \mathscr{C}_r^R(0) = \mathscr{C}^R(0)$. Thus $l_\Lambda(\varphi 1) \subset L_1$ which implies $L_1 = L_1\varphi^{-1}$. But then $\Lambda = [\mathscr{C}^{-1}] R_1$ (3.1d) so L_1 is an ideal of Λ, as claimed. Finally, because E is a faithful left Λ-module, $L_1 = 0$. Part (c) follows from (3.1). □

4. Localizations of FPF rings. We maintain the notations and conventions established in the previous sections with the exception that *R need not have finite right dimension.*

The results of this section are partial answers to a question raised by Faith and Page in **[Fa4**, page 187, problem 15] and **[FP**, pages 0.1–0.3]. Namely, for right FPF rings R when is $Q_m^r(R)$ an Ore localization? When is $Q_m^r(R)$ right self-injective? When does $Q_c^r(R)$ exist? Faith conjectures that for FPF rings R, $Q_c(R)$ exists and is self-injective. These questions have been settled for commutative FPF rings and for various permutations of the hypotheses left and right Noetherian, semi-perfect, and nonsingular. (See **[Fa4]**, **[FP]**, **[Bu]**, **[Pa2]**, **[Pa3]**, **[Ft1]**, **[Ft2]**.) In this section we will show that the injectivity of the maximal or classical ring of quotients is tied to the regularity condition $\mathscr{C}(0) = \mathscr{C}_r(0)$. This regularity condition is then used to investigate various kinds of self-injective quotient rings of R. Also, several results from the literature are revisited in light of new classifications.

We begin by showing that $Q_m^r(R)$ is also a left localization of R.

THEOREM 4.1. *Let R be a right FPF ring of finite right dimension, and let $\mathscr{F}$ be the filter of left ideals defined in* (3.4). *Then $R_{\mathscr{F}} = Q^r_m(R) \subset Q^l_m(R)$.*

Proof. Recall that $Q^r_m(R) = Q$ is the maximal rational extension of R. Then for $\lambda \in \Lambda$, $\lambda R = 0$ iff $\lambda Q = 0$, so that $Q = \{x \in E \mid L_1 \subset l_{R_1}(x)\}$. By (3.6a), $Q = R_{\mathscr{F}}$. Further, R is $\mathscr{F}$-torsion-free as a left R-module, so the elements of $\mathscr{F}$ are dense left ideals of R. (See **[St**, page 200, Example 1].) Thus $R_{\mathscr{F}} \subset Q^l_m(R)$. □

COROLLARY 4.2. *Let R be an FPF ring which has finite right and finite left dimension. Then $Q^r_m(R) = Q^l_m(R)$.* □

We do not have examples of right FPF rings for which $Q^r_m(R)$ is not a left Ore localization of R. Also, it is not known if $Q^r_m(R)$ is semi-local, whereas all known examples of localizations of finite dimensional FPF rings are semi-local. Semi-local perfect localizations of general rings are classified in **[FZ]** and **[Fa3**, page 54–55, Proposition 18.47]. Commutative FPF rings R possessing semi-local $Q_c(R)$ are classified in **[Fa1**, page 90].

The next result points out the connection between the regularity condition and the injective property.

THEOREM 4.3. *Let R be a right FPF ring. Then $Q^r_m(R)$ is a semi-perfect right self-injective ring iff R has finite right dimension and $\mathscr{C}^R_r(0) = \mathscr{C}^R(0)$. In this case, Z is a classically left localizable ideal of R and $R_Z = Q^l_c(R) = Q^r_m(R)$.*

Proof. Let $Q^r_m(R) = Q$. If Q is semi-perfect and right self-injective then R has finite right dimension by **[St**, Proposition XIV.4.3]. Recall that right regular elements in a semi-perfect right self-injective ring are units. Since R is right essential in Q, $\mathscr{C}^R_r(0) \subset \mathscr{C}^Q_r(0) = \mathscr{C}^Q(0)$, so $\mathscr{C}^R_r(0) = \mathscr{C}^R(0)$.

Conversely, assume R has finite right dimension and that $\mathscr{C}^R_r(0) = \mathscr{C}^R(0)$. Then by (3.1d) and (3.6c), $Z_1 = Z$ is a classically left localizable ideal of $R_1 = R$ and $E = \Lambda = R_Z = Q^l_c(R)$ is a semi-perfect ring. Also, $Q = \mathrm{End}_\Lambda(E) = \mathrm{End}_\Lambda(\Lambda) = \Lambda$. Then by **[St**, Proposition XIV.4.1], $Q = E$ is also right self-injective, which proves the theorem. □

Faith conjectures **[FP**, 0.1, Problem 6] that for FPF rings R, $Q_c(R)$ exists and is self-injective. In this case, $Q_c(R) = Q^r_m(R) = Q^l_m(R)$, as in (4.2). Observe that (4.3) partially resolves this conjecture. For FPF rings there is a significant simplification of (4.3) which provides a new insight to Faith's conjecture.

COROLLARY 4.4. *Let R be an FPF ring. Then R possesses a semi-perfect, self-injective, classical ring of quotients iff the left singular ideal of R equals the right singular ideal of R, and R has finite left dimension and finite right dimension.*

Proof. If $Q_c(R) = Q$ exists and is self-injective, then $J(Q)$ is the left singular ideal and the right singular ideal of Q, **[St**, Corollary XIV.1.3]. Because R is essential in Q, $R \cap J(Q)$ is the left singular ideal and right singular ideal of R. Further, R has finite dimension by (4.3). Conversely, if Z denotes the common singular ideal of R then $\mathscr{C}_r^R(0) = \mathscr{C}^R(Z) = \mathscr{C}_l^R(0) = \mathscr{C}^R(0)$ by (3.3b). Apply (4.3) to complete the proof. □

REMARK. Burgess **[Bu]** has shown that if $Z = 0$ then $Q_c(R)$ exists, while **[Pa1]** and **[FP]** show that a semi-prime right Goldie right FPF ring is left Goldie. We do not know of an example of an FPF ring in which the right singular ideal is not the left singular ideal.

In view of (4.4) it is natural to ask when Z is a classically localizable ideal of R.

COROLLARY 4.5. *The following are equivalent for a right FPF ring R.*

(a) *R possesses a semi-perfect, right self-injective, classical right ring of quotients.*

(b) *R has finite right dimension and Z is a classically localizable ideal in R.*

(c) *R possesses a classical right ring of quotients, R has finite right dimension, and $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$.*

(d) *R and R/Z have finite right dimension, $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$, and for $c, d \in \mathscr{C}^R(0)$, there are regular $c', d' \in \mathscr{C}^R(0)$ such that $cc' = dd'$.*

If R satisfies any of the above statements, then $Q_c(R)$ exists and $R_Z = Q_c(R) = Q_m^r(R)$.

Proof. (a) ⇒ (b) By (a) and **[St**, Proposition XIV.4.1], $E = Q_c^r(R) = Q_m^r(R)$. By (4.3) Z is a classically left localizable ideal in R, and by (3.3b) $\mathscr{C}^R(Z) = \mathscr{C}_r^R(0) = \mathscr{C}^R(0)$. Then $\mathscr{C}^R(Z)$ is a right denominator set in R, and Z is a classically localizable ideal in R.

(b) $\Rightarrow$ (c) By (b) and (3.3c) $\mathscr{C}^R(Z) = \mathscr{C}_r^R(0)$ is a denominator set in R. Hence $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$ and $R_Z = Q_c(R)$. This is (c).

(c) $\Rightarrow$ (d) Since $\mathscr{C}^R(0)$ is a right Ore set in R, it remains to prove that R/Z has finite right dimension. From (3.3b) and hypotheses $\mathscr{C}^R(Z) = \mathscr{C}_r^R(0) = \mathscr{C}^R(0)$ is a right denominator set in R, so $R_Z = Q_c^r(R)$. But then R_Z/Z_Z is canonically the semi-simple classical right ring of quotients of R/Z. By Goldie's Theorem, R/Z has finite right dimension.

(d) $\Rightarrow$ (a) By (4.3) $Q_m^r(R) = Q_c^l(R) = R_Z = Q$ is a semi-perfect right self-injective ring. Thus it suffices to show that $\mathscr{C}^R(0)$ is a right Ore set in R. Let $x \in R$ and $c \in \mathscr{C}^R(0) = \mathscr{C}^R(Z)$, (3.3b). By (3.3a) and hypotheses, R/Z is a semi-prime Goldie ring, so there are $c' \in \mathscr{C}^R(Z)$ and $x' \in R$ such that $xc' - cx' \in Z$. Now let $z = xc' - cx'$. Then by (3.3b) $z + c \in \mathscr{C}^R(Z) = \mathscr{C}^R(0)$. From (d) there are $c'', d'' \in \mathscr{C}^R(0)$ such that $(z+c)c'' = cd''$. Hence $zc'' = (z+c)c'' - cc'' = cd'' - cc'' = c(d'' - c'')$. But then $(xc' - cx')c'' = c(d'' - c'')$ implies $x(c'c'') = c(x'c'' + d'' - c'')$. This proves $\mathscr{C}^R(0)$ is a right Ore set in R, as required to end the proof. □

Remark. Observe that the proof of (d) $\Rightarrow$ (a) above is similar to that of [**Fa3**, Proposition 18.47] where Faith classifies those rings having semi-local Ore localizations. Other papers considering localization at the singular ideal of an FPF ring include [**Ft1**, **Ft2**, **Pa1**, **Pa2**, **Bu**] and [**Fa4**, Theorem 9B].

Since FPF rings are generalizations of PF and QF rings, it is natural to ask when the associated quotient rings of FPF rings are PF or QF. Right FPF rings with right PF localizations are quite symmetric.

Theorem 4.6. *The following are equivalent for a right FPF ring R.*

(a) *$Q_m^r(R)$ is a right PF ring.*

(b) *$Q_c^r(R)$ exists and is a right PF ring.*

(c) *R has finite right dimension, $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$, and each dense right ideal of R contains a right regular element.*

(d) *R has finite right dimension and a right ideal I of R is dense in R iff I contains a regular element of R.*

If R satisfies any of the above statements, then R has finite right and finite left dimension, $\mathscr{C}_r^R(0) = \mathscr{C}_l^R(0) = \mathscr{C}^R(0)$ is an Ore set in R, and $Q_c(R) = Q_m^r(R) = Q_m^l(R)$.

Proof. (b) $\Rightarrow$ (a) In general $Q_c^r(R) \subset Q_m^r(R) \subset E$. By (b) and [**St**, Proposition XV.5.2], $E = Q_c^r(R)$. This proves (a).

(a) $\Rightarrow$ (c) A right PF ring is semi-perfect and right self-injective, [**FP**, Chapter 1]. Then by (4.3), $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$ and $E = Q_m^r(R) = Q_c^l(R) = Q$. Now if I is a dense right ideal of R then $l_R(I) = 0 = l_R(IQ)$. Recall the projection of (1.3) and observe that because $Q = \mathrm{End}_R(E)$ the projections π_{ij} are idempotents of Q. Therefore $\pi_{ij}IQ = \pi_{ij}Q$ is a mixed right R-submodule of Q. Since R is right essential in Q, $\pi_{ij}I$ is a mixed right R-submodule of $Q = E$. By (2.10) R embeds in I. The image of 1 under this embedding is the (right) regular element we seek.

(c) $\Rightarrow$ (d) The "if" direction is part (c). Conversely, if I contains a (right) regular element of R then $l_R(I) = 0$. By (2.11b), I is dense in R.

(d) $\Rightarrow$ (b) By (d), cR is a dense right ideal of R for each $c \in \mathscr{C}_r^R(0)$. Hence $l_R(cR) = 0$ implying c is regular. i.e. $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$. Further, because cR is dense in R, for each $x \in R$ there is a dense right ideal I, and so an element $d \in \mathscr{C}^R(0)$, such that $xd \in xI \subset cR$. Thus $\mathscr{C}^R(0)$ is a right Ore set in R. But then $Q_c^r(R)$ exists and is semi-perfect right self-injective, (4.5c). Now in this case, the set of dense right ideals of R generates a perfect right topology on R, [**St**, Proposition XI.6.3], so [**St**, Proposition XI.5.2] shows $Q_c^r(R)$ is a cogenerator ring. i.e. $Q_c^r(R)$ is a right PF ring. This proves the equivalence of the statements (a) thru (d).

Now assume R satisfies any one (and hence all) of the above statements. By (4.3) and (4.5) R is left essential in $Q_m^r(R) = Q_c^l(R) = Q_c(R) = Q$. Because a right PF ring has finite left and finite right dimension [**FP**, Chapter 1] R has finite left and finite right dimension. Now by (4.1) $Q \subset Q_m^l(R)$. Because a right PF ring is its own maximal left ring of quotients, $Q = Q_m^l(R)$. Finally, for $c \in \mathscr{C}_l(0)$, $l_R(cR) = 0$, so by (2.11b) cR is a dense right ideal of R. By (4.6b) $cRQ = cQ = Q$, so that c is right invertible in the semi-perfect ring Q. Thus c is a unit of Q and $c \in \mathscr{C}^R(0)$. This completes the proof of the theorem. □

REMARK. [**Ft1**, page 94] determines those commutative FPF rings with PF classical ring of quotients. The regularity condition in (4.3), (4.5), and (4.6) appears in [**Pa2**] and [**Ft2**] (for localizations in semi-perfect right FPF rings) and in [**Ft1**] where Beachy's extension [**Be**] of Small's Theorem [**Sm**] is used to investigate localization in Noetherian FPF rings.

We now improve upon several results from the literature. S. Endo **[En]** initially proved that a commutative Noetherian FPF ring possesses a QF classical ring of quotients. Attempts to extend Endo's theorem to the noncommutative case include **[FP**, Chapter 5] where the semi-perfect, Noetherian, FPF rings are considered, **[Pa2]** where a QF quotient ring is produced using Krull dimension techniques, and **[Ft1]** where it is shown that an FPF ring R is an order in a QF ring iff R has the acc and dcc on right annihilators and his finite left and finite right reduced ranks. (See **[Be]** for the definition of reduced rank.) The latest in this series of results is

COROLLARY 4.7. *The following are equivalent for a right FPF ring R.*

(a) *R possesses a QF classical right ring of quotients.*
(b) *R possesses a right Artinian classical right ring of quotients.*
(c) *R is a Goldie ring.*
(d) *R is a right Goldie ring with the acc on left annihilators.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (d) and (a) $\Rightarrow$ (c) $\Rightarrow$ (d) are clear. For (d) $\Rightarrow$ (a) recall that a left Ore set of right regular elements in a ring with the acc on left annihilators is a set of regular elements. Then by (3.3c) and (4.3), $Q_m^r(R) = Q$ is a right self-injective classical left ring of quotients of R. Since R has the acc on left annihilators, so does Q. Thus Q is a QF ring and by (4.6a), (a) holds. □

The quotient ring results from **[En]**, **[Ft1]**, **[FP]**, and **[Pa3]** concerning Noetherian FPF rings extend as follows.

COROLLARY 4.8. *A Noetherian right FPF ring possesses a QF classical ring of quotients.* □

REMARK. The literature is rich in papers concerning orders in QF rings. (See **[Fa3**, page 222] for references.) We feel (4.7) and (4.8) stand apart from these references due to the constraints we place on the ring R.

With regard to semi-prime right FPF rings of finite right dimension, (3.3a) shows that these rings are left Goldie. It is known that semi-prime left Goldie rings have the acc on right annihilators, so a semi-prime right FPF ring of finite right dimension is actually left and right Goldie. This agrees with **[FP**, Corollary 3.1B].

A semi-perfect right FPF ring R has finite right dimension by [**FP**, Theorem 2.1A]. Thus, most of the material here will apply to such rings. The common result from [**Ft2**], [**FP**, Chapter 2], and [**Pa2**] shows that R possesses a right self-injective classical right ring of quotients iff $\mathscr{C}_r^R(0) = \mathscr{C}^R(0)$. For more general semi-perfect right FPF rings there is

PROPOSITION 4.9. *Let R be a semi-perfect right FPF ring. Then the classical left ring of quotients exists and $Q_m^r(R) = Q_c^l(R)$.*

Proof. Let $Q = Q_m^r(R)$ and recall the filter $\mathscr{F}$ defined in (3.4). By (4.1) $R_{\mathscr{F}} = Q$. Now using the proof of [**Ft2**, Lemma 3.2c] we can prove that R is a left order in $R_{\mathscr{F}}$. Let $\mathscr{C} = \{c \in \mathscr{C}_r^R(0) | R_{\mathscr{F}}c = R_{\mathscr{F}}\}$. Claim $\mathscr{C} = \mathscr{C}^R(0)$. By [**St**, Theorem XI.2.1, Proposition XI.6.4] $R_{\mathscr{F}} = [\mathscr{C}^{-1}]R$. Thus $\mathscr{C} \subset \mathscr{C}^R(0)$. On the other hand, $E = Ec$ for each $c \in \mathscr{C}^R(0)$ and $\{x \in E | xc = 0\} = 0$. (xR embeds in R by (2.6).) Hence right multiplication by c is an automorphism of E. i.e. c is a unit of $Q = \mathrm{End}_\Lambda(E)$, implying $\mathscr{C} = \mathscr{C}^R(0)$. Thus $Q = Q_c^l(R)$. □

COROLLARY 4.10. *For a semi-perfect FPF ring R, the classical ring of quotients exists and $Q_c(R) = Q_m^l(R) = Q_m^r(R)$.*

Proof. Use the left/right symmetry of (4.9) and (4.1). □

5. Some open questions.

(5.1) In a right FPF ring R (of finite right dimension) are the right regular elements regular?

(5.2) In an FPF ring R is the right singular ideal the left singular ideal?

(5.3) Let R be a right FPF ring of finite right dimension. We echo Faith and Page: Do the rings $Q_c^r(R)$, $Q_c^l(R)$ exist? Are the rings $Q_m^r(R)$ and $Q_m^l(R)$ Ore localizations of R? Which of these four rings is self-injective?

(5.4) We expand upon (5.3). Let R be a right FPF ring (of finite right dimension). Let $Q = Q_m^r(R)$. Are finitely presented right Q-modules torsionless? Equivalently, if M is a finitely generated right Q-submodule of Q^n is M then rationally closed in Q^n? Equivalently, is $Q_m^r(R)$ a left FP-injective ring [**Ja**]? See (2.6) in this regard.

(5.5) Let R be a right FPF ring of finite right dimension and let M and N be generators of minimal dimension in MOD-R. Is M a projective right R-module? Is there an integer k such that $M^k \cong N^k$? (i.e. Are M and N of the same genus, for some reasonable definition of genus?) Is M a genus summand of each generator of MOD-R?

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