Pacific Journal of Mathematics

FINITE-DIMENSIONAL REPRESENTATION OF CLASSICAL CROSSED-PRODUCT ALGEBRAS

IGAL MEGORY-COHEN

Vol. 135, No. 2 October 1988

PACIFIC JOURNAL OF MATHEMATICS
Vol. 135, No. 2, 1988

FINITE DIMENSIONAL REPRESENTATION OF CLASSICAL CROSSED-PRODUCT ALGEBRAS

IGAL MEGORY-COHEN

The paper describes the structure of finite dimensional representations of B_T, the crossed-product algebra of a classical dynamical system $(\alpha_T, \mathbb{Z}, C(X))$ where T is a homeomorphism on a compact space X. The results are used to describe the topology of $\mathrm{Prim}_n(B_T)$ and to partially classify the hyperbolic crossed-product algebras over the torus. One of the main results is that the number of orbits of any fixed length with respect to T is an invariant of B_T. A consequence of that is that the entropy of T is an invariant of B_T, for T a hyperbolic automorphism on the m-torus.

Introduction. The purpose of this paper is to study finite dimensional representations of classical crossed-product algebras. The results are used to describe the primitive ideal space of these algebras and partially classify them. The first two sections deal primarily with finite dimensional representations of B_T, the crossed-product algebra B_T of a classical dynamical system of the form $(\alpha_T, \mathbb{Z}, C(X))$ where T is a homeomorphism on a compact space X. In §1 we study the general form of an irreducible n-dimensional representation of B_T. We show how to adjoin an orbit of length n to each such representation. The idea of adjoining an orbit to each finite dimensional representation is then further explored in §2. We show that the number of connected components in $\mathrm{Prim}_n(B_T)$ is equal to the number of orbits of length n with respect to T. A consequence of this result is that the entropy of T, for T a hyperbolic automorphism on $\mathbf{T}^m$, is an invariant of B_T. In §3 we investigate the classification of the B_T's corresponding to automorphisms on the 2-torus.

Preliminaries. For any integer n we define $E_n\colon B_T \to C(X)$ to be the (continuous) transformation that takes C in B_T to its nth "Fourier" coefficient f_n, see [1] for details. Symbolically, we write each C in B_T as $\sum f_n U^n$ where $f_n = E_n(C)$. Let $(\hat{\alpha}, \mathbf{T}, B_T)$ be the C^*-dynamical system defined by the dual action $\hat{\alpha}_\lambda(C) = \sum \lambda^n U^n$, [2]. It is known that the Fejer sums of the function $\lambda \to \hat{\alpha}_\lambda(C)$ converge uniformly to

$\hat{\alpha}_\lambda(C)$, see [3] for an elementary proof. In other words,

$$\sum_{|k|<N} \left(1 - \frac{|k|}{N}\right) f_k U^k \lambda^k \to \hat{\alpha}_\lambda(C)$$

uniformly in λ, and in particular for $\lambda = 1$,

$$\sum_{|k|<N} \left(1 - \frac{|k|}{N}\right) f_k U^k \to C.$$

1. Finite dimensional representations of B_T.

NOTATION. Let Y be a subset of X. Then by J_Y we denote the closed ideal in B_T generated by $\{f \in C(X); f|_Y = 0\}$.

LEMMA (1.1). *If Y is an invariant set then*

$$J_Y = \left\{\sum f_n U^n \in B_T; f_n|_Y = 0\right\}.$$

Here $\sum f_n U^n$ stands for the element C in B_T whose $E_n(C)$ is equal to f_n.

Proof. Show $\{\dots\} \subseteq J_Y$. Let $C = \sum f_n U^n$ be in B_T such that $f_n|_Y = 0$ for all n. Since the Fejer sums of C converge to C, as was mentioned in the preliminaries, it follows that C is in J_Y. Conversely, show that $J_Y \subseteq \{\dots\}$. Note that the collection $I = \{\sum_{\text{finite}} f_n U^n; f_n|_Y = 0\ \forall n\}$ is an ideal, not closed, in $K(\mathbb{Z}, C(X))$. *Reason*: If $f|_Y = 0$ then $(\alpha_T)^n(f) = f(T^{-n}(\cdot))$ is zero on Y since Y is invariant and therefore I is closed under multiplication. It is clearly closed under addition and scalar multiplication. Since $K(\mathbb{Z}, C(X))$ is dense in B_T it follows at once that the closure of I is an ideal of B_T. Therefore, the closure of I is exactly J_Y. Let $C = \sum f_n U^n$ be in J_Y and let $\{C_k = \sum f_n^k U^n\}$ in I be such that $C_k \to C$. From the continuity of E_n it follows that $f_n^k \to f_n$ for all n whence f_n is 0 on Y for all n. □

We need some characterization of the J_Y's which is invariant under algebra isomorphism. This will be done by means of finite dimensional irreducible representations of B_T. The treatment of a general n-dimensional irreducible representation of B_T will be tailored after the 1-dimensional case which is described in what follows. Let $\rho: B_T \to \mathbb{C}$ be a non-degenerate representation. We know, [2], that $\rho = \pi \times W$ for some covariant representation $(\pi, W, \mathbb{C})$ of our dynamical system $(\alpha_T, \mathbb{Z}, C(X))$. Now, since π restricted to $C(X)$ is a representation of $C(X)$ on $\mathbb{C}$ it is known to be of the form $\pi(f) = f(x_0)$ for some x_0 in

X. Also, since W is unitary it is given by powers of some λ of absolute value 1. The covariant condition implies that $\pi(\alpha_1(f)) = W\pi(f)W^{-1}$ for all f in $C(X)$. As a result $T^{-1}x_0 = x_0$ whence x_0 is a fixed point.

Conversely, given any λ of absolute value 1 and x_0 a fixed point we can construct a covariant representation $(\pi, W, \mathbb{C})$ by defining $\pi(f) = f(x_0)$ for all f in $C(X)$ and $W(n) = \lambda^n$ for all n in $\mathbb{Z}$. We denote the dependence of ρ on x_0 and λ by $\rho_{x_0,\lambda}$. To summarize, the $\rho_{x_0,\lambda}$'s describe all the irreducible 1-dimensional representations of B_T.

We now turn to a general irreducible n-dimensional representation of B_T. First we describe some such representations and then we show that those are the only ones up to equivalence of representations. Let Y be the orbit of some periodic point of period n. Fix some $\lambda = \{\lambda_y\}_{y\in Y}$ where $|\lambda_y| = 1$ for all y in Y. As in the 1-dimensional case we will show that corresponding to Y and λ there is an n-dimensional representation $\rho_{Y,\lambda}$ of B_T. The representation $\rho_{Y,\lambda}$ will be constructed via a covariant representation $(\pi, W, l^2(Y))$ of our dynamical system. Let $\{e_y\}_{y\in Y}$ be the natural basis in $l^2(Y)$. Then for all f in $C(X)$, we define $\pi(f)$ as follows. For all y in Y, $\pi(f)e_y = f(y)e_y$. The unitary representation W is defined via the unitary W, with some abuse of notation, as follows. For all y in Y, $We_y = \lambda_y e_{Ty}$. Note that with respect to the basis $\{e_y\}$ the unitary W is the product of the unitaries W_0 and D, where W_0 is the unitary taking e_y to e_{Ty} and D is the diagonal unitary having λ_y's on the diagonal.

We check that the covariant condition is satisfied. Let n be an arbitrary integer. Then,

$$\pi(\alpha_n(f))e_y = \pi(f(T^{-n}(\cdot)))e_y = f(T^{-n}y)e_y.$$

On the other hand, $W^{-n}e_y = \mu e_{T^{-n}y}$ for some μ of absolute value 1. Therefore,

$$\begin{aligned} W^n\pi(f)W^{-n}e_y &= W^n\pi(f)(\mu e_{T^{-n}y}) = W^n\mu f(T^{-n}y)e_{T^{-n}y} \\ &= (\mu f(T^{-n}y))(W^n e_{T^{-n}y}) = (\mu f(T^{-n}y))(\mu^{-1}e_y) \\ &= f(T^{-n}y)e_y. \end{aligned}$$

Finally, we need to show that $\pi \times W$ is irreducible. Since the algebra $M_n(\mathbb{C})$ is simple it is sufficient to show that $\pi \times W$ contains all the elementary matrices in $M_n(\mathbb{C})$. Since Y is a finite orbit T acts on it transitively. Therefore, each elementary matrix in $M_n(\mathbb{C})$ will be equal to $\pi(f)W^m$ for appropriate f and m.

Next, we show that any n-dimensional representations of B_T must have, up to equivalence of representations, the form $\rho_{Y,\lambda}$ for some Y, λ

as described above. Let ρ be any irreducible representation of B_T on some n-dimensional vector space $\mathbb{C}^n$. Then, $\rho = \pi \times W$ for some covariant representation $(\pi, W, \mathbb{C}^n)$ of B_T. Since π reduced to $C(X)$ is a representation of that algebra, it is known that with respect to some orthonormal basis in $\mathbb{C}^n$, π is given by $f \to \text{diagonal}(f(y_0), \ldots, f(y_{n-1}))$. We index this basis by the y_i's so that $\{e_i\}$, $0 \leq i \leq n-1$, is the new basis. We may assume that the representation of π is with respect to this basis. Let Y be the collection $\{y_0, \ldots, y_{n-1}\}$. Note that for the time being we do not know that the y_i's are all distinct.

First, we show that Y is invariant. Since $(\pi, W, \mathbb{C}^n)$ is a covariant representation then for all f in $C(X)$, $\pi(\alpha_1(f)) = W\pi(f)W^{-1}$. If Y was not invariant under T then there would exist y in Y such that $T^{-1}y$ is not in Y. Choose f in $C(X)$ such that f is 0 on Y but is 1 on $T^{-1}y$. In that case $W\pi(f)W^{-1} = 0$ but $\pi(\alpha_1(f)) \neq 0$—contradiction.

Next, we show that Y is an orbit. Note that a priori we do not know that the y_i's are all distinct so that we also have to show that there is no duplication among the y_i's. Let Y_1 be the orbit of some arbitrary element y in Y. Let $\{i_j\}$ be a subsequence of $\{i\}$ such that the y_{i_j}'s are distinct and their union is Y_1. Also, let H_{Y_1} be the linear subspace of $\mathbb{C}^n$ generated by $\{e_{i_j}\}$ and let f in $C(X)$ be such that f is 1 on Y_1. The definition of π implies that $\pi(f)$ is the orthogonal projection P onto H_{Y_1} and moreover since Y_1 is invariant $\pi(\alpha_n(f)) = \pi(f)$. Therefore the covariant condition implies now that P commutes with W^j for all j whence H_{Y_1} is a reducing subspace for W. Since it is also a reducing subspace for $\pi(C(X))$ it follows that it is a reducing subspace for $(\pi \times W)(B_T)$ and as a result $H_{Y_1} = \mathbb{C}^n$. We may conclude that $Y = Y_1$ or in other words Y is an orbit and there is no duplication among the y_i's.

We summarize the previous discussion in the following

PROPOSITION (1.2). *The $\rho_{Y,\lambda}$'s describe, up to equivalence of representations, all the irreducible n-dimensional representations of B_T.*

In the next proposition we find a necessary and sufficient condition for two representations of the form $\rho_{Y,\lambda}$, Y is fixed, to be equivalent. Note that the previous discussion let us identify the representation space with $l^2(Y)$.

PROPOSITION (1.3). *Let $\rho_{Y,\lambda}$ and $\rho_{Y,\mu}$, where $\lambda = \{\lambda_y\}$ and $\mu = \{\mu_y\}$, be irreducible n-dimensional representations of B_T. Then, $\rho_{Y,\lambda}$ is equivalent to $\rho_{Y,\mu}$ if and only if $\prod_{y \in Y} \lambda_y = \prod_{y \in Y} \mu_y$.*

Proof. First assume that $\rho_{Y,\lambda}$ is equivalent to $\rho_{Y,\mu}$. Let U be the unitary in B_T induced by T and let $\{e_y\}$ be the natural basis in $l^2(Y)$, the representation space. Since $T^n y = y$, the definition of $\rho_{Y,\lambda}$ implies that

$$\rho_{Y,\lambda}(U^n)e_y = \lambda_y \lambda_{Ty} \cdots \lambda_{T^{n-1}y} e_y = \left(\prod_{y\in Y} \lambda_y\right) e_y.$$

What follows is that $\rho_{Y,\lambda}(U^n) = (\prod_{y\in Y} \lambda_y)I$. Hence, $U^n - (\prod_{y\in Y} \lambda_y)I$ is in $\ker(\rho_{Y,\lambda})$. Since we assumed that $\rho_{Y,\lambda}$ is equivalent to $\rho_{Y,\mu}$ it follows that $U^n - (\prod_{y\in Y} \lambda_y)I$ is also in $\ker(\rho_{Y,\mu})$. But the above calculation also shows that $\rho_{Y,\mu}(U^n) = (\prod_{y\in Y} \mu_y)I$ whence the first half of the proposition follows. Conversely, assume that λ and μ satisfy $\prod_{y\in Y} \lambda_y = \prod_{y\in Y} \mu_y$. We need to find a unitary W in $B(l^2(Y))$ such that $W\rho_{Y,\lambda}W^{-1} = \rho_{Y,\mu}$. Fix some y in Y. We then define W in the following way. We let $We_{T^i y} = \alpha_{T^i y} e_{T^i y}$, for $0 \le i \le n-1$, where $\alpha_y = 1$ and for $1 \le i \le n-1$,

$$\alpha_{T^i y} = \prod_{j=0}^{i-1} \mu_{T^j y} \prod_{j=0}^{i-1} \lambda^{-1}_{T^j y}.$$

First note that if f is in $C(X)$ then $W\rho_{Y,\lambda}(f)W^{-1} = \rho_{Y,\mu}(f)$. Therefore, since B_T is generated by U and $C(X)$ it follows that in order to show that $W\rho_{Y,\lambda}W^{-1} = \rho_{Y,\lambda}$ it is enough now to prove that for $0 \le i \le n-1$

$$W\rho_{Y,\lambda}(U)W^{-1}e_{T^i y} = \rho_{Y,\mu}(U)e_{T^i y} = \mu_{T^i y} e_{T^{i+1} y}.$$

Check the case $i = 0$:

$$W\rho_{Y,\lambda}(U)W^{-1}e_y = W\rho_{Y,\lambda}(U)e_y = W\lambda_y e_{Ty} = \lambda_y \mu_y \lambda_y^{-1} e_{Ty} = \mu_y e_{Ty}.$$

Check the case $0 < i < n-1$:

$$\begin{aligned} W\rho_{Y,\lambda}(U)W^{-1}e_{T^i y} &= W\rho_{Y,\lambda}(U)\left(\prod_{j=0}^{i-1} \mu^{-1}_{T^j y} \prod_{j=0}^{i-1} \lambda_{T^j y}\right) e_{T^i y} \\ &= W(\lambda_{T^i y})\left(\prod_{j=0}^{i-1} \mu^{-1}_{T^j y} \prod_{j=0}^{i-1} \lambda_{T^j y}\right) e_{T^{i+1} y} \\ &= \left(\prod_{j=0}^{i} \mu_{T^j y} \prod_{j=0}^{i} \lambda^{-1}_{T^j y}\right)(\lambda_{T^i y})\left(\prod_{j=0}^{i-1} \mu^{-1}_{T^j y} \prod_{j=0}^{i-1} \lambda_{T^j y}\right) e_{T^{i+1} y} \\ &= \mu_{T^i y} e_{T^{i+1} y}. \end{aligned}$$

Check the case $i = n-1$:

$$\begin{aligned} W\rho_{Y,\lambda}(U)W^{-1}e_{T^{n-1}y} &= W\rho_{Y,\lambda}(U)\left(\prod_{j=0}^{n-2}\mu_{T^jy}^{-1}\prod_{j=0}^{n-2}\lambda_{T^jy}\right)e_{T^{n-1}y} \\ &= W(\lambda_{T^{n-1}y})\left(\prod_{j=0}^{n-2}\mu_{T^jy}^{-1}\prod_{j=0}^{n-2}\lambda_{T^jy}\right)e_y \\ &= (\lambda_{T^{n-1}y})\left(\prod_{j=0}^{n-2}\mu_{T^jy}^{-1}\prod_{j=0}^{n-2}\lambda_{T^jy}\right)e_y \\ &= \left(\prod_{j=0}^{n-2}\mu_{T^jy}^{-1}\prod_{j=0}^{n-1}\lambda_{T^jy}\right)e_y = \mu_{T^{n-1}y}e_y. \end{aligned}$$

The last equality follows from the hypothesis that $\prod_{y\in Y}\lambda_y = \prod_{y\in Y}\mu_y$. □

2. The structure of $\mathrm{Prim}_n(B_T)$. In this section we use the description of irreducible representations of B_T to study the structure of $\mathrm{Prim}_n(B_T)$. The number of connected components of $\mathrm{Prim}_n(B_T)$ is proven to be equal to the number of orbits of length n.

Let ρ be a finite dimensional irreducible representation of B_T.

NOTATION. We denote by ρ_λ the composition $\rho\cdot\hat{\alpha}_\lambda$ where $\hat{\alpha}$ is the dual action.

LEMMA (2.1). *For any λ in $\mathbf{T}$, $\mu = \{\mu_y\}$ and Y a finite invariant set of T,*

$$(\rho_{Y,\mu})_\lambda = \rho_{Y,\lambda\mu}.$$

Proof. For any f in $C(X)$, $(\rho_{Y,\mu})_\lambda(f) = \rho_{Y,\lambda\mu}(f)$; therefore we only need to check that $(\rho_{Y,\mu})_\lambda(U) = \rho_{Y,\lambda\mu}(U)$. Let $\{e_y\}$ be the natural orthonormal basis in $l^2(Y)$. Then for any y in Y,

$$(\rho_{Y,\mu})_\lambda(U)e_y = \rho_{Y,\mu}(\lambda U)e_y = \lambda\rho_{Y,\mu}(U)e_y = \lambda\mu_y e_{Ty} = \rho_{Y,\lambda\mu}(U). \quad \square$$

PROPOSITION (2.2). *Let $\rho = \rho_{Y,\lambda}$ be an n-dimensional irreducible representation of B_T. Then,*

$$J_Y = \bigcap_{\lambda\in\mathbf{T}}\ker(\rho_\lambda).$$

Proof. Assume that $\rho = \pi \times W$. Let $C = \sum f_n U^n$ be in J_Y. By Lemma (1.1) the f_n's are 0 on Y and hence the $\pi(f_n)$'s are all 0. We noted in the preliminaries that

$$\sum_{|k|<N} \left(1 - \frac{|k|}{N}\right) f_k U^k \lambda^k \to \hat{\alpha}_\lambda(C)$$

uniformly in λ. As a result,

$$\sum_{|k|<N} \left(1 - \frac{|k|}{N}\right) \pi(f_k) W^k \lambda^k \to \rho \cdot \hat{\alpha}_\lambda(C) = \rho_\lambda(C)$$

for all λ in $\mathbf{T}$ and therefore C is in $\bigcap_\lambda \ker(\rho_\lambda)$.

Conversely, let $C = \sum f_n U^n$ be in $\bigcap_\lambda \ker(\rho_\lambda)$. By Lemma (1.1) we need to show that f_n is 0 on Y for all n. Let $\{C_k\} \subseteq K(\mathbb{Z}, C(X))$ be such that $C_k \to C$. Since $\rho_\lambda(C_k) \to \rho_\lambda(C)$ uniformly in λ it follows by our hypothesis that $\rho_\lambda(C_k) \to 0$ uniformly. Therefore for all ξ, η in $l^2(Y)$, $(\rho_\lambda(C_k)\xi, \eta) \to 0$ uniformly in λ. Let $\xi = e_y$ and $\eta = e_{y'}$. Assume that for all k, $\sum a_n^k \lambda^n$ is the Fourier expansion of $\lambda \to (\rho_\lambda(C_k)e_y, e_{y'})$. Then, $a_n^k \to 0$ for all n. Let $C_k = \sum f_n^k U^n$ for all k. Then, $a_n^k = (\pi(f_n^k)W^n e_y, e_{y'})$. Since $f_n^k \to f_n$ for all n, it follows that $(\pi(f_n)W^n e_y, e_{y'}) = 0$. But $W^n e_y = \delta e_{T^n y}$, for some δ of absolute value 1. Therefore what we have shown is that for all n and for all y, y' in Y, $(e_{T^n y}, \overline{f_n(y')} e_{y'}) = 0$. In particular if we pick $y = T^{-n} y'$ we get that $f_n(y') = 0$. Since n in y' are arbitrary it follows that f_n is 0 on Y for all n. □

Let $\{Y_i\}_{i \in I}$ be the set of all orbits of length n with respect to T. Assume that I is finite.

NOTATION. Let $F_{Y_i} = \{R \in \mathrm{Prim}_\pi(B_T);\ R \supseteq J_{Y_i}\}$.

By definition, F_{Y_i} is closed in $\mathrm{Prim}_n(B_T)$. Also by Proposition (2.2) each R in $\mathrm{Prim}_n(B_T)$ is in one of the F_{Y_i}'s. Since the Y_i's are mutually exclusive it follows that the F_{Y_i}'s are too. Consequently the F_{Y_i}'s are open and closed in $\mathrm{Prim}_n(B_T)$.

Finally, we show that if $\{\ker(\rho)\} \in F_{Y_i}$, then the connected component of $\{\ker(\rho)\}$ includes F_{Y_i}. Fix ρ such that $\{\ker(\rho)\} \in F_{Y_i}$. Now, the function $\lambda \to \{\ker(\rho_\lambda)\}$ is continuous with respect to the Jacobson topology on $\mathrm{Prim}_n(B_T)$. *Reason*: $\rho_\lambda = \rho \cdot \hat{\alpha}_\lambda$ and $\hat{\alpha}_\lambda$ is continuous with respect to the pointwise topology. Therefore, $\lambda \to \{\ker(\rho_\lambda)\}$ is a continuous function from $\mathbf{T}$ to $\mathrm{Prim}_n(B_T)$.

Conclusion. The connected component of $\{\ker(\rho)\}$ includes the set

$$\left\{R \in \mathrm{Prim}_n(B_T); R \supseteq \bigcap_{\lambda} \ker(\rho_\lambda)\right\}.$$

But by Proposition (2.2), $\bigcap_\lambda \ker(\rho_\lambda) = J_{Y_i}$ and therefore the connected component of $\{\ker(\rho)\}$ includes F_{Y_i}. Since the F_{Y_i}'s are open and closed it follows that the connected component of $\{\ker(\rho)\}$ is exactly F_{Y_i}.

We summarize the above discussion in the following theorem.

NOTATION. For any homeomorphism T we denote by $O(T)$ the set of all finite orbits of T.

THEOREM (2.3). *Let Θ: $B_T \to B_S$ be an isomorphism. Let Y be a finite orbit with respect to T. Then, $\Theta(J_Y) = J_Z$ for some Z a finite orbit with respect to S having the same cardinality as Y. The correspondence $Y \to Z$ defines a set theoretic isomorphism Θ' between $O(T)$ and $O(S)$. Moreover, $(\Theta')^{-1} = (\Theta^{-1})'$. Note that T and S may act on different spaces.*

Proof. We know that the map $\mathrm{Prim}_n(\Theta)$: $\mathrm{Prim}_n(B_T) \to \mathrm{Prim}_n(B_S)$, defined by $\{\ker(\rho)\} \to \{\ker(\rho \cdot \Theta^{-1})\}$ is a homeomorphism. Therefore, the image of F_Y under $\mathrm{Prim}(\Theta)$ must be equal to some F_Z where Z is a finite orbit of S having the same cardinality as Y. Now, $\Theta(J_Y) = J_Z$ because $\Theta(\ker(\rho)) = \ker(\rho \cdot \Theta^{-1})$ and $\bigcap_{\{R; R \in \mathrm{Prim}_n(B_T), R \supseteq J_Y\}} R = J_Y$. Finally, Θ' is a set theoretic isomorphism because $\mathrm{Prim}(\Theta)$ is a homeomorphism. □

THEOREM (2.4). *Let ρ be an irreducible n-dimensional representation of B_T. Assume that T has finitely many orbits of length n. Then the connected component of* $\{\ker(\rho)\}$ *in* $\mathrm{Prim}_n(B_T)$ *is equal to*

$$\{\ker(\rho_\lambda); 0 \le \arg(\lambda) < 2\pi/n\}.$$

The number of connected components in $\mathrm{Prim}_n(B_T)$ *is equal to the number of orbits of length n.*

Proof. The only part that was not proven is that the component of $\{\ker(\rho)\}$ in $\mathrm{Prim}_n(B_T)$ is equal to $\{\ker(\rho_\lambda); 0 \le \arg(\lambda) < 2\pi/n\}$. By Proposition (1.2) we know that ρ is equivalent to some $\rho_{Y,\mu}$, where Y is an orbit of length n and $\mu = \{\mu_y\}$, and the discussion preceding Theorem (2.3) shows that the connected component of ρ is equal to

$F_Y = \{R \in \mathrm{Prim}_n(B_T); R \supseteq J_{Y_i}\}$. Therefore, what is left to show is that for any $\nu = \{\nu_y\}$, $\ker(\rho_{Y,\nu})$ is equal to $\ker(\rho_\lambda)$ for some λ such that $0 \leq \arg(\lambda) < 2\pi/n$. By Proposition (1.3) the class of $\rho_{Y,\nu}$ is dependent only on $\prod_{y \in Y} \nu_y$ and by Lemma (2.1) $\rho_\lambda = (\rho_{Y,\mu})_\lambda = \rho_{Y,\lambda\mu}$. Therefore we are done because for $\{\lambda; 0 \leq \arg(\lambda) < 2\pi/n\}$ the range of the function $\lambda \to \prod_{y \in Y} \lambda\mu_y$ is $\mathbf{T}$. □

3. Partial classification of hyperbolic crossed-product algebras. We now specialize to the case $X = \mathbf{T}^m$ and T an automorphism on $\mathbf{T}^m$.

NOTATION. Denote by $N_p(T)$ the cardinality of the set $\{x \in X; T^p x = x\}$ and by $O_p(T)$ the cardinality of the set of all periodic points of period equal to p.

DEFINITION. An automorphism T is called hyperbolic if it has no eigenvalue of unit modulus.

THEOREM (3.1). *A partial classification of the B_T's. Let T and S be hyperbolic automorphisms on tori not necessarily of the same dimensions. If the algebra B_T is isomorphic to B_S, then for all $p \geq 1$, $N_p(T) = N_p(S)$. In particular, T and S must have the same entropy.*

Proof. If Θ: $B_T \to B_S$ is an isomorphism then it induces a homeomorphism between $\mathrm{Prim}_n(B_T)$ and $\mathrm{Prim}_n(B_S)$ for $n \geq 1$. Since the number of connected components is a topological invariant it must be the same for $\mathrm{Prim}_n(B_T)$ and $\mathrm{Prim}_n(B_S)$. On the other hand we know that the number of connected components in $\mathrm{Prim}_n(B_T)$ is equal to the number of orbits of length n. Therefore, $O_n(S) = O_n(T)$. Note that $N_n(T)$ is not quite the number of periodic points of period n because it includes all points of period m for m which divides n. But $N_n(T)$ can be recovered from the $O_m(T)$'s simply because

$$N_n(T) = \sum_{\{m \geq 1; m|n\}} O_m(T).$$

Let $\sigma(T) = \{\lambda_1, \dots, \lambda_k\}$ be the spectrum of T and $\sigma(S) = \{\mu_1, \dots, \mu_l\}$ be the spectrum of S. Recall that $N_p(T) = |\det(T^p - I)|$, **[4]**. By the above discussion we know that if B_T is isomorphic to B_S then for all p, $|\det(T^p - I)| = |\det(S^p - I)|$. Now,

$$|\det(T^p - I)| = \prod_{i=1}^{k} |\lambda_i^p - 1|.$$

Fix some $\varepsilon > 0$. Note that we can make the following estimations. If $\lambda \in \sigma(T)$ and $|\lambda| > 1$ then for p sufficiently large,

$$(1 - \varepsilon)|\lambda|^p \leq |\lambda^p - 1| \leq (1 + \varepsilon)|\lambda|^p$$

and if $\lambda \in \sigma(T)$ and $|\lambda| < 1$ then for p sufficiently large,

$$(1-\varepsilon) \leq |\lambda^p - 1| \leq (1+\varepsilon).$$

Denote by Λ the quantity $\prod_{\{i;|\lambda_i|>1\}} |\lambda_i|$. By the above estimation,

$$(1-\varepsilon)^k \Lambda^p \leq N_p(T) \leq (1+\varepsilon)^k \Lambda^p.$$

Repeating the above calculation for S we get that for any fixed $\varepsilon' > 0$ and for p sufficiently large

$$(1-\varepsilon')^l M^p \leq N_p(S) \leq (1+\varepsilon')^l M^p$$

where M is analogous to Λ. Claim: Λ must be equal to M. Proof: Assume without loss of generality that $\Lambda < M$. Then for any positive $\varepsilon, \varepsilon'$

$$(1+\varepsilon)^k \Lambda^p < (1-\varepsilon')^l M^p$$

for p sufficiently large. The last inequality implies that $N_p(T) < N_p(S)$—contradiction. We have completed the proof since the entropy of an automorphism T is equal to $\log(\Lambda)$, [**4**]. □

What can be now deduced about the classification of the crossed-product algebras over the 2-torus. Note that if T is an automorphism on the 2-torus then the equation for its characteristic polynomial, regarded as a linear transformation on the plane, is

$$x^2 - \text{trace}(T)x + \det(T) = 0.$$

From this relation we deduce that if T and S have the same trace and determinant then they have the same eigenvalues and conversely.

In the last section we showed that the entropy of T is an invariant of B_T. Since the product of the eigenvalues of T is 1 in absolute value it follows that if $B_T \cong B_S$ then

$$\{|\lambda|; \lambda \in \sigma(T)\} = \{|\mu|; \mu \in \sigma(T)\}.$$

Let us make the following notations. Let $\delta = \det(T)$, $\delta' = \det(S)$, $\tau = \text{trace}(T)$ and $\tau' = \text{trace}(S)$. Since the eigenvalues of T and S are real we now have that

$$\frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2} = \pm \frac{\tau' \pm \sqrt{\tau'^2 - 4\delta'}}{2}.$$

Claim. The above equation for the eigenvalues implies that $|\tau| = |\tau'|$ and $\tau^2 - 4\delta = (\tau')^2 - 4\delta'$. Therefore also $\delta = \delta'$. Proof: Recall that the eigenvalues of hyperbolic automorphisms are irrational, [**5**]. In general, if k, l, m, n are integers and $m + \sqrt{n}$, $k + \sqrt{l}$ are irrational numbers satisfying $m + \sqrt{n} = k + \sqrt{l}$ then $m = k$ and $n = l$. Reason: $\sqrt{n} = (k-m)+\sqrt{l}$ and therefore $n = (k-m)^2+l+2(k-m)\sqrt{l}$. If $k \neq m$ then $\sqrt{l}$ is rational whence $k + \sqrt{l}$ is also rational—contradiction.

Can we furthermore deduce that trace(T) = trace(S)? From the last section we know that $|\det(T^n - I)| = |\det(S^n - I)|$ for all $n \geq 1$. Observe that

$$|\det(T - I)| = |\det(T) + 1 - \operatorname{trace}(T)|.$$

Therefore if $\det(T) = \det(S) = 1$ then $|2-\tau| = |2-\tau'|$. Since $|\tau| = |\tau'|$ it follows that $\tau = \tau'$.

We may summarize the above discussion in the following

COROLLARY (3.2). *Let T and S be hyperbolic automorphisms on the 2-torus. If $B_T \cong B_S$ then:*

(i) $\det(T) = \det(S)$,

(ii) $|\operatorname{trace}(T)| = |\operatorname{trace}(S)|$.

If $\det(T)$ *or* $\det(S)$ *is equal to* 1 *then*

(iii) $\operatorname{trace}(T) = \operatorname{trace}(S)$.

REMARKS. In the case $\det(T) = \det(S) = -1$ it is not true that $B_T \cong B_S$ implies that trace(T) = trace(S). Example: Let T be a hyperbolic automorphism having determinant -1. Let $S = T^{-1}$. Note that trace$(S) = -$trace(T) but $B_T \cong B_{T^{-1}} = B_S$.

REFERENCES

[1] D. P. O'Donovan, *Weighted shifts and covariance algebras*, Trans. Amer. Math. Soc., **208** (1975), 1–25.

[2] G. K. Pedersen, *C*-Algebras and their automorphism groups*, London Math. Soc. Monographs 14, Academic Press, London, (1979).

[3] I. Megory-Cohen, *Properties of hyperbolic crossed-product algebras*, dissertation, University Microfilms International, (1985).

[4] P. Walters, *An Introduction of Ergodic Theory*, Springer-Verlag, New York, (1982).

[5] Z. Nitecki, *Differentiable Dynamics*, MIT Press, Cambridge, Mass., (1971).

Received June 8, 1987.

7231 MARTWOOD WAY
SAN JOSE, CA 95120

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $190.00 a year (5 Vols., 10 issues). Special rate: $95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) publishes 5 volumes per year. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 135, No. 2 October, 1988