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## **THE ISOMORPHISM PROBLEM FOR ORTHODOX SEMIGROUPS**

THOMAS ERIC HALL

# THE ISOMORPHISM PROBLEM FOR ORTHODOX SEMIGROUPS

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The author's structure theorem for orthodox semigroups produced an orthodox semigroup  $\mathcal{K}(E, T, \psi)$  from a band  $E$ , an inverse semigroup  $T$  and a morphism  $\psi$  between two inverse semigroups, namely  $T$  and  $W_E/\gamma$ , an inverse semigroup constructed from  $E$ . Here, we solve the isomorphism problem: when are two such orthodox semigroups isomorphic? This leads to a way of producing all orthodox semigroups, up to isomorphism, with prescribed band  $E$  and maximum inverse semigroup morphic image  $T$ .

**1. Preliminaries.** A semigroup  $S$  is called *regular* (in the sense of von Neumann for rings) if for each  $a \in S$  there exists  $x \in S$  such that  $axa = a$ ; and  $S$  is called an *inverse semigroup* if for each  $a \in S$  there is a unique  $x \in S$  such that  $axa = a$  and  $xax = x$ . A *band* is a semigroup in which each element is idempotent, and an *orthodox semigroup* is a regular semigroup in which the idempotents form a subsemigroup (that is, a band).

We follow the notation and conventions of Howie [4].

*Result 1 [3, Theorem 5]. The maximum congruence contained in Green's relation  $\mathcal{K}$  on any regular semigroup  $S$ ,  $\mu = \mu(S)$  say, is given by  $\mu = \{(a, b) \in \mathcal{K} : \text{for some [for each pair of]} \mathcal{K}\text{-related inverses } a' \text{ of } a \text{ and } b' \text{ of } b, a'ea = b'eb \text{ for each idempotent } e \leq aa'\}$ .*

A regular semigroup  $S$  is called *fundamental* if  $\mu$  is the identity relation on  $S$ . For each band  $E$ , the semigroup  $W_E$  is fundamental, orthodox, has its band isomorphic to  $E$ , and contains, for each orthodox semigroup  $S$  with band  $E$ , a copy of  $S/\mu$  as a subsemigroup: see the author [1] (or [3] with  $E = \langle E \rangle$  and  $W_E = T_{\langle E \rangle}$ ) or Howie [4, §VI.2].

Now take any inverse semigroup  $T$ , and, if such exist, any idempotent-separating morphism  $\psi: T \rightarrow W_E/\gamma$  whose range contains the semilattice of all idempotents of  $W_E/\gamma$ , where  $\gamma$  denotes the least inverse semigroup congruence on  $W_E$ . A semigroup  $\mathcal{K}(E, T, \psi)$  (see

$S(E, T, \psi)$  in the author [2], or see Howie [4, §VI.4]) is defined by

$$\mathcal{H}(E, T, \psi) = \{(w, t) \in W_E \times T : w\gamma^{\natural} = t\psi\};$$

that is,  $\mathcal{H}(E, T, \psi)$  occurs in the pullback diagram

$$\begin{array}{ccc} \mathcal{H}(E, T, \psi) & \xrightarrow{p_2} & T \\ p_1 \downarrow & & \downarrow \psi \\ W_E & \xrightarrow{\gamma^{\natural}} & W_E/\gamma \end{array}.$$

Here,  $p_1$  and  $p_2$  are projections.

The semigroup  $\mathcal{H}(E, T, \psi)$  is orthodox, has band isomorphic to  $E$ , and has its maximum inverse semigroup morphic image isomorphic to  $T$ ; conversely every such semigroup is obtained in this way (the author [2], or Howie [4, §VI.4]).

## 2. The isomorphism problem.

**LEMMA 1.** *Take any two morphisms  $\phi, \psi$  from a regular semigroup  $T$  to a regular semigroup  $S$  such that the range of each of  $\phi$  and  $\psi$  contains the set  $E(S)$  of all the idempotents of  $S$ . If  $\phi|E(T) = \psi|E(T)$  then  $(t\phi, t\psi) \in \mu$ , for all  $t \in T$ ; in particular, if also  $S$  is fundamental, then  $\phi = \psi$ .*

*Proof.* Take any  $t \in T$  and any inverse  $t'$  of  $t$  in  $T$ . Of course, in  $S$ ,  $t'\phi$  and  $t'\psi$  are inverses of  $t\phi$  and  $t\psi$  respectively and  $(t'\phi)(t\phi) = (t't)\phi = (t't)\psi = (t'\psi)(t\psi)$ . Likewise  $(t\phi)(t'\phi) = (t\psi)(t'\psi)$ , so  $(t\phi)\mathcal{H}(t\psi)$  and  $(t'\phi)\mathcal{H}(t'\psi)$ . Take any idempotent  $e$  of  $S$  such that  $e \leq (tt')\phi$  and any  $x \in T$  such that  $x\phi = e$ : then  $(tt'xtt')\phi = [(tt')\phi]e[(tt')\phi] = e$ , so  $e \in \text{range}(\phi|tt'Ttt')$ . Now  $tt'Ttt'$  is a regular semigroup, so by Lallement's Lemma [4, Lemma II.4.7] there is an idempotent  $f \in tt'Ttt'$  such that  $f\phi = e$ . Since  $t'ft$  is idempotent, we have

$$\begin{aligned} (t'\phi)e(t\phi) &= (t'\phi)(f\phi)(t\phi) = (t'ft)\phi = (t'ft)\psi \\ &= (t'\psi)(f\psi)(t\psi) = (t'\psi)e(t\psi). \end{aligned}$$

Thus  $(t\phi, t\psi) \in \mu$ , as required, completing the proof.

Take any isomorphism  $\alpha: E \rightarrow E'$  from a band  $E$  to a band  $E'$ . Consider  $W_E$  and  $W_{E'}$  and, as usual, identify  $E$  and  $E'$  with the bands of

$W_E$  and  $W_{E'}$  respectively. Since  $W_{E'}$  is constructed from  $E'$  precisely as  $W_E$  is constructed from  $E$ , there is an isomorphism from  $W_E$  to  $W_{E'}$  extending  $\alpha$ , say  $\alpha^*$  (in fact, there is a unique such isomorphism, by Lemma 1). Denote by  $\gamma$  and  $\gamma'$  the least inverse semigroup congruences on  $W_E$  and  $W_{E'}$  respectively: then the map  $\alpha^{**}: W_E/\gamma \rightarrow W_{E'}/\gamma'$ , given by  $w\gamma\alpha^{**} = w\alpha^*\gamma'$ , for all  $w \in W_E$ , is an isomorphism such that  $\gamma^{\natural}\alpha^{**} = \alpha^*\gamma'^{\natural}$ , and is the unique such isomorphism. Summarizing, we have that the following diagram commutes, and  $\alpha^*, \alpha^{**}$  are the unique morphisms making the diagram commute.

$$\begin{array}{ccc}
 E & \xrightarrow{\alpha} & E' \\
 \downarrow \subseteq & & \downarrow \subseteq \\
 W_E & \xrightarrow{\alpha^*} & W_{E'} \\
 \downarrow \gamma^{\natural} & & \downarrow \gamma'^{\natural} \\
 W_E/\gamma & \xrightarrow{\alpha^{**}} & W_{E'}/\gamma'
 \end{array}$$

**THEOREM 2.** *Take any bands  $E, E'$ , inverse semigroups  $T, T'$  and idempotent-separating morphisms  $\psi: T \rightarrow W_E/\gamma$  and  $\psi': T' \rightarrow W_{E'}/\gamma'$  whose ranges contain the idempotents of  $W_E/\gamma$  and  $W_{E'}/\gamma'$  respectively. Then  $\mathcal{H}(E, T, \psi)$  is isomorphic to  $\mathcal{H}(E', T', \psi')$  if and only if there exist isomorphisms  $\alpha: E \rightarrow E'$  and  $\beta: T \rightarrow T'$  such that the following diagram commutes*

$$\begin{array}{ccc}
 T & \xrightarrow{\beta} & T' \\
 \psi \downarrow & & \downarrow \psi' ; \\
 W_E/\gamma & \xrightarrow{\alpha^{**}} & W_{E'}/\gamma'
 \end{array}$$

that is, such that  $\psi' = \beta^{-1}\psi\alpha^{**}$ .

*Proof.* (a) *if* statement. Suppose such  $\alpha, \beta$  exist. Informally we could say that  $E', T', \psi'$  are a renaming of  $E, T, \psi$  respectively, obtained by renaming each  $e \in E$  by  $e\alpha$  and each  $t \in T$  by  $t\beta$ , and so  $\mathcal{H}(E', T', \psi')$  is isomorphic to  $\mathcal{H}(E, T, \psi)$ . More formally, we consider the isomorphism  $(\alpha^*, \beta): W_E \times T \rightarrow W_{E'} \times T'$  given by  $(w, t)(\alpha^*, \beta) = (w\alpha^*, t\beta)$  for all  $(w, t) \in W_E \times T$ , and we show that  $\mathcal{H}(E, T, \psi)(\alpha^*, \beta) = \mathcal{H}(E', T', \psi')$ .

Take any  $(w, t) \in \mathcal{H}(E, T, \psi)$ : then  $w\gamma^{\natural} = t\psi$ , and so

$$t\beta\psi' = t\beta\beta^{-1}\psi\alpha^{**} = t\psi\alpha^{**} = w\gamma^{\natural}\alpha^{**} = w\alpha^{*}\gamma'^{\natural},$$

so  $(w, t)(\alpha^{*}, \beta) = (w\alpha^{*}, t\beta) \in \mathcal{H}(E', T', \psi')$  and hence  $\mathcal{H}(E, T, \psi)(\alpha^{*}, \beta) \subseteq \mathcal{H}(E', T', \psi')$ .

From symmetry, we deduce that

$$\mathcal{H}(E', T', \psi')(\alpha^{*}, \beta)^{-1} = \mathcal{H}(E', T', \psi')(\alpha^{*-1}, \beta^{-1}) \subseteq \mathcal{H}(E, T, \psi),$$

whence  $\mathcal{H}(E, T, \psi)(\alpha^{*}, \beta) = \mathcal{H}(E', T', \psi')$ , as required.

(b) *only if* statement. Informally, we could say that, for any orthodox semigroup  $S$  with band  $E$  and least inverse semigroup congruence  $\mathcal{Y}$ , there is a unique morphism  $\psi$  making the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{(\rho, \lambda)} & W_E \\ \mathcal{Y}^{\natural} \downarrow & & \downarrow \gamma^{\natural} \\ S/\mathcal{Y} & \xrightarrow{\psi} & W_E/\gamma \end{array}.$$

Hence  $E$ ,  $S/\mathcal{Y}$ ,  $\psi$  are all determined to within isomorphisms (or renamings) by  $S$ . Formally, we proceed as follows.

Take any isomorphism  $\theta: S \rightarrow S'$ , where  $S = \mathcal{H}(E, T, \psi)$  and  $S' = \mathcal{H}(E', T', \psi')$ . Put  $\theta|E = \alpha$ , an isomorphism of  $E$  upon  $E'$ , by Lallement's Lemma [4, Lemma II.4.7]. Let  $\mathcal{Y}$  and  $\mathcal{Y}'$  denote the least inverse semigroup congruences on  $S$  and  $S'$  respectively. Clearly there is a unique isomorphism  $\beta: S/\mathcal{Y} \rightarrow S'/\mathcal{Y}'$  making the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{\theta} & S' \\ \mathcal{Y}^{\natural} \downarrow & & \downarrow \mathcal{Y}'^{\natural} \\ S/\mathcal{Y} & \xrightarrow{\beta} & S'/\mathcal{Y}' \end{array}.$$

Now  $T \cong S/\mathcal{Y}$  and  $T' \cong S'/\mathcal{Y}'$  ([2, Theorem 1] or [4, Theorem VI.4.6]), so we assume without loss of generality that  $T = S/\mathcal{Y}$  and  $T' = S'/\mathcal{Y}'$ ; it remains to show that  $\psi' = \beta^{-1}\psi\alpha^{**}$ .

We shall see that Diagram 1 commutes ( $p_1, p_2, p'_1, p'_2$  are projections).

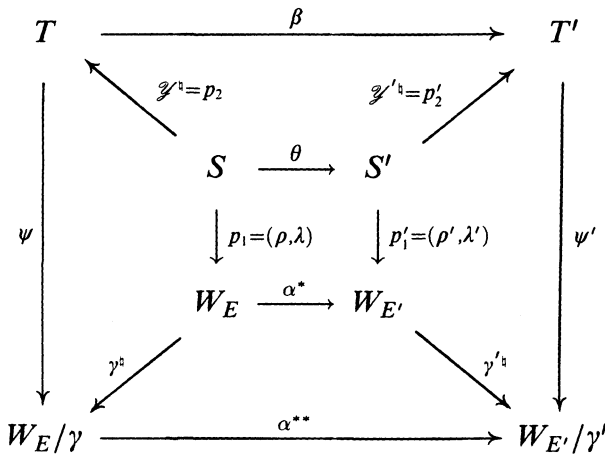


DIAGRAM 1

We have seen already that each of the four outer faces is a commuting diagram: we consider the central face. Now  $\theta p'_1$  and  $p_1 \alpha^*$  are morphisms which agree on  $E$  (with  $\alpha = \theta|_E$ ), and which map  $E$  (isomorphically) onto  $E'$ , the band of  $W_{E'}$ . Hence, by Lemma 1,  $\theta p'_1 = p_1 \alpha^*$ ; that is, the central face commutes.

Consideration of the external face leads us to the following diagram.

$$\begin{array}{ccc}
 S & \xrightarrow{p_1=(\rho, \lambda)} & W_E \\
 \mathscr{Z}^{\natural}=p_2 \downarrow & & \downarrow \gamma^{\natural} \\
 T & \xrightarrow[\beta \psi' \alpha^{** - 1}]{\psi} & W_E/\gamma
 \end{array}$$

The commuting of the five internal faces of Diagram 1 gives us that  $p_1 \gamma^{\natural} = p_2 \beta \psi' \alpha^{** - 1}$ . But the mapping  $s \mathscr{Z} \mapsto (\rho_S, \lambda_S) \gamma$  (for all  $s \in S$ ), namely  $\psi$ , is the unique morphism from  $T$  to  $W_E/\gamma$  making this diagram commute, and hence  $\psi = \beta \psi' \alpha^{** - 1}$  (that is, the external face commutes) and so  $\psi' = \beta^{-1} \psi \alpha^{**}$  as required.

**3. Orthodox semigroups, up to isomorphism.** Consider the following problem: given a band  $E$  and an inverse semigroup  $T$ , find, up to isomorphism, the orthodox semigroups with band  $E$  and with maximum inverse semigroup morphic image isomorphic to  $T$ .

The author's structure theorem ([2, Theorem 1] or [4, Theorem VI.4.6]) and Theorem 2 above together immediately yield a solution as follows.

Denote by  $\text{Aut}(S)$  the group of automorphisms of any semigroup  $S$ . From Lemma 1, for any  $\varphi \in \text{Aut}(W_E)$ , we see that  $\varphi = (\varphi|E)^*$ , so we have that  $\text{Aut}(E) \cong \text{Aut}(W_E)$  under the map  $\alpha \mapsto \alpha^*$  for each  $\alpha \in \text{Aut}(E)$ . The map  $\text{Aut}(W_E) \rightarrow \text{Aut}(W_E/\gamma)$ ,  $\alpha^* \mapsto \alpha^{**}$  (for each  $\alpha \in \text{Aut}(E)$ ), is a morphism; we denote its image by  $[\text{Aut}(E)]^{**}$ ; then  $[\text{Aut}(E)]^{**} = \{\alpha^{**} : \alpha \in \text{Aut}(E)\}$ .

Denote by  $M$  the set of idempotent-separating morphisms from  $T$  into  $W_E/\gamma$  whose ranges each contain the idempotents of  $W_E/\gamma$ . By [2, Corollary 1] or [4, Theorem VI.4.6], there exists an orthodox semigroup with band  $E$  and with maximum inverse semigroup morphic image isomorphic to  $T$ , if and only if  $M$  is nonempty. Assume henceforth that  $M$  is nonempty. Define an action on  $M$  by the group  $\text{Aut}(T) \times [\text{Aut}(E)]^{**}$  as follows:

$$\psi(\beta, \alpha^{**}) = \beta^{-1} \psi \alpha^{**},$$

for all  $\psi \in M, \beta \in \text{Aut}(T), \alpha \in \text{Aut}(E)$ .

The orbits of  $M$  under  $\text{Aut}(T) \times [\text{Aut}(E)]^{**}$  are the sets

$$\psi(\text{Aut}(T) \times [\text{Aut}(E)]^{**}) = \{\beta^{-1} \psi \alpha^{**} : \beta \in \text{Aut}(T), \alpha \in \text{Aut}(E)\},$$

for each  $\psi \in M$  (thus these sets partition  $M$ ). By Theorem 2, we have, for all  $\psi, \psi' \in M$ ,  $\mathcal{H}(E, T, \psi) \cong \mathcal{H}(E, T, \psi')$  if and only if  $\psi$  and  $\psi'$  are in the same orbit. Thus, if  $\{\psi_i : i \in I\}$  is a transversal of the set of orbits (that is, a selection of precisely one morphism from each orbit) then  $\mathcal{H}(E, T, \psi_i)$ ,  $i \in I$ , is a list of all the orthodox semigroups with band  $E$  and maximum inverse semigroup morphic image isomorphic to  $T$ , and the semigroups are pairwise nonisomorphic.

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