A SIMPLE FORMULA FOR CONDITIONAL WIENER INTEGRALS WITH APPLICATIONS

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Yeh's inversion formula for conditional Wiener integrals is very complicated to apply when the conditioning function is vector-valued. This paper gives a very simple formula for such integrals. In particular, we express the conditional Wiener integral directly in terms of an ordinary (i.e., nonconditional) Wiener integral. Using this new formula, it is very easy to generalize the Kac-Feynman formula and also to obtain a Cameron-Martin type translation theorem for general conditional Wiener integrals.

1. Introduction. Consider the Wiener measure space $(C[0, T], \mathscr{F}^*, m_w)$ where $C[0, T]$ is the space of all continuous functions x on $[0, T]$ vanishing at the origin. For each partition $\tau = \tau_n = \{t_1, \ldots, t_n\}$ of $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$, let $X_\tau: C[0, T] \to \mathbf{R}^n$ be defined by $X_\tau(x) = (x(t_1), \ldots, x(t_n))$. Let $\mathscr{B}^n$ be the σ-algebra of Borel sets in $\mathbf{R}^n$. Then a set of the type

$$I = \{x \in C[0, T]: X_\tau(x) \in B\} \equiv X_\tau^{-1}(B), \qquad B \in \mathscr{B}^n,$$

is called a Wiener interval (or a Borel cylinder). It is well known that

$$(1.1) \qquad m_w(I) = \int_B K(\tau, \vec{\xi}) \, d\vec{\xi},$$

where

$$(1.2) \quad K(\tau, \vec{\xi}) = \left\{ \prod_{j=1}^{n} 2\pi(t_j - t_{j-1}) \right\}^{-1/2} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(\xi_j - \xi_{j-1})^2}{t_j - t_{j-1}} \right\},$$
$$\vec{\xi} = (\xi_1, \ldots, \xi_n), \quad \text{and} \quad \xi_0 = 0.$$

m_w is a probability measure defined on the algebra $\mathscr{F}$ of all Wiener intervals and m_w is extended to the Carathéodory extension $\mathscr{F}^*$ of $\mathscr{F}$. Let $\mathscr{F}_\tau$ be the σ-algebra generated by the set $\{X_\tau^{-1}(B): B \in \mathscr{B}^n\}$ with τ fixed. Then, by the definition of conditional expectation (see Tucker

[6] and Yeh [8]), for each Wiener integrable function $F(x)$

$$(1.3) \quad \mu_\tau(B) \equiv \int_{X_\tau^{-1}(B)} F(x)m_w(dx) = \int_{X_\tau^{-1}(B)} E(F|\mathscr{F}_\tau)m_w(dx)$$
$$= \int_B E(F(x)|X_\tau(x) = \vec{\xi})P_{X_\tau}(d\vec{\xi}), \qquad B \in \mathscr{B}^n,$$

where $P_{X_\tau}(B) = m_w(X_\tau^{-1}(B))$, and $E(F(x)|X_\tau(x) = \vec{\xi})$ is a Borel measurable function of $\vec{\xi}$ which is unique up to Borel null sets in $\mathbf{R}^n$. According to Yeh [8], the conditional expectation $E(F(x)|X_\tau(x) = \vec{\xi})$ can be found by the Fourier transform

$$(1.4) \quad E(F(x)|X_\tau(x) = \vec{\xi})$$
$$= \left[\frac{dp_{X_\tau}}{dm_L}(\vec{\xi})\right]^{-1} (2\pi)^{-n} \int_{\mathbf{R}^n} e^{-i\langle \vec{U}, \vec{\xi}\rangle} E\left[e^{i\langle \vec{U}, \vec{X}_\tau\rangle}F\right] m_L(d\vec{U})$$

where m_L is Lebesgue measure on $\mathbf{R}^n$. Using the above inversion formula (changing the conditional expectation into non-conditional expectation) for the special case $X_\tau(x) = x(T)$, i.e., τ consists of the endpoint only, Yeh [9, 10] obtains very useful results including the Kac-Feynman integral equation and the conditional Cameron-Martin translation theorem. However, Yeh's inversion formula (1.4) is very difficult and complicated to apply for general τ's. Some results are obtained by Chang and Chang [3] for the latter case using Yeh's formula (1.4), but these results are somewhat limited in scope and the calculations are quite lengthy in general.

The main purpose of this paper is to develop a new simple formula for the conditional Wiener integral when the conditioning function is vector-valued and then to obtain a general form of the Kac-Feynman formula and a general conditional Cameron-Martin translation theorem.

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2. A simple formula for conditional Wiener integrals. For a given partition $\tau = \tau_n$ of $[0, T]$ and $x \in C[0, T]$, define the polygonal function $[x]$ on $[0, T]$ by

$$(2.1) \quad [x](t) = x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})),$$

$$t_{j-1} \leq t \leq t_j, \; j = 1, \ldots, n.$$

Similarly, for each $\vec{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$, define the polygonal function $[\vec{\xi}]$ of $\vec{\xi}$ on $[0, T]$ by

$$(2.2) \quad [\vec{\xi}](t) = \xi_{j-1} + \frac{t - t_{j-1}}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}),$$

$$t_{j-1} \leq t \leq t_j, \quad j = 1, \ldots, n \text{ and } \xi_0 = 0.$$

Then both $[x]$ and $[\vec{\xi}]$ are continuous on $[0, T]$, their graphs are line segments on each subinterval $[t_{j-1}, t_j]$, and $[x](t_j) = x(t_j)$ and $[\vec{\xi}](t_j) = \xi_j$ at each $t_j \in \tau$.

The following theorem gives an interesting observation for the process $x(t) - [x](t)$. While the process $[x](t)$ has been widely used to approximate the Brownian motion $x(t)$, recognizing $x(t) - [x](t)$ as a Brownian bridge on each subinterval is new as far as the authors know.

THEOREM 1. *If $\{x(t), 0 \leq t \leq T\}$ is the standard Wiener process, then the process $\{x(t)-[x](t), 0 \leq t \leq T\}$ and $X_\tau(x)=(x(t_1), \ldots, x(t_n))$ are (stochastically) independent.*

Proof. In view of (2.1), we may write for $t_{j-1} \leq t \leq t_j$

$$(2.3) \quad x(t) - [x](t) = (x(t) - x(t_{j-1})) - \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})).$$

Note that $x(t) - [x](t)$ is a Brownian bridge process on $[t_{j-1}, t_j]$ vanishing at t_{j-1} and t_j. Thus $Y \equiv \{x(t) - [x](t), t_{j-1} \leq t \leq t_j\}$ is independent of the process $\{x(t): t \in [0, t_{j-1}] \cup [t_j, T]\}$. In particular, Y is independent of $X_\tau(x) = (x(t_1), \ldots, x(t_n))$. Since j was arbitrary, the proof is complete.

COROLLARY 1. *If $\{x(t), 0 \leq t \leq T\}$ is the standard Wiener process, then $\{x(t) - [x](t), t_{j-1} \leq t \leq t_j\}$, $j = 1, 2, \ldots, n$ are independent Brownian bridge processes.*

The following theorem is one of our main results:

THEOREM 2. *Let F be any Wiener integrable function on $C[0, T]$. Then, for every Borel measurable set B in $\mathbf{R}^n$,*

$$(2.4) \qquad \mu_\tau(B) \equiv \int_{X_\tau^{-1}(B)} F(x) m_w(dx)$$

$$= \int_B E[F(x - [x] + [\vec{\xi}])] P_{X_\tau}(d\vec{\xi}).$$

Proof. First assume that F is the characteristic function of a Wiener measurable set A, i.e., $F(x) = \chi_A(x)$. Then

$$\int_{X_\tau^{-1}(B)} \chi_A(x) m_w(dx) = m_w(A \cap X_\tau^{-1}(B))$$

$$= \int_B m_w(x \in A | X_\tau(x) = \vec{\xi}) P_{X_\tau}(d\vec{\xi})$$

$$= \int_B m_w(x - [x] + [\vec{\xi}] \in A | X_\tau(x) = \vec{\xi}) P_{X_\tau}(d\vec{\xi}).$$

But, $x - [x]$ and $X_\tau(x)$ are independent by Theorem 1. Thus,

$$\int_{X_\tau^{-1}(B)} \chi_A(x) m_w(dx) = \int_B m_w(x - [x] + [\vec{\xi}] \in A) P_{X_\tau}(d\vec{\xi})$$

$$= \int_B E[\chi_A(x - [x] + [\vec{\xi}])] P_{X_\tau}(d\vec{\xi}).$$

Thus, the result holds for the characteristic function of any Wiener measurable set. The general case follows by the usual arguments in integration theory.

If F is a Wiener integrable function on $C[0, T]$, then from (2.4), the Radon-Nikodym derivative of μ_τ with respect to P_{X_τ} is

$$\frac{d\mu_\tau}{dP_{X_\tau}}(\vec{\xi}) = E[F(x - [x] + [\vec{\xi}])]$$

for a.e. $\vec{\xi}$ in $\mathbf{R}^n$. But (1.3) yields

$$\frac{d\mu_\tau}{dP_{X_\tau}}(\vec{\xi}) = E(F(x) | X_\tau(x) = \vec{\xi})$$

for a.e. $\vec{\xi}$ in $\mathbf{R}^n$. Thus $E(F(x) | X_\tau(x) = \vec{\xi})$ and $E[F(x - [x] + [\vec{\xi}])]$ are equal a.e. on $\mathbf{R}^n$. However, while $E(F(x) | X_\tau(x) = \vec{\xi})$ is always Borel measurable in $\vec{\xi}$ by definition, $E[F(x - [x] + [\vec{\xi}])]$ is Lebesgue measurable in $\vec{\xi}$ but not necessarily Borel measurable as the following example shows.

EXAMPLE 1. Let G be a Legesgue measurable null set in $\mathbf{R}$ that is not Borel measurable. Then the function $f(\xi) = \chi_G(\xi)$ is Lebesgue measurable but is not Borel measurable. Let $A = \{x \in C[0, T]: x(T) \in G\}$. Then $m_w(A) = 0$. Define $F: C[0, T] \to \mathbf{R}$ by $F(x) = \chi_A(x)$. Then $F(x) = \chi_G(x(T))$ and

$$E[F(x - [x] + [\xi])] = E[\chi_G(x(T) - [x](T) + [\xi](T))]$$
$$= E[\chi_G(\xi)] = \chi_G(\xi) = f(\xi).$$

However, if $f(\vec{\xi})$ is Lebesgue measurable on $\mathbf{R}^n$, then there exists a Borel measurable function $\hat{f}(\vec{\xi})$ such that $\hat{f}(\vec{\xi}) = f(\vec{\xi})$ almost everywhere. Moreover $\hat{f}(\vec{\xi})$ is unique up to Borel null sets. Thus the following definition makes sense.

DEFINITION 1. If F is a Wiener integrable function on $C[0, T]$, then by $\hat{E}[F(x - [x] + [\vec{\xi}])]$ we mean any Borel measurable function of $\vec{\xi}$ which is equal to $E[F(x - [x] + [\vec{\xi}])]$ for a.e. $\vec{\xi}$ in $\mathbf{R}^n$.

We thus have the following useful formula, which is much simpler to apply than the inversion formula (1.4).

THEOREM 3. *If F is a Wiener integrable function on $C[0, T]$, then*

$$(2.5) \qquad E(F(x)|X_\tau(x) = \vec{\xi}) = \hat{E}[F(x - [x] + [\vec{\xi}])].$$

In particular, if F is also Borel measurable then

$$(2.6) \qquad E(F(x)|X_\tau(x) = \vec{\xi}) = E[F(x - [x] + [\vec{\xi}])].$$

The equality in (2.5) (and in (2.6)) means that both sides are Borel measurable functions of $\vec{\xi}$ and they are equal except for Borel null sets.

Proof. Equation (2.5) is obvious by the discussion following Theorem 2 and Definition 1. If F is also Borel measurable, then by the Fubini Theorem, $E[F(x - [x] + [\vec{\xi}])]$ is a Borel measurable function of $\vec{\xi}$, and hence (2.6) holds.

3. **Examples.** The following examples show that Theorem 3 above is indeed quite powerful and very simple to apply.

EXAMPLE 2. Let $F(x) = \int_0^T x(t)\, dt$, $x \in C[0, T]$. Then by (2.6) we have:

$$E\left(\int_0^T x(t)\, dt | X_\tau(x) = \vec{\xi}\right) = E\left[\int_0^T (x(t) - [x](t) + [\vec{\xi}](t))\, dt\right]$$

$$= \int_0^T E[x(t) - [x](t) + [\vec{\xi}](t)]\, dt = \int_0^T [\vec{\xi}](t)\, dt$$

$$= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left(\xi_{j-1} + \frac{t - t_{j-1}}{t_j - t_{j-1}}(\xi_j - \xi_{j-1})\right) dt$$

$$= \frac{1}{2}\sum_{j=1}^n (\xi_j + \xi_{j-1})(t_j - t_{j-1}).$$

In the above calculation, we used the Fubini Theorem to interchange the order of integration. In particular when $n = 1$,

$$E\left(\int_0^T x(t)\,dt|x(T) = \xi\right) = \frac{1}{2}\xi T$$

which was computed by Yeh [9, pp. 629–631] using (1.4) with $n = 1$; the computation was rather lengthy.

EXAMPLE 3. Let $F(x) = \int_0^T (x(t))^2\,dt$, $x \in C[0, T]$. Then, again by (2.6)

$$E\left(\int_0^T (x(t))^2\,dt|X_\tau(x) = \vec{\xi}\right)$$

$$= E\left[\int_0^T (x(t) - [x](t) + [\vec{\xi}](t))^2\,dt\right]$$

$$= \int_0^T E[(x(t))^2 + ([x](t))^2 + ([\vec{\xi}](t))^2 - 2x(t)[x](t)$$

$$+ 2x(t)[\vec{\xi}](t) - 2[x](t)[\vec{\xi}](t)]\,dt.$$

Using the properties $E[x(t)] = 0$ and $E[x(s)x(t)] = \min\{s, t\}$ repeatedly, we obtain:

$$E\left(\int_0^T (x(t))^2\,dt|X_\tau(x) = \vec{\xi}\right)$$

$$= \int_0^T \left\{t - \sum_{j=1}^n \chi_{[t_{j-1}, t_j]}(t)\left[t_{j-1} + \frac{(t - t_{j-1})^2}{t_j - t_{j-1}}\right] + ([\vec{\xi}](t))^2\right\}\,dt$$

$$= \frac{T^2}{2} - \frac{1}{3}\sum_{j=1}^n (t_j + 2t_{j-1})(t_j - t_{j-1})$$

$$+ \frac{1}{3}\sum_{j=1}^n (\xi_j^2 + \xi_j\xi_{j-1} + \xi_{j-1}^2)(t_j - t_{j-1}).$$

In particular when $n = 1$, $E(\int_0^T (x(t))^2\,dt|x(T) = \xi) = T^2/6 + \xi^2 T/3$, which agrees with Yeh's computation [9, 631–632].

EXAMPLE 4. Let $F(x) = \exp\{\int_0^T x(t)\, dt\}$, $x \in C[0, T]$. Then

$$E\left(\exp\left\{\int_0^T x(t)\, dt\right\} | X_\tau(x) = \vec{\xi}\right)$$

$$= E\left[\exp\left\{\int_0^T (x(t) - [x](t) + [\vec{\xi}](t))\, dt\right\}\right]$$

$$= \exp\left\{\int_0^T [\vec{\xi}](t)\, dt\right\} E\left[\exp\left\{\int_0^T (x(t) - [x](t))\, dt\right\}\right].$$

It is interesting to note that for each fixed $u \in C[0, T]$, we have, as expected,

$$\lim_{\|\tau\| \to 0} E\left(\exp\left\{\int_0^T x(t)\, dt\right\} | X_\tau(x) = X_\tau(u)\right) = \exp\left\{\int_0^T u(t)\, dt\right\}.$$

Some of the above examples are also considered by Chang and Chang [3] using the inversion formula (1.4); their computations are very complicated.

4. A generalization of the Kac-Feynman formula. Under the assumption that V is a nonnegative continuous function on $\mathbf{R}$ satisfying the condition

$$\int_{\mathbf{R}} V(\xi) e^{-\xi^2/2t}\, d\xi < \infty \quad \text{for every } t > 0,$$

Yeh [9] has shown that the function U defined on $\mathbf{R} \times (0, \infty)$ by

$$(4.1) \qquad U(\xi, t) = E\left(\exp\left\{-\int_0^t V(x(s))\, ds\right\} | x(t) = \xi\right)$$

$$\cdot (2\pi t)^{-1/2} \exp\left(-\frac{\xi^2}{2t}\right)$$

satisfies the Kac-Feynman integral equation

$$(4.2) \qquad U(\xi, t) = \frac{1}{\sqrt{2\pi t}} e^{-\xi^2/2t}$$

$$- \int_0^t \left[\frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(\xi - \eta)^2}{2(t-s)}\right\}\right.$$

$$\left. \cdot V(\eta) U(\eta, s)\, d\eta\right]\, ds,$$

whose solution is given by

$$U(\xi, t) = \sum_{k=0}^{\infty} (-1)^k U_k(\xi, t), \qquad (\xi, t) \in \mathbf{R} \times (0, \infty),$$

where the sequence $\{U_k\}$ is defined inductively by

$$U_0(\xi, t) = \frac{1}{\sqrt{2\pi t}} e^{-\xi^2/2t},$$

$$U_{k+1}(\xi, t)$$
$$= \int_0^t \left(\frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^\infty \exp\left\{ -\frac{(\xi-\eta)^2}{2(t-s)} \right\} V(\eta) U_k(\eta, s)\, d\eta \right) ds$$

for $k = 0, 1, 2, \ldots$. Thus $E(\exp\{-\int_0^t V(x(s))\, ds\} | x(t) = \xi)$ can be found using (4.1).

Now, under the same assumptions of V, using Theorem 3, we proceed to find the multi-conditional expectation

$$E\left(\exp\left\{ -\int_0^t V(x(s))\, ds \right\} \bigg| x(t_i) = \xi_i, \ i = 1, \ldots, n \right),$$

where $0 < t_1 < t_2 < \cdots < t_n = t$. First consider the conditional expectation for $0 < t_1 < t$,

$$(4.3) \qquad I \equiv E\left(\exp\left\{ -\int_{t_1}^t V(x(s))\, ds \right\} \bigg| x(t_1) = \xi_1, x(t) = \xi \right).$$

Then, by Theorem 3, (2.1) and (2.2),

$$(4.4) \quad I = E\left[\exp\left\{ -\int_{t_1}^t V\left(x(s) - x(t_1) - \frac{s-t_1}{t-t_1}(x(t) - x(t_1)) \right. \right. \right.$$
$$\left. \left. \left. + \xi_1 + \frac{s-t_1}{t-t_1}(\xi - \xi_1) \right) ds \right\} \right].$$

Since the Brownian motion $x(t)$ has stationary increments, it follows from (4.4) that

$$I = E\left[\exp\left\{ -\int_0^{t-t_1} V\left(x(u) - \frac{u}{t-t_1}(x(t-t_1)) + \xi_1 \right. \right. \right.$$
$$\left. \left. \left. + \frac{u}{t-t_1}(\xi - \xi_1) \right) du \right\} \right]$$
$$= E\left(\exp\left\{ -\int_0^{t-t_1} V(x(u) + \xi_1)\, du \right\} \bigg| x(t-t_1) = \xi - \xi_1 \right)$$
$$= E\left(\exp\left\{ -\int_0^{t-t_1} W(x(u))\, du \right\} \bigg| x(t-t_1) = \xi - \xi_1 \right),$$

where $W(x(u)) = V(x(u) + \xi_1)$. Next, let

$$U(\xi_1, \xi - \xi_1, t - t_1) = I \cdot [2\pi(t-t_1)]^{-1/2} \exp\left\{ -\frac{(\xi-\xi_1)^2}{2(t-t_1)} \right\}.$$

Then, applying (4.1) and (4.2) to (4.5), we obtain that

$$U(\xi_1, \xi - \xi_1, t - t_1)$$
$$= [2\pi(t - t_1)]^{-1/2} \exp\left\{-\frac{(\xi - \xi_1)^2}{2(t - t_1)}\right\}$$
$$- \int_0^{t-t_1} \left[[2\pi(t - t_1 - u)]^{-1/2} \right.$$
$$\left. \cdot \int_{-\infty}^{\infty} \exp\left\{-\frac{(\xi - \xi_1 - \eta)^2}{2(t - t_1 - u)}\right\} W(\eta) U(\xi_1, \eta, u) \, d\eta \right] du.$$

Since $W(\eta) = V(\eta + \xi_1)$, we may use the transformation $\gamma = \eta + \xi_1$ to obtain

$$U(\xi_1, \xi - \xi_1, t - t_1)$$
$$= [2\pi(t - t_1)]^{-1/2} \exp\left\{-\frac{(\xi - \xi_1)^2}{2(t - t_1)}\right\}$$
$$- \int_0^{t-t_1} \left[[2\pi(t - t_1 - u)]^{-1/2} \right.$$
$$\left. \cdot \int_{-\infty}^{\infty} \exp\left\{-\frac{(\xi - \gamma)^2}{2(t - t_1 - u)}\right\} V(\gamma) U(\xi_1, \gamma - \xi_1, u) \, d\gamma \right] du.$$

But this equation is similar to the Kac-Feynman integral equation, and hence its solution is given by

$$U(\xi_1, \xi - \xi_1, t - t_1) = \sum_{k=0}^{\infty} (-1)^k U_k(\xi, \xi - \xi_1, t - t_1),$$

where

$$U_0(\xi_1, \xi - \xi_1, t - t_1) = [2\pi(t - t_1 - u)]^{-1/2} \exp\left\{-\frac{(\xi - \xi_1)^2}{2(t - t_1)}\right\},$$

and

$$U_{k+1}(\xi_1, \xi - \xi_1, t - t_1)$$
$$= \int_0^{t-t_1} \left[[2\pi(t - t_1 - u)]^{-1/2} \int_{-\infty}^{\infty} \exp\left\{-\frac{(\xi - \gamma)^2}{2(t - t_1 - u)}\right\} \right.$$
$$\left. \cdot V(\gamma) U_k(\xi_1, \gamma - \xi_1, u) \, d\gamma \right] du$$

for $k = 0, 1, 2, \ldots$. In particular the solution $U(\xi, t)$ of the Kac-Feynman integral equation (4.2) is a special case, namely

$$U(\xi, t) = U(0, \xi, t).$$

We are now ready to write out an expression for the multi-conditional expectation. Using the Markov property of the conditional process and the multiplicative nature of the exponential function, we may write

$$
\text{(4.6)} \quad E\left(\exp\left\{-\int_0^t V(x(s))\,ds\right\}\Big|x(t_i) = \xi_i, i = 1, 2, \ldots, n\right)
$$

$$
= \prod_{i=1}^N E\left(\exp\left\{-\int_{t_{i-1}}^{t_i} V(x(s))\,ds\right\}\Big|x(t_{i-1}) = \xi_{i-1}, x(t_i) = \xi_i\right)
$$

$$
= \prod_{i=1}^N [2\pi(t_i - t_{i-1})]^{1/2} \exp\left\{\frac{(\xi_i - \xi_{i-1})^2}{2(t_i - t_{i-1})}\right\}
$$

$$
\cdot U(\xi_{i-1}, \xi_i - \xi_{i-1}, t_i - t_{i-1}).
$$

Also, we have

$$
\text{(4.7)}\, E\left(\exp\left\{-\int_0^t V(x(s))\,ds\right\}\Big|a_i < x(t_i) < b_i,\ i = 1, 2, \ldots, n\right)
$$

$$
= \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^n U(\xi_{i-1}, \xi_i - \xi_{i-1}, t_i - t_{i-1})\,d\xi_1 \cdots d\xi_n.
$$

5. Translation of multi-conditional Wiener integrals.

The Cameron-Martin Translation Theorem [2, 7] states that if $x_0(t) = \int_0^t h(s)\,ds$ for all $t \in [0, T]$ with $h \in L^2[0, T]$, and if T_1 is the transformation of $C[0, T]$ into itself defined by

$$
T_1(x) = x + x_0 \quad \text{for } x \in C[0, T],
$$

then for any Wiener integrable function F on $C[0, T]$ and any Wiener measurable set Γ

$$
\text{(5.1)} \qquad \int_\Gamma F(y)m_w(dy) = \int_{T_1^{-1}(\Gamma)} F(x + x_0)J(x_0, x)m_w(dx)
$$

where

$$
\text{(5.2)} \quad J(x_0, x) = \exp\left\{-\frac{1}{2}\int_0^T (h(t))^2\,dt\right\}\exp\left\{-\int_0^T h(t)\,dx(t)\right\}
$$

and $\int_0^T h(t)\,dx(t)$ is a stochastic integral with mean zero, and variance $\int_0^T (h(t))^2\,dt$. In particular, if $\Gamma = C[0, T]$, then (5.1) becomes:

$$
\text{(5.3)} \qquad E[F(y)] = E[F(x + x_0)J(x_0, x)].
$$

Yeh [10] gives a conditional version of (5.3) which states that under the same conditions as above,

$$(5.4) \quad E(F(y)|y(T) = \xi) = E(F(x + x_0)J(x_0, x)|x(T) = \xi - x_0(T))$$
$$\cdot \exp\left\{-\frac{x_0^2(T)}{2T}\right\} \exp\left\{\frac{x_0(T)}{T}\xi\right\}.$$

The following is the multi-conditional version of (5.3). Its proof is very simple and straightforward because our "inversion formula", Theorem 3, is so simple and easy to apply.

THEOREM 4. *Let* $x_0(t) = \int_0^t h(s)\, ds$ *for all* $t \in [0, T]$ *with* $h \in L^2[0, T]$. *If* F *is any Wiener integrable function on* $C[0, T]$, *then for any partition* $\tau = \{t_1, \ldots, t_n\}$ *of* $[0, T]$,

$$E(F(y)|y(t_j) = \xi_j, \ j = 1, \ldots, n)$$
$$= E(F(x + x_0)J(x_0, x)|x(t_j) = \xi_j - x_0(t_j), \ j = 1, \ldots, n)$$
$$\cdot \prod_{j=1}^n \exp\left\{\frac{-(x_0(t_j) - x_0(t_{j-1}))^2}{2(t_j - t_{j-1})} + \frac{(x_0(t_j) - x_0(t_{j-1}))(\xi_j - \xi_{j-1})}{t_j - t_{j-1}}\right\}$$

where $J(x_0, x)$ *is given by* (5.2).

Proof. By use of Theorem 3 we see that

$$(5.5) \qquad E(F(y)|y(t_j) = \xi_j, \ j = 1, \ldots, n) = \hat{E}[F(y - [y] + [\vec{\xi}])].$$

Now applying (5.3) and noting that $[x + x_0](t) = [x](t) + [x_0](t)$ we have

$$(5.6) \quad \hat{E}[F(y - [y] + [\vec{\xi}])] = \hat{E}[F(x + x_0 - [x] - [x_0] + [\vec{\xi}])J(x_0, x)].$$

Next we rewrite $J(x_0, x)$ in the form

$$(5.7) \quad J(x_0 x,) = \exp\left\{-\frac{1}{2}\int_0^T (h(t))^2\, dt\right\}$$
$$\cdot \exp\left\{-\int_0^T h(t)\, d(x(t) - [x](t) + [\vec{\xi}](t) - [x_0](t))\right\}$$
$$\cdot \exp\left\{-\int_0^T h(t)\, d[x](t)\right\}$$
$$\cdot \exp\left\{\int_0^T h(t)\, d([\vec{\xi}](t) - [x_0](t))\right\}.$$

But

(5.8) $\int_0^T h(t)\,d[x](t)$

$$= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} h(t)\,d\left(x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1}))\right)$$

$$= \sum_{j=1}^n \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})).$$

Similarly,

(5.9) $\int_0^T h(t)\,d([\vec{\xi}](t) - [x_0](t))$

$$= \sum_{j=1}^n \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}[(\xi_j - \xi_{j-1}) - (x_0(t_j) - x_0(t_{j-1}))].$$

On the other hand,

$$E\left[\exp\left\{-\int_0^T h(t)\,d[x](t)\right\}\right]$$

$$= E\left[\exp\left\{-\sum_{j=1}^n \frac{x_0(t_j) - x_0(t_{j-1})}{\sqrt{t_j - t_{j-1}}} \cdot \frac{x(t_j) - x(t_{j-1})}{\sqrt{t_j - t_{j-1}}}\right\}\right].$$

Since

$$\left\{\frac{x(t_j) - x(t_{j-1})}{\sqrt{t_j - t_{j-1}}},\ j = 1,\ldots,n\right\}$$

is a set of independent identically distributed standard normal random variables, we have that

(5.10) $\qquad E\left[\exp\left\{-\int_0^T h(t)\,d[x](t)\right\}\right]$

$$= \prod_{j=1}^n \exp\left\{\frac{(x_0(t_j) - x_0(t_{j-1}))^2}{2(t_j - t_{j-1})}\right\}.$$

Now using the expression (5.8), one can easily see that $\int_0^T h(t)\,d[x](t)$ is independent of the process $\{x(t) - [x](t) : 0 \le t \le T\}$. Thus,

$$F(x + x_0 - [x] - [x_0] + [\vec{\xi}])$$

$$\cdot \exp\left\{-\int_0^T h(t)\,d(x(t) - [x](t) + [\vec{\xi}](t) - [x_0](t))\right\}$$

and $\exp\{-\int_0^T h(t)\, d[x](t)\}$ are independent. Hence, using (5.7), (5.9), and (5.10), it follows that

$$\hat{E}[F(x + x_0 - [x] - [x_0] + [\vec{\xi}])J(x_0, x)]$$

$$= \hat{E}\left[F(x + x_0 - [x] - [x_0] + [\vec{\xi}])\right.$$

$$\cdot \exp\left\{-\int_0^T h(t)\, d(x(t) - [x](t) + [\vec{\xi}](t) - [x_0](t))\right\}\right]$$

$$\cdot \exp\left\{-\frac{1}{2}\int_0^T (h(t))^2\, dt\right\} \prod_{j=1}^n \exp\left\{\frac{(x_0(t_j) - x_0(t_{j-1}))^2}{2(t_j - t_{j-1})}\right\}$$

$$\cdot \prod_{j=1}^n \exp\left\{\frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}[(\xi_j - \xi_{j-1}) - (x_0(t_j) - x_0(t_{j-1}))]\right\}.$$

Therefore, by using (2.5)

$$\hat{E}[F(x + x_0 - [x] - [x_0] + [\vec{\xi}])J(x_0, x)]$$

$$= E\left(F(x + x_0)\exp\left\{-\int_0^T h(t)\, dx(t)\right\}\,\bigg|\, x(t_j) = \xi_j - x_0(t_j),\right.$$

$$\left. j = 1, \ldots, n\right)$$

$$\cdot \exp\left\{-\frac{1}{2}\int_0^T (h(t))^2\, dt\right\}$$

$$\cdot \prod_{j=1}^n \exp\left\{-\frac{(x_0(t_j) - x_0(t_{j-1}))^2}{2(t_j - t_{j-1})}\right.$$

$$\left. +\frac{(x_0(t_j) - x_0(t_{j-1}))(\xi_j - \xi_{j-1})}{t_j - t_{j-1}}\right\}.$$

This, together with (5.6) and (5.5), completes the proof.

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