Pacific Journal of Mathematics

A LOCALIZED ERDŐS-WINTNER THEOREM

P. D. T. A. Elliott

Vol. 135, No. 2 October 1988

PACIFIC JOURNAL OF MATHEMATICS
Vol. 135, No. 2, 1988

A LOCALIZED ERDÖS-WINTNER THEOREM

P.D.T.A. ELLIOTT

In this paper I show that a form of the well-known Erdős-Wintner theorem for additive arithmetic functions holds, even if the information is only given on widely separated intervals.

For $y \geq x \geq 2$ let

$$\nu_{x,y}(n; f(n) \leq z) \tag{1}$$

denote the frequency amongst the integers n in the interval $(x-y, x]$, of those for which the real additive function $f(n)$ does not exceed z.

THEOREM. *Let $c > 1$. Let N_j be an increasing sequence of positive integers for which $N_{j+1} \leq N_j^c$. Let M_j be a further sequence of integers, $M_j \leq N_j$, $\log M_j / \log N_j \to 1$, as $j \to \infty$.*

In order that the frequencies

$$\nu_{N_j, M_j}(n; f(n) \leq z) \tag{2}$$

converge weakly, as $j \to \infty$, it is necessary and sufficient that the three series

$$\sum_{|f(p)|>1} \frac{1}{p}, \quad \sum_{|f(p)|\leq 1} \frac{f(p)}{p}, \quad \sum_{|f(p)|\leq 1} \frac{f(p)^2}{p} \tag{3}$$

converge.

When $N_j = j$, $M_j = j$ this is the well-known theorem of Erdös, Erdös and Wintner [5]. For $N_j = j$ and any M_j which satisfies $M_j/N_j \to 0$, together with the above condition $\log M_j \sim \log N_j$, it was proved by Hildebrand [7].

The present argument differs from theirs in many respects.

2. Preliminary results. It is convenient to introduce the Lévy-distance $\rho(F, G)$ between distributions $F(z)$ and $G(z)$ on the line, defined as the greatest lower bound of those real h for which

$$F(z-h) - h \leq G(z) \leq F(z+h) + h$$

for all z. Convergence in the topology which this induces on the space of distribution functions, is equivalent to the usual weak-convergence of measures.

For primes $p \le x$ let Y_p be independent random variables distributed according to

$$Y_p = \begin{cases} f(p^\alpha) \text{ with probability } \dfrac{1}{p^\alpha}\left(1 - \dfrac{1}{p}\right), & 0 \le \alpha < \gamma_p, \\ f(p^{\gamma_p}) \text{ with probability } \dfrac{1}{p^{\gamma_p}} & \end{cases}$$

where $\gamma_p = [\log x / \log p]$.

Let

$$G_x(z) = P\left(\sum_{p \le x} Y_p \le z\right),$$

and let $F_x(z)$ denote the frequency distribution function (1).

LEMMA 1. *There is a positive absolute constant c so that*

$$\rho(F_x, G_x) \le c\left(\sum_{\substack{y^\varepsilon < q \le y \\ |f(q)| > u}} \frac{1}{q} + \frac{u}{\varepsilon} + \exp\left(-\frac{1}{80\varepsilon}\log\frac{1}{\varepsilon}\right) + \frac{1}{\log y} + \frac{\log\frac{x}{y}}{\log x}\right)$$

holds uniformly for all $u > 0$, $x \ge y \ge x^{2/3} \ge 3$, $x^\varepsilon \ge (\log x)^3$, $0 < \varepsilon \le 1$, and $f(q)$, where q denotes a prime-power.

Proof. Inequalities of this type are obtained in Elliott [**1**] Chapter 12, [**2**] Lemma 6. In the main they depend upon the application of a finite probability model constructed with the aid of Selberg's sieve method. The necessary background results can be found in Elliott [**1**], Chapter 3.

For an arithmetic function g, $M(g, x)$ will denote

$$\sum_{n \le x} g(n).$$

For real α, g_α will denote the modified arithmetic function $n \mapsto g(n)n^{i\alpha}$.

LEMMA 2. *Let g be a complex-valued multiplicative function, $|g(n)| \le 1$ for positive n; and $x \ge y \ge 3$. Then*

$$M(g, x) - M(g, x - y) = \frac{M(g_\alpha, x)}{x}\int_{x-y}^{x} t^{-i\alpha}\, dt + O(yR(x, y))$$

where α is any real number, $|\alpha| \le x$, for which

$$|M(g_\alpha, x)| = \max_{|\beta| \le x} |M(g_\beta, x)|$$

and

$$R(x, y) = \left(\log \frac{\log 2x}{\log 2x/y}\right)^{-1/4}$$

Proof. This is Theorem 4 of Hildebrand [7].

LEMMA 3. *In the notation of Lemma 2, define the Dirichlet series*

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

Then

$$M(g, x) \ll x \left(T^{-1} + \frac{1}{\log x} \max_{|\tau| \le T} \left| G\left(1 + \frac{1}{\log x} + i\tau\right)\right|\right)^{1/5}$$

uniformly in all multiplicative functions g with $|g(n)| \le 1$, and in x, $T \ge 2$.

Proof. This result is due essentially to Halász [6], a detailed proof may be found in Elliott [1], Lemma (6.10).

LEMMA 4. *If*

$$\operatorname{Re} \sum_{p \le x} p^{-1}(1 - p^{i\lambda}) \ll 1$$

for some real λ, $|\lambda| \le x$, then $\lambda \ll (\log x)^{-1}$.

Proof. If $\delta = 1 + 1/\log x$, then the hypothesis of this lemma asserts that the Riemann-function $\zeta(s)$ satisfies

$$\log \left| \frac{\zeta(\delta)}{\zeta(\delta + i\lambda)} \right| \ll 1$$

uniformly in $x \ge 3$. The conclusion now follows from application of the bounds

$$\zeta(\sigma + it) = \begin{cases} \dfrac{1}{\sigma + it - 1} + O(1) & \text{if } \sigma > 1, |t| \le 2, \\ O((\log |t|)^{2/3}) & \text{if } \sigma > 1, |t| > 2, \end{cases}$$

the proofs of which may be found in Ellison and Mendès-France [4].

LEMMA 5. *Let the bounded function u, defined on the interval* $[-1, 1]$, *satisfy*

$$|u(t_1 + t_2) - u(t_1) - u(t_2)| \le K$$

whenever t_1, t_2 *and* $t_1 + t_2$ *belong to the interval. Then*

$$|u(t) - u(1)t| \le 3K.$$

Proof. This is established in Ruzsa [**9**]. It extends an earlier result of Hyers [**8**].

LEMMA 6. *Suppose that for a sequence of real numbers* α_n *the limit (as* $n \to \infty$*) of* $\exp(it\alpha_n)$ *exists uniformly on some open interval of real t-values including* $t = 0$. *Then* $\lim \alpha_n$ *exists (finitely).*

Proof. (Cf. Elliott and Ryavec [**3**].) Since $(e^{it\alpha_n})^2 = \exp(i2t\alpha_n)$, we see that the hypothesis holds on every bounded set of t-values. Here $\exp(it\alpha_n)$ is the characteristic function of the improper distribution function $H_n(z)$ which has a jump at the point α_n. It follows from a standard theorem in the theory of probability that the $H_n(z)$ converge weakly to a distribution function $J(z)$, say.

It is now not difficult to deduce that the α_n are bounded uniformly for all n, that $J(z)$ is itself improper, with a jump at β, say; and that $\alpha_n \to \beta$ as $n \to \infty$.

LEMMA 7. *Let* $P_j(x)$ *be polynomials in* x *with complex coefficients, and* d_j *distinct real numbers,* $j = 1, \dots, k$. *If*

$$\theta(t) = \sum_{j=1}^{k} P_j(t)e^{id_jt} = 0$$

on a proper interval of real t-values, then the polynomials are identically zero.

Proof. Without loss of generality $0 = d_1 > d_2 > \cdots > d_k$. As a function of the *complex-variable* t, $\theta(t)$ is everywhere analytic. After the hypothesis, analytic continuation shows that $\theta(t)$ is identically zero. We set $t = -iy$ for real y, and consider

$$\lim_{y\to\infty} y^{-m}\theta(-iy)$$

where m is the degree of P_1.

The terms $P_j(-iy)\exp(d_jy)$ with $j \ge 2$ converge exponentially to zero, whilst $y^{-m}P_1(-iy)$ approaches $(-i)^m$ times the coefficient of x^m

in P_1. Since the value of this limit is zero, $P_1(x)$ is identically zero. An argument by induction completes the proof of the lemma.

3. Proof of the theorem: (3) *implies* (2). Define independent random variables Z_p by

$$Z_p = \begin{cases} Y_p & \text{if } Y = f(p), \\ 0 & \text{otherwise.} \end{cases}$$

The convergence of the three series at (3) is precisely Kolmogorov's condition that the series $Z_2 + Z_3 + \cdots$ be almost surely convergent. Moreover,

$$\sum_p P(Z_p \neq Y_p) \leq \sum_p \sum_{m=2}^{\infty} \frac{1}{p^m} < \infty,$$

so that by the Borel-Cantelli lemma, $Y_2 + Y_3 + \cdots$ is also almost surely convergent. This is equivalent to the weak convergence of the distribution functions $G_x(z)$ appearing in Lemma 1. The relevant background results from the theory of probability may be found in Elliott [1], Lemma (1.18).

We apply Lemma 1 with $x = N_j$, $y = M_j$. Since the series $\sum p^{-1}$ taken over those primes p for which $|f(p)| > u$ converges for each positive u,

$$\limsup_{j\to\infty} \rho(F_{N_j}, G_{N_j}) \leq c\left(\frac{u}{\varepsilon} + \exp\left(-\frac{1}{80\varepsilon}\log\frac{1}{\varepsilon}\right)\right)$$

for all $u > 0$, $0 < \varepsilon < 1$. Letting $u \to 0+$, $\varepsilon \to 0+$ we obtain the weak convergence of the frequencies (2).

In this direction no restriction upon the rate of growth of the N_j need be assumed.

4. Proof of the theorem: (2) *implies* (3). The characteristic function of a typical frequency (2) is given by

$$\phi_j(t) = M_j^{-1} \sum_{N_j - M_j < n \leq N_j} g(n),$$

where $g(n) = \exp(itf(n))$ is a multiplicative function, and t is real. If the frequencies (2) converge weakly to a distribution function with characteristic function $\phi(t)$, then by a standard result in the theory of probability, $\phi_j(t) \to \phi(t)$ as $j \to \infty$, uniformly on any bounded interval of t-values.

If we temporarily use x, y to denote N_j, M_j respectively, then it follows from Lemma 2 that

$$\phi(t) = x^{-1}M(g_\alpha, x)y^{-1}\int_{x-y}^{x} v^{-i\alpha}\,dv + o(1), \quad x \to \infty, \tag{4}$$

for some real α, $|\alpha| \le x$. Since $\phi(t)$ is continuous in t, and $\phi(0) = 1$, there is a proper interval $|t| \le \tau$, on which $|\phi(t)| \ge 1/2$. On this same interval $|M(g_\alpha, x)| \ge x/4$ for all sufficiently large x $(= N_j)$. The parameter α may depend upon both t and x.

Applying Lemma 3 with $T = \log x$ gives

$$M(g_\alpha, x) \ll x \exp\left(-\frac{1}{5} \operatorname{Re} \sum_{p \le x} \frac{1 - g(p) p^{i\psi}}{p}\right) + x(\log x)^{-1/5}$$

for some real ψ, $|\psi(x) - \alpha| \le \log x$. Thus $|\psi(x)| \le x + \log x$. In view of the lower bound for $|M(g_\alpha, x)|$

$$\operatorname{Re} \sum_{p \le x} \frac{1 - g(p) p^{i\psi}}{p} \ll 1.$$

We first show that $\psi = \psi(t)$ is essentially linear in t.

Let

$$S(f) = \sum_{p \le x} p^{-1} \left(\operatorname{Sin} \frac{f(p)}{2}\right)^2.$$

Then since $|\operatorname{Sin}(a + b)| \le |\operatorname{Sin} a| + |\operatorname{Sin} b|$,

$$S(f_1 + f_2) \le 2(S(f_1) + S(f_2)). \tag{5}$$

With $g(p) = \exp(itf(p))$,

$$\begin{aligned} \operatorname{Re}(1 - g(p) p^{i\psi}) &= \operatorname{Re}(1 - \exp(i(tf(p) + \psi(t) \log p))) \\ &= 2\left(\operatorname{Sin} \frac{1}{2}(tf(p) + \psi(t) \log p)\right)^2 \end{aligned}$$

so that

$$S(tf + \psi(t) \log) \ll 1$$

uniformly for $|t| \le \tau$.

In view of the inequality (5), whenever $|t_j| \le \tau$, $j = 1, 2$, $|t_1 + t_2| \le \tau$,

$$S((\psi(t_1 + t_2) - \psi(t_1) - \psi(t_2)) \log) \ll 1,$$

so that by Lemma 4

$$\psi(t_1 + t_2) - \psi(t_1) - \psi(t_2) \ll (\log x)^{-1}.$$

We can now apply Lemma 5, to deduce that

$$\psi(t) = t\psi(\tau)/\tau + O((\log x)^{-1}).$$

Then

$$\sum_{p\le x}\frac{1}{p}|p^{i\psi(t)}-p^{it\psi(\tau)/\tau}|\le|\psi(t)-t\psi(\tau)/\tau|\sum_{p\le x}\frac{\log p}{p}\ll 1$$

uniformly for $|t|\le\tau$. Thus

$$S(t(f-\omega(x)\log))\ll 1 \tag{6}$$

holds, uniformly for $|t|\le\tau$, for some function $\omega(x)$ of x alone.

Up until this point the proof has followed Elliott [2]. The relative sizes of the N_j now comes into play.

For all sufficiently large integers j, the interval $(2^{c^j},2^{c^{j+1}}]$ contains at least one member, r_j say, of the sequence of N_i. Since $r_{j+2}\ge r_j^c$, by induction

$$\frac{\log r_m}{\log r_n}\ge(\sqrt{c})^{m-n-1} \tag{7}$$

for all $m\ge n\ge$ (some fixed) n_0.

From their definition $r_{m+1}\le r_m^{c^2}$. By an elementary estimate from number theory

$$\sum_{r_m<p\le r_{m+1}}\frac{1}{p}=\log\left(\frac{\log r_{m+1}}{\log r_m}\right)+O((\log r_m)^{-1})\ll 1,$$

so that

$$\operatorname{Re}\sum_{p\le r_m}\frac{1}{p}(1-g(p)p^{it\omega})\ll 1$$

holds for both $\omega=\omega(r_m)$, and $\omega=\omega(r_{m+1})$. Another application of Lemma 4 yields

$$|\omega(r_{m+1})-\omega(r_m)|\le\frac{D}{\log r_m}$$

for some D and all positive m.

Employing our lower bound (7), an argument by induction shows that

$$|\omega(r_m)-\omega(r_n)|\le\frac{2D}{\log r_n}\sum_{n<k\le m}c^{-(k-n-1)/2} \tag{8}$$

uniformly for $m\ge n\ge n_0$. In particular the $\omega(r_m)$ form a Cauchy sequence, and converge to a limit, A say. Letting $m\to\infty$ in (8) gives

$$\omega(r_n)-A\ll(\log r_n)^{-1}$$

for $n\ge n_0$.

Since every large enough N_j lies in an interval $(r_m, r_{m+1}]$,

$$\omega(N_j) - A \ll (\log N_j)^{-1}$$

for all j. In the way that we replaced $\psi(t)$ by $t\psi(\tau)/\tau$ we replace $\omega(N_j)$ by A, to obtain

$$S(t(f - A\log)) \ll 1$$

uniformly for $|t| \le \tau$, for all sufficiently large (underlying) N_j.

Again we argue as in Elliott [2]. Let d denote $\pi/|\tau|$. The inequality $|\operatorname{Sin}\theta| \ge 2|\theta|/\pi$ holds for $|\theta| \le \pi/2$. With $h(p) = f(p) - A\log p$, $x = N_j$, we deduce that

$$\frac{\tau^2}{\pi^2} \sum_{\substack{p\le x \\ |h(p)|\le d}} \frac{|h(p)|^2}{p} \le S(\tau h) \ll 1.$$

Moreover,

$$\left(1 - \frac{1}{\pi}\right) \sum_{\substack{p\le x \\ |h(p)|> d}} \frac{1}{p} \le \sum_{p\le x} \frac{1}{p}\left(1 - \frac{\operatorname{Sin}\tau h(p)}{\tau h(p)}\right)$$
$$= \frac{1}{2\tau}\int_{-\tau}^{\tau} S(th)\,dt \ll 1.$$

Together these inequalities imply the convergence of the series

$$\sum_{|h(p)|>u} \frac{1}{p}, \quad \sum_{|h(p)|\le u} \frac{h(p)^2}{p} \tag{9}$$

for each positive u. We shall use this to estimate $M(g_\alpha, x)$ for all large x, whether of the form N_j or not.

Let

$$\mu(x) = \sum_{\substack{p\le x \\ |h(p)|\le 1}} \frac{h(p)}{p}.$$

If $x^{1/2} \le w \le x$, $u > 0$,

$$|\mu(x) - \mu(w)| \le \sum_{\substack{w<p\le x \\ |h(p)|>u}} \frac{1}{p} + u \sum_{\substack{w<p\le x \\ |h(p)|\le u}} \frac{1}{p}$$
$$= o(1) + O\left(u\log\left(\frac{\log x}{\frac{1}{2}\log x}\right)\right)$$

as $x \to \infty$. Since u may be chosen arbitrarily small, $\mu(x) - \mu(w) \to 0$ as $x \to \infty$, uniformly for $x^{1/2} \le w \le x$.

In the same way that the convergence of the three series (3) implies the weak convergence of the distribution functions $G_x(z)$, the convergence of the two series at (9) implies the weak convergence of

$$P\left(\sum_{p \le x} Z_p - \mu(x) \le z\right),$$

where the random variables Z_p are defined like the Y_p, but with $f(p^\alpha)$ everywhere replaced by $f(p^\alpha) - A \log p^\alpha$.

Another application of Lemma 1, this time with $y = x$, and to the function $f(n) - A \log n$, shows that

$$\nu_{x,x}(n; f(n) - A \log n - \mu(x) \le z) \Rightarrow H(z), \qquad x \to \infty,$$

for some distribution function $H(z)$. If $h(t)$ is the characteristic function of $H(z)$, we can express this last assertion in the form of the asymptotic estimate:

$$x^{-1} M(g_{-A}, x) e^{-it\mu(x)} \to h(t), \qquad x \to \infty,$$

uniformly on every bounded set of t-values.

An integration by parts shows that

$$M(g_\alpha, x) = x^{i(\alpha + At)} M(g_{-A}, x) - i(\alpha + At) \int_{1-}^{x} v^{i(\alpha+At)-1} M(g_{-A}, v)\, dv.$$

The integral term is small. In fact, from our hypothesis (4) (with $x = N_j$),

$$\operatorname{Re} \sum_{p \le x} p^{-1}(1 - g(p)p^{i\alpha}) \ll 1,$$

and we have shown that a similar relation holds with α replaced by $-At$. Arguing with the function S (as earlier), we see that $\alpha + At \ll (\log x)^{-1}$, $x = N_j$. Thus as x $(= N_j) \to \infty$,

$$M(g_\alpha, x) = x h(t) \exp(i(\alpha + At) \log x + it\mu(x)) + o(x).$$

Combining this result with that of (4),

$$e^{it(\mu(x) + A \log x)} \left(\frac{1 - (1 - y/x)^{1-i\alpha}}{(1 - i\alpha) y/x}\right) \to \frac{\phi(t)}{h(t)}, \qquad x \to \infty, \tag{10}$$

uniformly on a proper interval $|t| \le t_0$. Here $x = N_j$, $y = M_j$.

Suppose now that for a sequence of j-values, $M_j/N_j \to \rho$. Then for this sequence of values the coefficient of the exponential at (10) converges to

$$\rho^{-1}(1-(1-\rho)^{1+iAt}) \text{ if } \rho \neq 0; \quad 1+iAt \text{ if } \rho = 0.$$

This convergence is uniform on some bounded interval of t-values which includes $t = 0$. Here we have again applied the estimate $\alpha + At \ll (\log x)^{-1}$. It follows from this and an application of Lemma 6, that on this same sequence of j-values, $\beta(\rho) = \lim(\mu(x) + A\log x)$ exists. Moreover, for all sufficiently small t,

$$e^{it\beta(\rho)}\rho^{-1}(1-(1-\rho)^{1+iAt}) = \phi(t)h(t)^{-1}$$

if $\rho > 0$, with a similar (modified) relation if $\rho = 0$.

We next show that the value of $\beta(\rho)$ does not depend upon ρ.

Assume that for an interval of real t-values

$$\rho_1^{-1}e^{it\beta_1}(1-(1-\rho_1)^{1+iAt}) = \rho_2^{-1}e^{it\beta_2}(1-(1-\rho_2)^{1+iAt}), \tag{11}$$

where each ρ_j is positive and < 1. Suppose that $\beta_1 \neq \beta_2$. Then $A \neq 0$, and the coefficient of $e^{it\beta_2}$ on the right-hand side is ρ_2^{-1}. It follows from Lemma 7 that

$$\beta_2 = \beta_1 + A\log(1-\rho_1), \qquad \beta_1 = \beta_2 + A\log(1-\rho_2),$$

which is impossible. A similar argument may be made when the restrictions upon the values of ρ_1, ρ_2 are removed.

We have now proved that

$$\lim_{j\to\infty}(\mu(N_j) - A\log N_j)$$

exists, the variable j running through all positive integers. By an elementary estimate

$$|\mu(N_j)| \leq \sum_{p\leq N_j}\frac{1}{p} \ll \log\log N_j,$$

so that $A\log N_j \ll \log\log N_j$ for all j, and $A = 0$. A look back at (11) shows that $A = 0$ removes the possibility of comparing the values of ρ_1 and ρ_2.

Thus the series

$$\sum_{|f(p)|>1}\frac{1}{p}, \quad \sum_{|f(p)|\leq 1}\frac{f(p)^2}{p}$$

converge, and

$$\lim_{j\to\infty} \sum_{\substack{p\le N_j \\ |f(p)|\le 1}} \frac{f(p)}{p}$$

exists. Since every sufficiently large real w lies in an interval $(N_j, N_{j+1}]$, and (now with $A = 0$) $\mu(N_{j+1}) - \mu(w) \to 0$ as $j \to \infty$, uniformly for $N_j < w \le N_{j+1}$, the series

$$\sum_{|f(p)|\le 1} \frac{f(p)}{p}$$

also converges.

The proof of the theorem is complete.

References

[1] P.D.T.A. Elliott, *Probabilistic Number Theory*, Grund. der math. Wiss., **239** (1979), **240** (1980), Springer-Verlag, New York, Heidelberg, Berlin.

[2] ———, *Localised value-distribution of additive arithmetic functions*, J. für die reine und ang. Math., **379** (1987), 64–75.

[3] P.D.T.A. Elliott and C. Ryavec, *The distribution of the values of additive arithmetical functions*, Acta Math., **126** (1971), 143–164.

[4] W. J. Ellison and M. Mendès-France, *Les Nombres Premiers*, Hermann, Paris, 1975.

[5] P. Erdös and A. Wintner, *Additive arithmetical functions and statistical independence*, Amer. J. Math., **61** (1939), 713–721.

[6] G. Halász, *On the distribution of additive arithmetic functions*, Acta Arith., **27** (1975), 143–152.

[7] A. Hildebrand, *Multiplicative functions in short intervals*, Canad. J. Math., **39** (1987), 646–672.

[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224.

[9] I. Z. Ruzsa, *On the concentration of additive functions*, Acta Math. Acad. Sci. Hungar., **36** (1980), 215–232.

Received May 22, 1986 and in revised form March 15, 1988. Partially supported by N.S.F. contract DMS-8500949.

University of Colorado
Boulder, CO 80309

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$190.00 a year (5 Vols., 10 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) publishes 5 volumes per year. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 135, No. 2 October, 1988

Waleed A. Al-Salam and Mourad Ismail, q-beta integrals and the q-Hermite polynomials 209
Johnny E. Brown, On the Ilieff-Sendov conjecture 223
Lawrence Jay Corwin and Frederick Paul Greenleaf, Spectrum and multiplicities for restrictions of unitary representations in nilpotent Lie groups 233
Robert Jay Daverman, 1-dimensional phenomena in cell-like mappings on 3-manifolds 269
P. D. T. A. Elliott, A localized Erdős-Wintner theorem 287
Richard John Gardner, Relative width measures and the plank problem 299
F. Garibay, Peter Abraham Greenberg, L. Reséndis and Juan José Rivaud, The geometry of sum-preserving permutations 313
Shanyu Ji, Uniqueness problem without multiplicities in value distribution theory 323
Igal Megory-Cohen, Finite-dimensional representation of classical crossed-product algebras 349
Mirko Navara, Pavel Pták and Vladimír Rogalewicz, Enlargements of quantum logics 361
Claudio Nebbia, Amenability and Kunze-Stein property for groups acting on a tree 371
Chull Park and David Lee Skoug, A simple formula for conditional Wiener integrals with applications 381
Ronald Scott Irving and Brad Shelton, Correction to: "Loewy series and simple projective modules in the category $\mathcal{O}_S$" 395
Robert Tijdeman and Lian Xiang Wang, Correction to: "Sums of products of powers of given prime numbers" 396