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CHARACTERIZATION OF C^*-ALGEBRAS WITH CONTINUOUS TRACE BY PROPERTIES OF THEIR PURE STATES

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We characterize C^*-algebras with continuous trace among all C^*-algebras by a condition on the set $P(A)$ of pure states. The condition is that (1) the graph $R(A)$ of the unitary equivalence relation on $P(A)$ is closed in $P(A) \times P(A)$, and (2) transition probabilities are continuous for the product topology on $R(A)$ (i.e. that inherited from $P(A) \times P(A)$). If $R(A)$ is given the quotient topology, these conditions are equivalent to properness of the inclusion map from $R(A)$ into $P(A) \times P(A)$. We show the product and quotient topologies on $R(A)$ coincide iff transition probabilities are continuous for the product topology, and this in turn is equivalent to Fell's condition. Transition probabilities are always continuous for the quotient topology on $R(A)$.

Introduction. In [**15**] it was shown that the set $P(A)$ of pure states of a C^*-algebra A determines A up to *-isomorphism. Here $P(A)$ carries the structure of a uniform space (for the weak* uniformity), transition probabilities (or equivalently, the distance given by the norm on $P(A) \subseteq A^*$), and orientation.

In §2 of the current paper we explore a connection between two of these structures by studying weak* continuity of transition probabilities. Continuity on all of $P(A) \times P(A)$ is rare: it occurs only for A equal to a c_0 direct sum of elementary C^*-algebras. (Equivalently: A is type I and $\hat{A}$ is discrete.) The set $R(A)$ of pairs of unitarily equivalent pure states provides a more interesting domain for transition probabilities. For the topology inherited as a subspace of $P(A) \times P(A)$ (which we call the product topology on $R(A)$), continuity of transition probabilities restricted to $R(A)$ is equivalent to Fell's condition. (Roughly following the terminology in [**4**], we say a C^*-algebra A satisfies Fell's condition if for each π_0 in $\hat{A}$ there is an element b in A^+ such that $\pi(b)$ is a projection of rank one for all π in some neighborhood of π_0.) This gives the characterization of continuous trace algebras described in the abstract above.

There is a second topology on $R(A)$ that is of great interest. Let $G(A)$ be the set of extreme points of the unit ball A^*. If ϕ is in $G(A)$, then $(|\phi|, |\phi^*|)$ is in $R(A)$, and the map $\phi \to (|\phi|, |\phi^*|)$ is a continuous surjection onto $R(A)$ with the product topology. In general the product and quotient topologies on $R(A)$ do not agree. The quotient topology played an important role in Renault's applications of the dual groupoid **[13]**.

One virtue of the quotient topology is that $G(A) \to R(A)$ becomes a locally trivial principal S^1-bundle, as observed in **[13]**. Another virtue is discussed in §3: transition probabilities are always continuous for the quotient topology on $R(A)$. It is shown that the product and quotient topologies coincide iff transition probabilities are continuous for the product topology. (It follows that these topologies coincide if A has continuous trace; this result is in **[11]** and **[13]**.) This result can be rephrased by saying A has continuous trace iff the canonical map from $R(A)$ into $P(A) \times P(A)$ is proper. In this form it is closely related to Green's **[9]** characterization of free actions of groups giving rise to continuous trace algebras: in each case the key property is properness of the map $s \times r$ where s and r are respectively the source and range maps on a principal groupoid.

Finally, we show continuity of transition probabilities at points in the diagonal of $R(A)$ implies continuity on all of $R(A)$ (for the product topology).

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1. Preliminaries. Let A be a C^*-algebra. If ϕ and ψ are pure states on A, let p and q be their respective support projections. Then the transition probability between ϕ and ψ is denoted $\langle \phi, \psi \rangle$ and is defined by

$$\langle \phi, \psi \rangle = \phi(q) = \psi(p).$$

If ϕ and ψ are unitarily equivalent, there will be an irreducible representation $\pi: A \to B(H)$ and unit vectors ξ and η such that

$$\phi(a) = (\pi(a)\xi, \xi) \quad \text{and} \quad \psi(a) = (\pi(a)\eta, \eta)$$

for all a in A. Then $\langle \phi, \psi \rangle = |(\xi, \eta)|^2$. (If ϕ and ψ are inequivalent their transition probability is zero). If $\langle \phi, \psi \rangle = 0$ we say ϕ and ψ are orthogonal.

We denote by $P(A)$ the set of pure states on A, with the weak* topology. The symbol $G(A)$ denotes the set of all extreme points of the unit ball of A^*, with the weak*-topology. (We call elements of $G(A)$ "pure functionals".)

For b in A and ϕ in A^* we define functionals $b\phi$ and ϕb in A^* by

$$(b\phi)(a) = \phi(ba), \quad \text{and} \quad (\phi b)(a) = \phi(ab).$$

We recall the definition of $|\phi|$, following the conventions in [4, §12]. For each ϕ in A^* there is a unique pair (u, ψ) with u being a partial isometry in A^{**}, ψ in A^* satisfying $\psi \geq 0$, $\|\psi\| = \|\phi\|$, $\phi = u\psi$, $\psi = u^*\phi$, and with the final projection of u being dominated by the support of ψ. The functional ψ is denoted $|\phi|$ and is called the absolute value of ϕ. The equation $\phi = u|\phi|$ is called the polar decomposition of ϕ. We note that some authors define $b\phi$, ϕb, and $|\phi|$ differently, so that what they call the absolute value of ϕ would be denoted $|\phi^*|$ here, where ϕ^* is the adjoint of ϕ, i.e.

$$\phi^*(a) = \phi(a^*)^-.$$

By [5, Lemma 3.5] the map $\phi \to |\phi|$ is weak* continuous if restricted to the set of ϕ in A^* of norm one.

The result below describes the set $G(A)$ of pure functionals more explicitly, and describes the map $\phi \to |\phi|$. We make no claim of originality. Although we know of no explicit reference, it seems likely this is known by many.

PROPOSITION 1.1. *The pure functionals on a C^*-algebra A are precisely those maps ϕ of the form*

$$\phi(a) = (\pi(a)\xi, \eta) \tag{1}$$

for $\pi: A \to B(H)$ an irreducible representation and for unit vectors ξ and η. The absolute value of ϕ is given by

$$|\phi|(a) = (\pi(a)\xi, \xi).$$

Proof. Suppose first that ϕ is any pure functional on A, and let $\phi = u|\phi|$ be its polar decomposition. By [1, Theorem 2.1], ϕ in A is a pure functional iff $|\phi|$ is a pure state. Let (π, H, ξ) be the GNS representation for $|\phi|$. Then

$$\phi(a) = (u|\phi|)(a) = (\pi(ua)\xi, \xi) = (\pi(a)\xi, \pi(u^*)\xi).$$

If we put $\eta = \pi(u^*)\xi$, then ϕ has the specified form.

Conversely, suppose ϕ has the form in (1). Let c be the minimal central projection in A^{**} such that $\pi(c) = 1$, so that $\pi: cA^{**} \to B(H)$ is a *-isomorphism. Let u be the partial isometry in cA^{**} such that

$\pi(u)$ has initial space $\mathbf{C}\eta$, final space $\mathbf{C}\xi$, and maps η to ξ. Let ψ in A^* be defined by

$$\psi(a) = (\pi(a)\xi, \xi).$$

Then ψ is a pure state on A. By [**1**, Theorem 2.1], ϕ is a pure functional iff $|\phi|$ is a pure state, so we'll be done if we show $\psi = |\phi|$. It is readily verified that $u\psi = \phi$, $u^*\phi = \psi$, and $uu^* =$ support projection of ψ. Thus $\phi = u\psi$ is the polar decomposition of ϕ, completing the proof.

Note that for ϕ as in equation (1),

$$\phi^*(a) = \phi(a^*)^- = (\pi(a)\eta, \xi)$$

so that $|\phi^*|(a) = (\pi(a)\eta, \eta)$. In particular, $|\phi|$ and $|\phi^*|$ will be equivalent pure states. Conversely any pair of equivalent pure states are of the form $a \to (\pi(a)\xi, \xi)$ and $a \to (\pi(a)\eta, \eta)$ and thus coincide with $(|\phi|, |\phi^*|)$ for some ϕ in $G(A)$.

We let $R(A)$ denote the set of pairs of equivalent pure states, i.e. the graph of the unitary equivalence relation on $P(A)$. By the product topology on $R(A)$ we mean the topology as a subspace of $P(A) \times P(A)$.

There is another topology on $R(A)$ that is of interest, namely the quotient topology with respect to the map $G(A) \to R(A)$ given by $\phi \to (|\phi|, |\phi^*|)$. As mentioned previously, this map is continuous from $G(A)$ to $R(A)$ if the latter is given the product topology, so the identity map on $R(A)$ is continuous from the quotient topology to the product topology. However, as we shall see these topologies need not coincide.

Finally, we observe that if ϕ and ψ are in $G(A)$, then it follows from Proposition 1.1 that $(|\phi|, |\phi^*|) = (|\psi|, |\psi^*|)$ iff $\phi = \lambda\psi$ for some λ in S^1. Thus $R(A)$ with the quotient topology can also be identified as the quotient of $G(A)$ by the natural action of S^1.

2. Transition probabilities and the product topology. We will use the symbol $\langle\ \rangle$ to denote the map from $R(A)$ into $\mathbf{R}$ given by transition probabilities, i.e. $(\phi, \psi) \to \langle\phi, \psi\rangle$. The purpose of this section is to investigate for which C^*-algebras A this map is continuous if $R(A)$ is given the product topology.

We will need the following result; see [**8**, Corollary 9], [**10**, p. 146], and [**12**, Lemma 2.4]:

LEMMA 2.1. *If ϕ and ψ are pure states of a C^*-algebra A, then*

$$\|\phi - \psi\| = 2(1 - \langle\phi, \psi\rangle)^{1/2}.$$

For $\pi \in \hat{A}$ let $S(\pi)$ be the set of those σ in $\hat{A}$ such that σ and π cannot be separated by disjoint open subsets of $\hat{A}$. Since $\hat{A} - S(\pi)$ is open, $S(\pi)$ is a closed subset of $\hat{A}$ containing π.

LEMMA 2.2. *Let A be a liminal C^*-algebra such that $\langle\ \rangle$ is continuous for the product topology. Then for each π in $\hat{A}$, π is an isolated point of the subspace $S(\pi)$.*

Proof. Let $\pi \in \hat{A}$ and suppose that π is not isolated in $S(\pi)$. Let ξ be a unit vector in the Hilbert space H_π and let $\phi = \omega_\xi \circ \pi \in P(A)$. Let $\mathcal{N}$ be a base of weak*-open neighborhoods of zero in A^*. Given $N \in \mathcal{N}$ choose $M \in \mathcal{N}$ with $M + M \subseteq N$. The canonical image of $(\phi + M) \cap P(A)$ in $\hat{A}$ is an open neighborhood of π [**4**, 3.4.11] and hence contains some element σ of $S(\pi) - \{\pi\}$. Thus there exists a unit vector $\eta \in H_\sigma$ such that if $\psi = \omega_\eta \circ \sigma$ then $\psi \in (\phi + M) \cap P(A)$.

Since $\sigma \in S(\pi)$ and A is liminal, it follows from the construction in [**6**, pp. 603–4] that there exists $(\phi_N, \psi_N) \in R(A)$ such that

$$\phi_N \in \phi + N, \qquad \psi_N \in \psi + M \subseteq \varphi + N$$

and $\langle \phi_N, \psi_N \rangle \leq 1/2$. The nets (ϕ_N) and (ψ_N), indexed by $\mathcal{N}$ in the obvious way, are both convergent to ϕ. However

$$\langle \phi_N, \psi_N \rangle \nrightarrow \langle \phi, \psi \rangle,$$

contradicting the continuity hypothesis.

THEOREM 2.3. *Let A be a C^*-algebra. Then $\langle\ \rangle$ is continuous for the product topology on $R(A)$ iff A satisfies Fell's condition.*

Proof. Assume A satisfies Fell's condition, and suppose $\{(\phi_\alpha, \psi_\alpha)\}$ is a net in $R(A)$ which converges (in the product topology) to $(\phi, \psi) \in R(A)$. Let π_ϕ be the GNS representation for ϕ and π_ψ for ψ. Note that (ϕ, ψ) in $R(A)$ means that π_ϕ and π_ψ represent the same element of $\hat{A}$. Choose x and y in A^+ such that $\pi(x)$ and $\pi(y)$ are projections of rank one for all π in a neighborhood N of π_ϕ. Kadison's transitivity theorem implies that we can also arrange $\phi(x) = 1$ by replacing x by u^*ux for a suitable unitary in $\tilde{A}$, and we may similarly arrange $\psi(y) = 1$.

Now choose an index α_0 such that $\alpha \geq \alpha_0$ implies that ϕ_α and ψ_α are in the inverse image of N for the canonical map from $P(A)$ onto $\hat{A}$. For each $\alpha \geq \alpha_0$ let ϕ'_α and ψ'_α be the unique pure states equivalent to both ϕ_α and ψ_α and satisfying $\phi'_\alpha(x) = \psi'_\alpha(y) = 1$. Let π_α be the GNS representation for ϕ'_α. Since ϕ'_α is equivalent to ϕ_α and $\phi_\alpha \to \phi$, then $\pi_\alpha \to \pi_\phi$. Furthermore for $\alpha \geq \alpha_0$

$$\langle \phi'_\alpha, \psi'_\alpha \rangle = \operatorname{tr} \pi_\alpha(xyx). \tag{2}$$

Since $x \to \operatorname{tr} \pi(x)$ is constant (and thus continuous) in a neighborhood of π_ϕ, and since

$$0 \leq xyx \leq \|y\| x^2 \leq \|y\| \, \|x\| x,$$

by [4, 4.4.2] the function $\to \operatorname{tr} \pi(xyx)$ is continuous at π_ϕ. By equation (2), it follows that

$$\langle \phi'_\alpha, \psi'_\alpha \rangle \to \operatorname{tr} \pi_\phi(xyx) = \langle \phi, \psi \rangle.$$

Finally, since

$$\langle \phi_\alpha, \phi'_\alpha \rangle = \phi_\alpha(x) \to \phi(x) = 1$$

by Lemma 2.1 we have $\|\phi_\alpha - \phi'_\alpha\| \to 0$. Similarly $\|\psi_\alpha - \psi'_\alpha\| \to 0$. Thus

$$\begin{aligned} &|\langle \phi_\alpha, \psi_\alpha \rangle - \langle \phi'_\alpha, \psi'_\alpha \rangle| \\ &\quad \leq |\langle \phi_\alpha, \psi_\alpha \rangle - \langle \phi_\alpha, \psi'_\alpha \rangle| + |\langle \phi_\alpha, \psi'_\alpha \rangle - \langle \phi'_\alpha, \psi'_\alpha \rangle| \\ &\quad \leq \|\psi_\alpha - \psi'_\alpha\| + \|\phi_\alpha - \phi'_\alpha\| \to 0 \end{aligned}$$

which proves that $\langle \phi_\alpha, \psi_\alpha \rangle \to \langle \phi, \psi \rangle$, finishing the proof that $\langle\ \rangle$ is continuous for the product topology on $R(A)$.

Now suppose $\langle\ \rangle$ is continuous for the product topology on $R(A)$. We first show A is type I, and then that it is liminal.

Suppose A is not of type I. Let $\pi: A \to B(H)$ be an irreducible representation such that $\pi(A) \cap K = \{0\}$, where K is the algebra of compact operators on H. Let ω_ξ be a vector state on $\pi(A)$. Let $\phi = \omega_\xi \circ \pi$ be the associated pure state of A, and let ψ be the extension of ω_ξ to the C^*-algebra $B = \pi(A) + K$ which satisfies $\psi(K) = 0$. By [7, Theorem 2] there is a net $\{\psi_\alpha\}$ of vector states on B such that $\psi_\alpha \to \psi$ on B. Let $\phi_\alpha = \psi_\alpha \circ \pi$, and note that $\{\phi_\alpha\}$ is a net in $P(A)$ which converges to $\phi = \psi \circ \pi$. Let q be the rank one projection onto $\mathbf{C}\xi$. Then if ψ_α is the vector state corresponding to a unit vector ξ_α, we have

$$\langle \phi_\alpha, \phi \rangle = |(\xi, \xi_\alpha)|^2 = \psi_\alpha(q) \to \psi(q) = 0.$$

On the other hand, the net $\{(\phi_\alpha, \phi)\}$ converges to (ϕ, ϕ) in $R(A)$ for the product topology, which shows that $\langle\ \rangle$ is discontinuous in that topology. This contradiction proves that A is of type I.

To show that A is liminal it suffices to show that each point of $\hat{A}$ is closed: [4, 4.7.15] or [6, Theorem 4]. Since $\hat{A}$ has the quotient topology from $P(A)$, we must show each equivalence class in $P(A)$ is closed in $P(A)$. Suppose $\{\phi_\alpha\}$ is a net of equivalent pure states of A which converges to a pure state ϕ. By continuity of $\langle\ \rangle$, since the net which to (α, β) assigns $(\phi_\alpha, \phi_\beta)$ converges to (ϕ, ϕ), then $\langle \phi_\alpha, \phi_\beta \rangle \to 1$. From Lemma 2.1, it follows that $\|\phi_\alpha - \phi_\beta\| \to 0$. Thus $\{\phi_\alpha\}$ is a Cauchy net for the norm, and so $\{\phi_\alpha\}$ converges in norm to some state ψ, so $\|\phi_\alpha - \phi\| \to 0$. In particular, for α large enough, $\|\phi_\alpha - \phi\| < 2$, and so ϕ_α and ϕ are equivalent by [**8**, Corollary 9]. Thus equivalence classes are closed in $P(A)$, and so A is liminal.

We now establish that Fell's condition holds. Let $\pi \in \hat{A}$. Since $S(\pi)$ is closed, there exists a closed two-sided ideal J of A such that $S(\pi) = (A/J)^\wedge$. By Lemma 2.2, π is isolated in $(A/J)^\wedge$ and so there exists a positive element b of norm one in A/J such that $\pi(b)$ is a projection of rank one and $\sigma(b) = 0$ for all other $\sigma \in (A/J)^\wedge$. Let A be a positive element of A such that $a + J = b$. Using functional calculus if necessary we may arrange that $\|a\| = 1$. By lower semicontinuity [**4**, 3.3.2], $\|\sigma(a)\| \to 1$ as $\sigma \to \pi$ in $\hat{A}$. Note that for each $\sigma \in \hat{A}$, $\sigma(a)$ is a positive compact operator with largest eigenvalue $\|\sigma(a)\|$.

We claim that there exists $\delta \in (0, 1)$ and an open neighborhood U of π such that, for each $\sigma \in U$, $\|\sigma(a)\|$ is the only eigenvalue of $\sigma(a)$ greater than $1 - \delta$ and furthermore the eigenspace for $\|\sigma(a)\|$ has dimension one. Supposing otherwise, for each $\delta \in (0, 1)$, and open neighborhood U of π there exists $\pi_{U,\delta} \in U$, orthogonal unit vectors $\xi_{U,\delta}$ and $\eta_{U,\delta}$ in the Hilbert space for $\pi_{U,\delta}$ and real numbers $\lambda_{U,\delta}, \mu_{U,\delta} > 1 - \delta$ such that

$$\pi_{U,\delta}(a)\xi_{U,\delta} = \lambda_{U,\delta}\xi_{U,\delta} \quad \text{and} \quad \pi_{U,\delta}(a)\eta_{U,\delta} = \mu_{U,\delta}\eta_{U,\delta}.$$

We direct the set of all such (U, δ) by defining $(U, \delta) \geq (U', \delta')$ if and only if $U \subseteq U'$ and $\delta \geq \delta'$. Then, writing $\alpha = (U, \xi)$, we have

$$\pi_\alpha \to \pi, \quad \pi_\alpha(a)\xi_\alpha = \lambda_\alpha \xi_\alpha, \quad \pi_\alpha(a)\eta_\alpha = \mu_\alpha \eta_\alpha \quad \text{and} \quad \lambda_\alpha, \mu_\alpha \to 1.$$

Let

$$\phi_\alpha = \omega_{\xi_\alpha} \circ \pi_\alpha \quad \text{and} \quad \psi_\alpha = \omega_{\eta_\alpha} \circ \pi_\alpha.$$

Since $\langle \xi_\alpha, \eta_\alpha \rangle = 0$ we have $\langle \phi_\alpha, \psi_\alpha \rangle = 0$. Passing to subnets if necessary, we may assume by compactness that $\phi_\alpha \to \phi$ and $\psi_\alpha \to \psi$ where $\phi, \psi \geq 0$ and $\|\phi\|, \|\psi\| \leq 1$. Since $\phi_\alpha(a) = \lambda_\alpha \to 1$, ϕ (and similarly ψ) is a state.

Let I be the kernel of the GNS representation π_ϕ. Let $\sigma \in (A/I)^\wedge$; we claim $\pi_\alpha \to \sigma$. Let K be a closed two-sided ideal of A such that $\sigma(K) \neq \{0\}$. Then $\phi(K) \neq \{0\}$ and so there exists α_0 such that $\phi_\alpha(K) \neq \{0\}$ and hence $\pi_\alpha(K) \neq \{0\}$ for all $\alpha \geq \alpha_0$. Thus $\pi_\alpha \to \sigma$ (see also [**2**, Lemma 3.8]). Hence

$$(A/I)^\wedge \subseteq S(\pi) = (A/J)^\wedge$$

and so $I \supseteq J$. Thus ϕ factors through A/J. Since π is isolated in $(A/J)^\wedge$ and $\phi(a) = 1$, $\phi = \rho \circ \pi$ where ρ is the unique pure state of

$\pi(A)$ supported by the rank one projection $\pi(a)$. The same applies to ψ and so $\phi = \psi \in P(A)$. Thus $(\phi_\alpha, \psi_\alpha) \to (\phi, \phi)$ whereas $\langle \phi_\alpha, \psi_\alpha \rangle \to 0$. This contradicts the continuity of $\langle\ \rangle$ and so δ and U exist as claimed.

Since $\|\sigma(a)\| \to 1$ as $\sigma \to \pi$ we may assume, by shrinking U if necessary, that $\|\sigma(a)\| > 1 - \delta/2$ for $\sigma \in U$. Then for $\sigma \in U$ we have

$$\sigma(a) = \|\sigma(a)\| e_\sigma + x_\sigma$$

where e_σ is a projection of rank one, $e_\sigma x_\sigma = x_\sigma e_\sigma = 0$, and $\|x_\sigma\| \leq 1-\delta$. Define a function f to be zero on $(-\infty, 1-\delta]$, one on $[1-\delta/2, \infty)$ and linear on $[1-\delta, 1-\delta/2]$. Then $f(a)$ is a positive element of A and $\sigma(f(a)) = e_\sigma$ for each $\sigma \in U$.

Recall that a C^*-algebra A has continuous trace iff $\hat{A}$ is Hausdorff and A satisfies Fell's condition: [**4**, 4.5.4]. Furthermore, $\hat{A}$ is Hausdorff iff the graph $R(A)$ of the unitary equivalence relation on $P(A)$ is closed in $P(A) \times P(A)$ [**3**, I.8.3.8]. This combined with Theorem 2.3 gives us a characterization of continuous trace algebras in terms of $P(A)$ and $R(A)$.

COROLLARY 2.4. *A C^*-algebra A has continuous trace iff $\langle\ \rangle$ is continuous for the product topology on $R(A)$ and $R(A)$ is closed in $P(A) \times P(A)$.*

REMARKS. (1) Let $T: P(A) \times P(A) \to [0, 1]$ be defined by $T(\phi, \psi) = \langle \phi, \psi \rangle$. If A has continuous trace, even if A is abelian, T need not be continuous on $P(A) \times P(A)$. In fact, continuity of T on $P(A) \times P(A)$ is equivalent to A being the c_0 direct sum of elementary C^*-algebras. For if A is a c_0-sum $\bigoplus K_\alpha$ where each K_α is isomorphic to the algebra of compact operators on a Hilbert space H_α, then $R(A)$ is the disjoint union of $P(K_\alpha) \times P(K_\alpha)$, and T is continuous on $P(K_\alpha) \times P(K_\alpha) = R(K_\alpha)$ by Theorem 2.3. On the other hand, if T is continuous on $P(A) \times P(A)$, then A is liminal by (the proof of) Theorem 2.3. Furthermore, $\hat{A}$ must be discrete: if $\pi_\alpha \to \pi$ in $\hat{A}$ with each π_α distinct from π, then by openness of the map $P(A) \to \hat{A}$ there is a net $\{\phi_\beta\}$ in $P(A)$ converging to some ϕ in $P(A)$ with no ϕ_β equivalent to ϕ. Then each ϕ_β is orthogonal to ϕ, and $(\phi_\beta, \phi) \to (\phi, \phi)$ in $P(A) \times P(A)$ shows T is discontinuous on $P(A) \times P(A)$. It now follows that A is the c_0-sum of liminal algebras A_α with each $\hat{A}_\alpha$ singleton. Thus each A_α is elementary, as claimed.

(2) It follows from Lemma 2.1 and the fact that the unit ball of A^* is weak*-closed that T is upper semi-continuous. Hence T is automatically continuous at points (ϕ, ψ) for which $\langle \phi, \psi \rangle = 0$. Thus T

is continuous on $P(A) \times P(A)$ if and only if T is continuous at every point of $R(A)$. Of course, this can be quite different from continuity of T restricted to $R(A)$, and the reader should keep in mind that we are using the symbol $\langle\ \rangle$ to denote a map with domain $R(A)$.

(3) The requirement in Corollary 2.4 that $R(A)$ be closed in $P(A) \times P(A)$ is not redundant, as can be seen by considering the algebra of continuous functions f from $[0, 1]$ into M_2 with $f(1)$ diagonal.

3. Transition probabilities and the quotient topology on $R(A)$. Recall from §1 that $G(A)$ is the set of extreme points of the unit ball of A, with the weak* topology. The quotient topology on $R(A)$ is that determined by the map $\phi \to (|\phi|, |\phi^*|)$ from $G(A)$ onto $R(A)$.

Each element ϕ of $G(A)$ can be viewed as a σ-weakly continuous functional on A^{**}. In the lemma below, 1 is the identity of A^{**} (and of A, if A has an identity).

LEMMA 3.1. *The map $\phi \to \phi(1)$ is weak*-continuous on $G(A)$.*

Proof. Suppose $\phi_\alpha \to \phi$ in $G(A)$. Choose irreducible representations $\{\pi_\alpha\}$ and π, and unit vectors $\{\xi_\alpha\}$, $\{\eta_\alpha\}$, ξ, and η such that for all a in A,

$$\phi_\alpha(a) = (\pi_\alpha(a)\xi_\alpha, \eta_\alpha), \quad \text{and} \quad \phi(a) = (\pi(a)\xi, \eta).$$

Since $\phi_\alpha \to \phi$, then $|\phi_\alpha| \to |\phi|$, and so

$$|\phi_\alpha|(a) = (\pi_\alpha(a)\xi_\alpha, \xi_\alpha) \to (\pi(a)\xi, \xi) = |\phi|(a). \tag{3}$$

Now fix $\varepsilon > 0$ and choose u in a with $0 \le u \le 1$ and with $|\phi|(u) > 1 - \varepsilon$. (Any approximate unit for A converges σ-weakly in A^{**} to the identity element; let u be a suitable member of this approximate unit.)

Note that

$$\begin{aligned}\|\pi(u)\xi - \xi\|^2 &= \|\pi(1-u)\xi\|^2 = (\pi(1-u)^2\xi, \xi) \\ &\le (\pi(1-u)\xi, \xi) = |\phi|(1-u) < \varepsilon.\end{aligned} \tag{4}$$

Now for some α_0, $\alpha \ge \alpha_0$ will imply that $|\phi_\alpha|(u) > 1 - \varepsilon$, and so as above

$$\|\pi_\alpha(u)\xi_\alpha - \xi_\alpha\| \le \sqrt{\varepsilon}, \quad \text{for all } \alpha \ge \alpha_0. \tag{5}$$

Since $\phi_\alpha \to \phi$, then for some index $\alpha_1 > \alpha_0$ and for all $\alpha \ge \alpha_1$:

$$|\phi_\alpha(u) - \phi(u)| = |(\pi_\alpha(u)\xi_\alpha, \eta_\alpha) - (\pi(u)\xi, \eta)| < \varepsilon. \tag{6}$$

If we combine (4), (5), and (6) we get

$$|\phi_\alpha(1) - \phi(1)| = |(\xi_\alpha, \eta_\alpha) - (\xi, \eta)| < \varepsilon + \sqrt{\varepsilon} + \sqrt{\varepsilon} \quad \text{for } \alpha \ge \alpha_1.$$

Since ε was arbitrary, this shows $\phi_\alpha(1) \to \phi(1)$.

PROPOSITION 3.2. *The map $\langle\ \rangle$ on $R(A)$ is continuous for the quotient topology.*

Proof. The composition of the quotient map from $G(A)$ onto $R(A)$ followed by the map $\langle\ \rangle$ is given by

$$\phi \to \langle|\phi|, |\phi^*|\rangle = |(\xi, \eta)|^2 = |(\pi(1)\xi, \eta)|^2 = |\phi(1)|^2$$

where $\phi(a) = (\pi(a)\xi, \eta)$. By Lemma 3.1 this composition is continuous, and it follows that $\langle\ \rangle$ is continuous for the quotient topology.

LEMMA 3.3. *Let $\{(\phi_\alpha, \psi_\alpha)\}$ be a net in $R(A)$ which converges to (ϕ, ϕ) for some ϕ in $P(A)$ for the product topology. Then $(\phi_\alpha, \psi_\alpha) \to (\phi, \phi)$ in the quotient topology iff $\langle\phi_\alpha, \psi_\alpha\rangle \to 1$.*

Proof. Continuity of $\langle\ \rangle$ for the quotient topology proves the forward implication. For the converse, suppose $\langle\phi_\alpha, \psi_\alpha\rangle \to 1$. For each α choose a pure functional σ_α such that $|\sigma_\alpha| = \phi_\alpha$, $|\sigma_\alpha^*| = \psi_\alpha$, and such that $\sigma_\alpha(1)$ is real and nonnegative. We are going to show that $\|\phi_\alpha - \sigma_\alpha\| \to 0$. For a fixed index α, let π be an irreducible representation of A and ξ and η unit vectors such that

$$\sigma_\alpha(x) = (\pi(x)\xi, \eta) \quad \text{for all } x \text{ in } A.$$

Then

$$|\sigma_\alpha|(x) = \phi_\alpha(x) = (\pi(x)\xi, \xi)$$

and

$$\begin{aligned} |\phi_\alpha(x) - \sigma_\alpha(x)| &= |(\pi(x)\xi, \xi) - (\pi(x)\xi, \eta)| \\ &\le \|x\|\,\|\xi - \eta\| = \|x\|(2 - 2\,\mathrm{Re}(\xi, \eta))^{1/2}. \end{aligned}$$

Since we arranged that $\phi_\alpha(1) = (\xi, \eta)$ is real and nonnegative, then

$$\langle\phi_\alpha, \psi_\alpha\rangle = |(\xi, \eta)|^2$$

implies $\langle\phi_\alpha, \psi_\alpha\rangle^{1/2} = (\xi, \eta)$. Thus

$$\|\phi_\alpha - \sigma_\alpha\| \le \left(2 - 2\langle\phi_\alpha, \psi_\alpha\rangle^{1/2}\right)^{1/2}$$

which approaches zero as $\langle\phi_\alpha, \psi_\alpha\rangle \to 1$. Then $\phi_\alpha \to \phi$ (weak*) implies $\sigma_\alpha \to \phi$, and now $|\sigma_\alpha| = \phi_\alpha$, $|\sigma_\alpha^*| = \psi_\alpha$ implies that $(\phi_\alpha, \psi_\alpha) \to (|\phi|, |\phi^*|) = (\phi, \phi)$ in the quotient topology.

PROPOSITION 3.4. *The quotient and product topologies on $R(A)$ coincide iff transition probabilities are continuous on $R(A)$ for the product topology.*

Proof. Let $(\phi_\alpha, \psi_\alpha) \to (\phi, \psi)$ in $R(A)$ for the product topology, and assume that $\langle\ \rangle$ is continuous on $R(A)$ for the same topology. By Kadison's transitivity theorem, we can choose a unitary u in $\tilde{A}$ such that $u^*\psi u = \phi$. Then $(\phi_\alpha, u^*\psi_\alpha u) \to (\phi, \phi)$ in the product topology, and for the quotient topology as well (by Lemma 3.3). By the openness of the map from $G(A)$ onto $G(A)/S^1$ (or by the proof of Lemma 3.3), there is a net $\{\sigma_\beta\}$ in $G(A)$ such that $\sigma_\beta \to \phi$ and such that the image of $\{\sigma_\beta\}$ in $R(A)$ is a subnet of $\{(\phi_\alpha, u^*\psi_\alpha u)\}$. (Thus $|\sigma_\beta| = \phi_\beta$ and $|\sigma_\beta^*| = u^*\psi_\beta u$.)

From the description of pure functionals in Proposition 1.1, it follows that multiplication of functionals in A^* by a unitary in $\tilde{A}$ maps pure functionals to pure functionals. Furthermore, Proposition 1.1 gives the following results for absolute values:

$$|u\sigma_\beta| = |\sigma_\beta| = \phi_\beta,$$
$$|(u\sigma_\beta)^*| = u|\sigma_\beta^*|u^* = u(u^*\psi_\beta u)u^* = \psi_\beta.$$

Thus $\{u\sigma_\beta\}$ is a net in $G(A)$ which converges to the pure functional $u\phi$, and the image under the quotient map $G(A) \to R(A)$ is a subnet $\{(\phi_\beta, \psi_\beta)\}$ of the original net $\{(\phi_\alpha, \psi_\alpha)\}$. Thus this subnet converges to (ϕ, ψ) in the quotient topology. Since this same argument can be applied to any subnet in place of $\{(\phi_\alpha, \psi_\alpha)\}$, it follows that $(\phi_\alpha, \psi_\alpha) \to (\phi, \psi)$ in the quotient topology.

Conversely, suppose the quotient and product topologies coincide. Then $\langle\ \rangle$ is continuous for both topologies by Proposition 3.2.

Note that in the proof of the reverse implication of Proposition 3.4, we only used continuity of $\langle\ \rangle$ at points (ϕ, ϕ) in the diagonal of $R(A)$. Thus we have the following consequence.

COROLLARY 3.5. *Let A be a C^*-algebra. Then $\langle\ \rangle$ is continuous for the product topology on $R(A)$ iff $\langle\ \rangle$ is continuous at each point in the diagonal of the equivalence relation $R(A)$.*

The following rephrasing of the previous results was pointed out to us by Dana Williams and Jean Renault.

COROLLARY 3.6. *Let A be a C^*-algebra. These are equivalent.*

(i) *A has continuous trace.*

(ii) *The inclusion $R(A) \to P(A) \times P(A)$ is proper (where $R(A)$ has the quotient topology).*

(iii) *The canonical map from $G(A)$ into $P(A) \times P(A)$ is proper.*

Proof. By [**3**, I.10.1, Prop. 2] the inclusion map $R(A) \to P(A) \times P(A)$ is proper iff (1) its image is closed and (2) it is a homeomorphism onto its image. The latter property is equivalent to continuity of $\langle\ \rangle$ for the product topology by Proposition 3.4, and so equivalence of (i) and (ii) follows from Corollary 2.4. Since $R(A)$ is the quotient of $G(A)$ by the action of the compact group S^1, the map $G(A) \to R(A)$ is proper by [**3**, III.4.1.2]. By [**3**, I.10.1, Cor. 4] the map $G(A) \to P(A) \times P(A)$ will be proper iff $G(A) \to R(A)$ and $R(A) \to P(A) \times P(A)$ are proper, and this shows (iii) and (ii) are equivalent.

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