LATTICE PATHS AND MULTIPLE BASIC HYPERGEOMETRIC SERIES

A. K. AGARWAL AND DAVID BRESSOUD

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Certain basic hypergeometric series with multiple indices of summation are interpreted as generating functions for weighted lattice paths. The approach uses ideas of William Burge and gives rise to identities analogous to the Rogers-Ramanujan identities.

Many of the generating function identities that have arisen in partition theory have natural extensions to multiple series. The classical example of this consists of the Rogers-Ramanujan identities:

$$(0.1) \qquad \sum_{m \geq 0} \frac{q^{m^2}}{(q)_m} = \prod (1 - q^n)^{-1}, \qquad n \geq 1, n \equiv \pm 1 \pmod 5,$$

$$(0.2) \qquad \sum_{m \geq 0} \frac{q^{m^2+m}}{(q)_m} = \prod (1 - q^n)^{-1}, \qquad n \geq 2, n \equiv \pm 1 \pmod 5,$$

where $(a)_m (a; q)_m = (1 - a)(1 - aq) \cdots (1 - aq^{m-1})$.

A generalization of these identities to arbitrary odd modulus was found by George Andrews [And 1]: $(1 \leq i \leq k + 1, \ 1 \leq k)$

$$(0.3) \qquad \sum \frac{q^{m_1^2 + \cdots + m_k^2 + m_i + \cdots + m_k}}{(q)_{m_1 - m_2} \cdots (q)_{m_{k-1} - m_k} (q)_{m_k}} = \prod (1 - q^n)^{-1},$$
$$m_1 \geq m_2 \geq \cdots \geq m_k \geq 0; n \geq 1, n \not\equiv 0, \pm i \pmod{2k + 3}.$$

While each side of equations (0.1) and (0.2) is easily interpreted as a generating function for certain restricted partitions, the interpretation of the left-hand side of equation (0.3) presents more of a problem. It was William Burge [Bur 1, Bur 2] who demonstrated that the most natural interpretation is in terms of weighted binary words. He was then able to establish one-to-one correspondences between his binary words of weight n and several different sets of partitions of n, thus providing new proofs of partition identities of Basil Gordon [Gor] and George Andrews [And 2].

Burge also gave interpretations of two other multiple basic hypergeometric series, and Andrews and Bressoud [A&B] have interpreted a

fourth multiple series and restated all of the conditions on the binary words as restrictions on weighted lattice paths. The switch to lattice paths was made because the restrictions become easier to describe and visualize.

In §1, we shall define our terminology and give Burge's interpretations of the three multiple basic hypergeometric series he considered. We shall also interpret some recently discovered series found by Agarwal, Andrews and Bressoud [AA&B]. The proof that these new series are the generating functions for the appropriate classes of weighted lattice paths is given in §§2 through 5. In the sixth section, we establish a bijection between the appropriate class of lattice paths of weight n and a set of partitions of n. This bijection provides a new proof of a partition identity of Agarwal and Andrews [A&A].

1. The lattice paths. We shall be considering lattice paths of finite length lying in the first quadrant. All our paths will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:

northeast :   from (i, j) to $(i + 1, j + 1)$,
southeast :   from (i, j) to $(i + 1, j - 1)$, only allowed if $j > 0$,
horizontal :   from $(i, 0)$ to $(i + 1, 0)$, only allowed along x−axis.

All our lattice paths are either empty or terminate with a southeast step: from $(i, 1)$ to $(i + 1, 0)$.

In describing lattice paths, we shall use the following terminology:

PEAK: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.

VALLEY: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.

MOUNTAIN: A section of the path which starts on either the x- or y-axis, which ends on the x-axis, and which does not touch the x-axis anywhere in between the endpoints. Every mountain has at least one peak and may have more than one.

RANGE: A section of the path which starts either on the y-axis or at a vertex preceded by a horizontal step, which ends either at the end of the path or at a vertex followed by a horizontal step, and which does not contain any horizontal steps. Every range includes at least one mountain and may have more than one.

PLAIN: A section of path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.

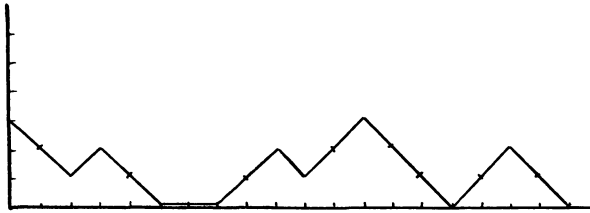EXAMPLE. The following path has five peaks, three valleys, three mountains, two ranges and one plain.

FIGURE 1

The HEIGHT of a vertex is its y-coordinate. The WEIGHT of a vertex is its x-coordinate. The WEIGHT OF A PATH is the sum of the weights of its peaks.

In the example given above, there are two peaks of height three and three of height two, two valleys of height one and one of height zero. The weight of this path is $0 + 3 + 9 + 12 + 17 = 41$.

We sum up what has been proved in [**Bur 1, Bur 2**].

PROPOSITION 1.1 [**Bur 1**]. *The multiple summation on the left side of equation* (0.3) *is the generating function for* $A(k, i, n) :=$ *the number of lattice paths of weight n which start at* $(0, k + 1 - i)$, *and have no peak above height k.*

COROLLARY 1.1. $A(k, i, n) =$ *the number of partitions of n into parts which are not congruent to 0 or* $\pm i$ *modulo* $2k + 3$.

The next identity was first proved in [**Bre 1**]. For $1 \leq i \leq k$:

$$(1.1) \quad \sum \frac{q^{m_1^2 + \cdots + m_k^2 + m_i + \cdots + m_k}}{(q)_{m_1 - m_2} \cdots (q)_{m_{k-1} - m_k}(q^2; q^2)_{m_k}} = \prod (1 - q^n)^{-1},$$

$$m_1 \geq \cdots \geq m_k \geq 0; n \geq 1, n \not\equiv 0, \pm i \pmod{2k + 2}.$$

PROPOSITION 1.2 [**Bur 1**]. *The multiple summation on the left side of equation* (1.1) *is the generating function for* $B(k, i, n) :=$ *the number of lattice paths of weight n which start at* $(0, k + 1 - i)$, *have no peak*

above height k and are such that every peak of height k has weight congruent to i − 1 modulo 2.

COROLLARY 1.2. *$B(k, i, n) =$ the number of partitions of n into parts not congruent to 0 or $\pm i$ modulo $2k + 2$.*

The next identity was first proved in [**Bre 2**], equation (3.8). For $i \leq i \leq k + 1$,

$$(1.2) \sum \frac{q^{2(m_1^2 + \cdots + m_k^2 + m_1 + \cdots + m_k)}(-q^{1-2m_1}; q^2)_{m_1}}{(q^2; q^2)_{m_1 - m_2} \cdot (q^2; q^2)_{m_{k-1} - m_k}(q^2; q^2)_{m_k}} = \prod (1 - q^n)^{-1},$$

$$m_1 \geq \cdots \geq m_k \geq 0; n \geq 1, n \not\equiv 2 \pmod 4,$$
$$\not\equiv 0, \pm(2i - 1) \pmod{4k + 4}.$$

PROPOSITION 1.3 [**Bur 2**]. *The multiple summation on the left side of equation (1.2) is the generating function for $C(k, i, n) :=$ the number of lattice paths of weight n which start at $(0, 2k + 2 - 2i)$, have maximum height 2k and are such that all plains have even length and all valleys have even weight.*

COROLLARY 1.3. *$C(k, i, n) =$ the number of partitions of n into parts not congruent to 2 modulo 4 nor congruent to 0 or $\pm(2i - 1)$ modulo $4k + 4$.*

Our final set of identities was proved in [**AA&B**]. If $k \geq 3$ is odd, $1 \leq i \leq (k + 1)/2$, and $r = (k - 1)/2$, then

$$(1.3) \sum \frac{q^{m_1^2 + \cdots + m_r^2 + m_1 + \cdots + m_{i-1} + 2m_i + \cdots + 2m_r}}{(q)_{m_1 - m_2} \cdots (q)_{m_{r-1} - m_r}(q)_{m_r}(q; q^2)_{m_r + 1}} = \prod (1 - q^n)^{-1},$$

$$m_1 \geq \cdots \geq m_r \geq 0; n \geq 1, n \not\equiv 0, \pm 2i \pmod{4k + 2}.$$

If $k \geq 2$ is even, $1 \leq i \leq (k + 2)/2$, and $r = k/2$, then

$$(1.4) \sum \frac{q^{m_1^2 + \cdots + m_r^2 + m_1 + \cdots + m_{i-1} + 2m_i + \cdots + 2m_r + m_r(m_r - 1)/2}}{(q)_{m_1 - m_2} \cdots (q)_{m_{r-1} - m_r}(q)_{m_r}(q; q^2)_{m_r + 1}}$$

$$= \prod (1 - q^n)^{-1}, \qquad m_1 \geq \cdots \geq m_r \geq 0;$$
$$n \geq 1, n \not\equiv 0, \pm 2i \pmod{4k + 2}.$$

If $k \geq 3$ is odd, $(k + 1)/2 \leq i \leq k - 1$, and $r = (k - 1)/2$, then

$$(1.5) \sum \frac{q^{m_1^2 + \cdots + m_r^2 - m_1 \cdots - m_{k-i}}(1 - q^{m_{k-i}})}{(q)_{m_1 - m_2} \cdots (q)_{m_{r-1} - m_r}(q)_{m_r}(q; q^2)_{m_r}} = \prod (1 - q^n)^{-1},$$

$$m_1 \geq \cdots \geq m_r \geq 0; n \geq 1, n \not\equiv 0, \pm 2i \pmod{4k + 2}.$$

If $k \geq 2$ is even, $k/2 \leq i \leq k - 1$, and $r = k/2$, then

$$(1.6) \quad \sum \frac{q^{m_1^2+\cdots+m_r^2-m_1-\cdots-m_{k-i}+m_r(m_r-1)/2}(1 - q^{m_{k-i}})}{(q)_{m_1-m_2} \cdots (q)_{m_{r-1}-m_r}(q)_{m_r}(q;q^2)_{m_r}}$$
$$= \prod (1 - q^n)^{-1}, \qquad m_1 \geq \cdots \geq m_r \geq 0;$$
$$n \geq 1, n \not\equiv 0, \pm 2i \pmod{4k + 2}.$$

If $k \geq 3$ is odd, $i = k$, and $r = (k - 1)/2$, then

$$(1.7) \quad \sum \frac{q^{m_1^2+\cdots+m_r^2}}{(q)_{m_1-m_2} \cdots (q)_{m_{r-1}m_r}(q)_{m_r}(q;q^2)_{m_r}} = \prod (1 - q^n)^{-1},$$
$$m_1 \geq \cdots \geq m_r \geq 0; n \geq 1, n \not\equiv 0, \pm 2k \pmod{4k + 2}.$$

If $k \geq 2$ is even, $i = k$, and $r = k/2$, then

$$(1.8) \quad \sum \frac{q^{m_1^2+\cdots+m_r^2+m_r(m_r-1)/2}}{(q)_{m_1-m_2} \cdots (q)_{m_{r-1}-m_r}(q)_{m_r}(q;q^2)_{m_r}} = \prod (1 - q^n)^{-1},$$
$$m_1 \geq \cdots \geq m_r \geq 0; n \geq 1, n \not\equiv 0, \pm 2k \pmod{4k + 2}.$$

The following proposition will be proved in the next sections.

PROPOSITION 1.4. *The multiple summations on the left-hand sides of equations (1.3)–(1.8) can each be interpreted, in the appropriate range, as the generating function for $D(k, i, n) :=$ the number of lattice paths of weight n which start at $(0, k - i)$ and have no valley above height $k - 3$. When $k = 2$ this means there are no valleys.*

COROLLARY 1.4. *$D(k, i, n) =$ the number of partitions of n into parts not congruent to 0 or $\pm 2i$ modulo $4k + 2$.*

2. Proof of Proposition 1.4 when $r = 1$. We shall start with equation (1.7) in the special case where $k = 3$ (and thus $i = 3$):

$$(2.1) \qquad \sum \frac{q^{m^2}}{(q)_m(q;q^2)_m} = \prod (1 - q^n)^{-1},$$
$$n \geq 1, \ n \not\equiv 0, \pm 6 \pmod{14}.$$

The factor q^{m^2} generates the lattice path of m peaks starting at $(0, 0)$ and terminating at $(2m, 0)$. If $m = 4$, then the path begins as:

FIGURE 2

The factor $(q)_m^{-1}$ generates m non-negative parts, say $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$, which are encoded by inserting a_m horizontal steps in front of the first mountain and $a_i - a_{i+1}$ horizontal steps in front of the $(m - i + 1)$st mountain, $1 \leq i \leq m$. If $a_1 = 3$, $a_2 = 1$, $a_3 = 1$, $a_4 = 0$, then our example becomes:

FIGURE 3

The factor $(q; q^2)_m^{-1}$ generates non-negative multiples of $2i - 1$, $1 \leq i \leq m$, say $b_1 * 1, b_2 * 3, \ldots, b_m * (2m - 1)$. This is encoded by having the ith peak grow to height $b_{m-i+1} + 1$. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two. If $b_1 = 3$, $b_2 = 1$, $b_3 = 2$, $b_4 = 0$, then our example becomes:
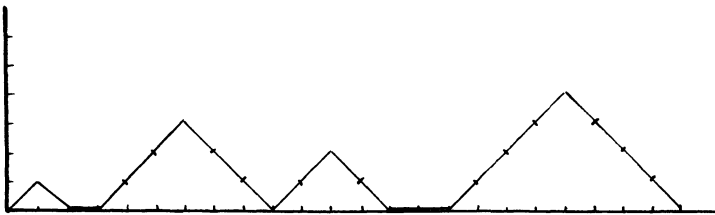
FIGURE 4

Every lattice path starting at $(0,0)$ with all valleys at height 0 is uniquely generated in this fashion.

Equation (1.8) with $k = 2$ (and thus $i = 2$)

$$(2.2) \quad \sum \frac{q^{m^2+m(m-1)/2}}{(q)_m(q; q^2)_m} = \prod (1 - q^n)^{-1},$$

$$n \geq 1, n \not\equiv 0, \pm 4 \pmod{10},$$

is treated in exactly the same manner except that the factor of $q^{m(m-1)/2}$ inserts a horizontal step between each of the peaks so that there are no valleys. If we keep the same values of the a's and b's, the

lattice path generated is
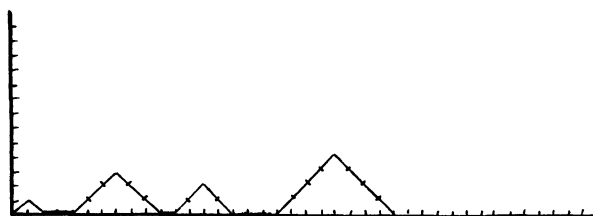
FIGURE 5

The case $k = i = 2$ also arises from equation (1.4) where the generating function becomes

$$(2.3) \qquad \sum \frac{q^{m^2+m+m(m-1)/2}}{(q)_m(q;q^2)_{m+1}} = \prod(1 - q^n)^{-1},$$

$$n \geq 1, n \not\equiv 0, \pm 4 \pmod{10}.$$

The extra factor of q^m puts a horizontal step in front of the first peak. The extra factor of $(1 - q^{2m+1})^{-1}$ introduces a non-negative multiple of $2m+1$, say $b_{m+1}*(2m+1)$, which is encoded as a mountain of height b_{m+1} inserted at the beginning of the lattice path.

The difference between the generating functions of (2.2) and (2.3) is that in (2.2) m counts the number of ranges while in (2.3) m counts the number of plains.

In the case $k = 2$, $i = 1$ of equation (1.4)

$$\sum \frac{q^{m^2+2m+m(m-1)/2}}{(q)_m(q;q^2)_{m+1}} + \prod(1-q^n)^{-1}, \qquad n \geq 1, n \not\equiv 0, \pm 2 \pmod{10},$$

we are bringing in an extra factor of q^m which introduces a southeast step from $(0, 1)$ to $(1, 0)$ at the front of the lattice path.

We next consider equation (1.3) with $k = 3$ and $i = 1$ or 2.

$$(2.5) \qquad \sum \frac{q^{m^2+(3-i)m}}{(q)_m(q;q^2)_{m+1}} = \prod(1 - q^n)^{-1},$$

$$n \geq 1, n \not\equiv 0, \pm 2i \pmod{14}.$$

This is interpreted essentially the same as the previous examples. We no longer have to have horizontal steps between peaks. If $i = 2$ we insert a southeast step $(0, 1)$ to $(1, 0)$ at the beginning of the path. If $i = 1$ then we insert two southeast steps $(0, 2)$ to $(1, 1)$ and $(1, 1)$ to $(2, 0)$ at beginning of the path.

In equation (1.5) with $k = 3$ and $i = 2$, the generating function becomes

$$(2.6) \quad \sum \frac{q^{m^2-m}(1-q^m)}{(q)_m(q;q^2)_m} = \sum \frac{q^{m^2-m}}{(q)_m(q;q^2)_m} - \sum \frac{q^{m^2}}{(q)_m(q;q^2)_m}$$

$$= \prod (1 - q^n)^{-1}, \quad n \geq 1, n \not\equiv 0, \pm 4 \pmod{14}.$$

As we have seen, the function

$$\frac{q^{m^2}}{(q)_m(q;q^2)_m}$$

generates paths with m peaks which start at $(0, 0)$ and have no valleys above height 0. To introduce the factor q^{-m} we eliminate the first step and consider two cases: if the path began with a horizontal step, we now have a path with m peaks which starts at $(0, 0)$; if the path began with a northeast step, it still has m peaks but now begins at $(0, 1)$. When we subtract off the generating function for paths starting at $(0, 0)$, we are left with the desired generating function.

Equation (1.6) with $k = 2$ and $i = 1$ is analyzed in exactly the same manner, except that now we have at least one horizontal step between each peak.

3. Proof of Proposition 1.4 for equations (1.7) **and** (1.8). The multiple summation in equation (1.7) is given by

$$\sum \frac{q^{m_1^2+\cdots+m_r^2}}{(q)_{m_1-m_2}\cdots(q)_{m_{r-1}-m_r}(q)_{m_r}(q;q^2)_{m_r}}.$$

To build this up inductively, we shall need to specify what the m_j's are counting.

For each mountain in our path, we define as follows a multi-set of ordered pairs called the SET OF RELATIVE PEAKS of the mountain. We choose a peak of maximal height, the first element of the set is the ordered pair whose first coordinate is the height of the peak and whose second coordinate is 0. As we proceed we ignore all peaks that have previously been chosen.

If there are any unchosen peaks left, we cut the mountain off at height one. This may have the effect of subdividing our mountain into several mountains. In fact, the number of mountains relative to height one will be one more than the number of valleys at height one. For each mountain relative to height one in which no peaks have been chosen, we choose a peak of maximal height and create a new element

of the set of relative peaks whose first coordinate is the height of that peak and whose second coordinate is one.

Inductively, after creating all elements whose second coordinate is $H-1$, if any unchosen peaks remain we cut the mountains off at height H. The number of mountains relative to height H will be one more than the number of valleys at or below height H minus the number of peaks at or below height H. For each mountain relative to height H in which no peaks have been chosen, we choose a peak of maximal height and create a new element of the set of relative peaks whose first coordinate is the height of that peak and whose second coordinate is H. Thus the mountain given below has $\{(5,0),(5,1),(2,1),(3,2)\}$ as its set of relative peaks.
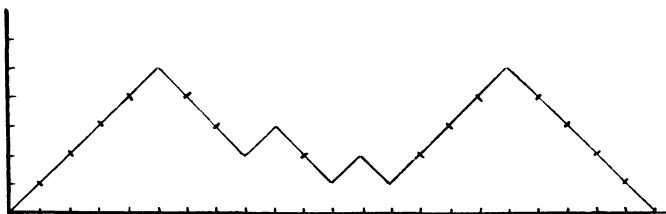
FIGURE 6

Note that while the order in which peaks are chosen is not unique, the set generated is unique.

The set of relative peaks of a lattice path is the multi-set union of the sets of relative peaks of all mountains in the lattice path. For the path given below, the set of relative peaks is $\{(3,0),(3,0),(2,0),(2,1),(2,1)\}$.
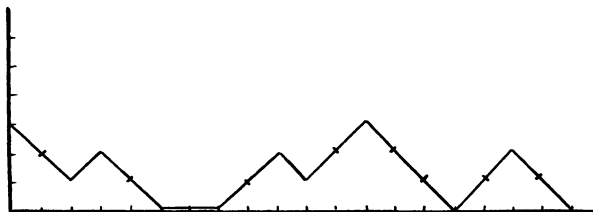
FIGURE 7

LEMMA 3.1. *The function*

$$\frac{q^{m_1^2+\cdots+m_r^2}}{(q)_{m_1-m_2}\cdots(q)_{m_{r-1}-m_r}(q)_{m_r}(q;q^2)_{m_r}}$$

is the generating function for lattice paths starting at $(0,0)$ with no valleys above height $2r - 2 = k - 3$ and such that for $1 \leq i \leq r$ there are exactly m_i relative peaks for which the difference of the coordinates is at least i and the second coordinate is at most $2r - 2i = k - 2i - 1$.

Proof. We observe that this lemma implies that m_1 counts the number of peaks and that $m_1 \geq m_2 \geq \cdots \geq m_r$. When $r = 1$, it agrees with the interpretation given for equation (2.1). The proof will proceed by establishing the inductive step on r.

Let $r' = r - 1, k' = k - 2$, and assume that

$$\frac{q^{m_2^2 + \cdots + m_r^2}}{(q)_{m_2 - m_3} \cdots (q)_{m_{r-1} - m_r}(q)_{m_r}(q;q^2)_{m_r}}$$

is the generating function for lattice paths starting at $(0,0)$, with no valleys above height $k' - 3$ and such that for $2 \leq i \leq r$ there are exactly m_i relative peaks for which the difference of the coordinates is at least $i - 1$ and the second coordinate is at most $k' - 2(i - 1) - 1 = k - 2i - 1$.

We introduce the factor

$$\frac{q^{m_1^2}}{(q)_{m_1 - m_2}}.$$

The effect of this factor on the lattice path is first to increase the height of each peak by one so that, for example, the path
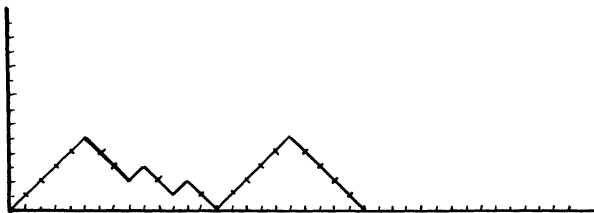
FIGURE 8

becomes
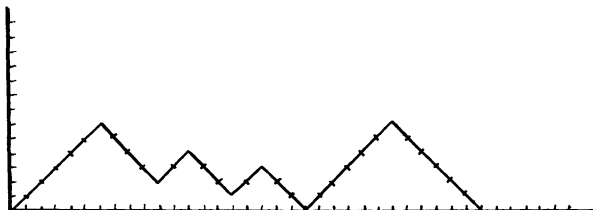
FIGURE 9

This "volcanic action" increases the weight of the path by $1+3+\cdots+(2m_2-1)=m_2^2$. It also has the effect of increasing the first coordinate of all relative peaks by one so that the parameter m_i now counts the number of relative peaks for which the difference of the coordinates is at least i and the second coordinate is at most $k-2i-1$.

We next insert the minimal path starting at $(0,0)$ with m_1-m_2 peaks of height one in front our our path. This increases the weight of the path by an additional $(m_1-m_2)^2+2*(m-1-m_2)*m_2$, so that the total amount of increase to the weight of the path is

$$m_2^2+(m_1-m_2)^2+2*(m_1-m_2)*m_2=m_1^2.$$

Our example becomes ($m_1-m_2=3$):
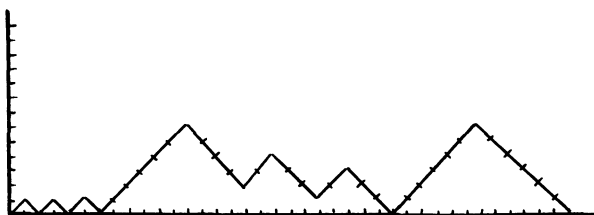
FIGURE 10

We note that m_1 now counts the total number of peaks and that none of the new peaks are counted by m_2.

The factor $(q)^{-1}_{m_1-m_2}$ gives us m_1-m_2 non-negative integers, say $a_1 \geq a_2 \geq \cdots \geq a_{m_1-m_2} \geq 0$. Beginning with the right-most of the new peaks, we move it to the right a_1 times according to the following rules. After each move the peak will still not be counted by m_2 which means that when viewed as a relative peak either the difference in the coordinates is one or the second coordinate is $k-3$ or $k-4$.

(1) If the peak is followed by a southeast step followed by a horizontal step, we change the preceding northeast step to a horizontal step, the southeast step to northeast and the horizontal step to southeast.
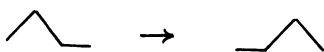
FIGURE 11

(2) If the peak is followed by two southeast steps and it is preceded by a northeast step preceded by a southeast step, then we change the

preceding northeast step to southeast and the following southeast step to northeast.
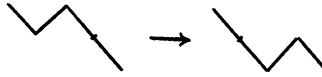
FIGURE 12

(3) If the peak is followed by a southeast step followed by two northeast steps and its height is not more than $k - 3$, we change the southeast step to northeast and the first of the following northeast steps to southeast.

FIGURE 13

(4) If the peak is followed by a southeast step followed by two northeast steps and its height is at least $k - 2$, then we change the first northeast step following the southeast step to southeast and change the first southeast step after the next peak to a northeast step. Note that since all valleys have height at most $k - 3$, the next peak is followed by at least two southeast steps. Also note that while the peak being moved is now a relative peak whose difference of coordinates is at least two, the second coordinate is $k - 4$ so that it is not counted by m_2.
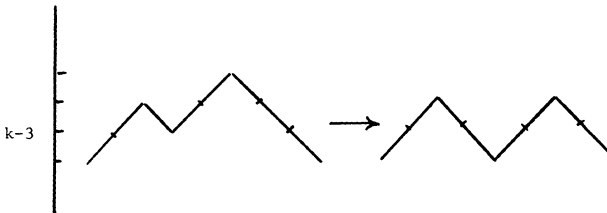
FIGURE 14

(5) If the peak is preceded by at least two northeast steps and followed by at least two southeast steps, then there is a valley on at least one side of the peak which has height $k - 3$ or $k - 4$. We consider five subcases:

(a) If the valley to the right has height $k - 3$ and the next peak is higher than the peak being moved, then we change the first northeast

step between the peaks to southeast, and the first southeast step after the second peak to northeast.

(b) If the valley to the right has height $k - 4$ and the next peak is higher than the peak being moved, then we change the first southeast step after the first peak to northeast and the first northeast step between the peaks to southeast.

(c) If the valley to the right has height $k - 3$ or $k - 4$ and the next peak is the same height as the peak being moved, we shift our attention to the peak on the right and instead move it.

(d) If the valley to the right has height less than $k - 4$ and the valley to the left has height $k - 3$, then we change the first northeast step between the peak to the left and the peak being moved to southeast and the first southeast step after the peak being moved to northeast.

(e) If the valley to the right has height less than $k - 4$ and the valley to the left has height $k - 4$, then we change the first southeast step after the peak to the left to northeast and the first northeast step between the peak to the left and peak being moved to southeast.

All of these possibilities are illustrated in the next example where we move the peak by four. For purposes of identification, we place a dot in the peak being moved.
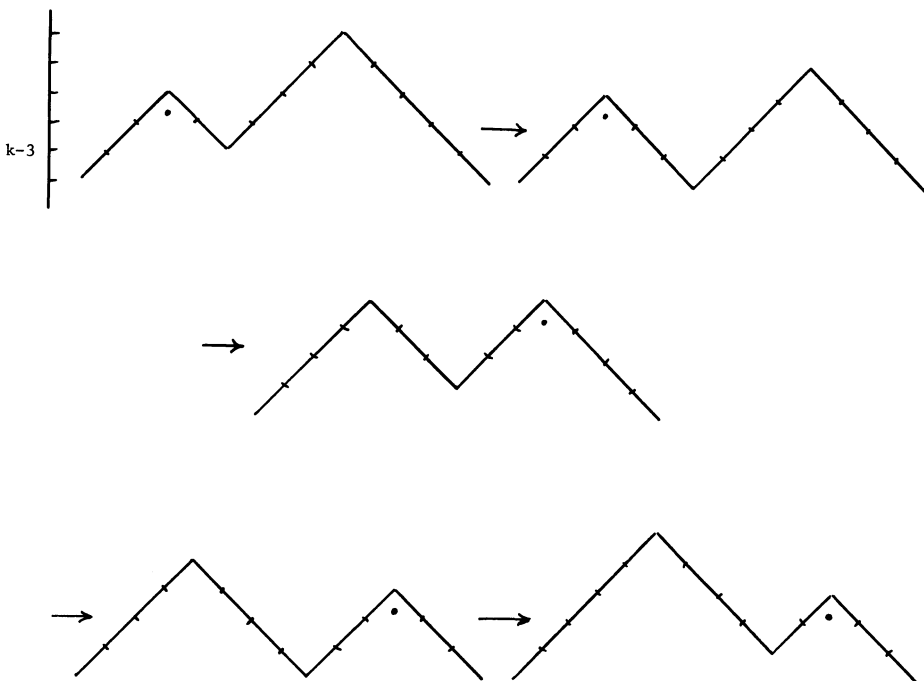
FIGURE 15

(6) If the peak is followed by a southeast step followed by a northeast step followed by a southeast step, we shift our attention to the peak on the right and instead move it.

In our example, we assume that $k = 7$ and that $a_1 = 10$. The dot marks the final position of the peak·being moved.
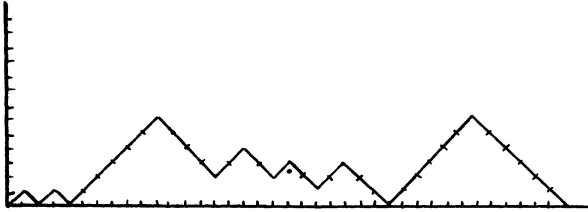
FIGURE 16

We proceed in similar manner for each a_i. After moving the $(i-1)$st new peak, counted from the right, a_{i-1} steps to the right, we move the next new peak a_i steps to the right. Since the a_i's are weakly decreasing, no new peak ever moves over another new peak.

It should be clear that every path starting at $(0,0)$, with no valleys above height $k - 3$ and with constants $m_1, m_2, \ldots, m_r$ as described above can be uniquely created in this way, thus concluding the proof of this lemma.   □

It follows from Lemma 3.1 that summing over all choices of $m-1 \geq m_2 \geq \cdots \geq m_r \geq 0$ yields the generating function for all lattice paths starting at $(0,0)$ which have no valleys above height $k-3$, thus proving Proposition 1.4 for equation (1.7).

To prove the proposition for equation (1.8) we use an analogous lemma.

LEMMA 3.2. *The function*

$$\frac{q^{m_1^2+\cdots+m_r^2+m_r(m_r-1)/2}}{(q)_{m_1-m_2}\cdots(q)_{m_{r-1}-m_r}(q)_{m_r}(q;q^2)_{m_r}}$$

is the generating function for lattice paths starting at $(0,0)$ with no valleys above height $2r - 3 = k - 3$ and such that for $1 \leq i \leq r$ there are exactly m_i relative peaks for which the difference of the coordinates is at least i and the second coordinate is at most $2r - 2i - 1 = k - 2i - 1$.

Proof. The proof proceeds essentially the same as the proof of Lemma 3.1. The only difference is that the factor of $q^{m_r(m_r-1)/2}$ will

introduce a horizontal step between each pair of original peaks. Note that in the inductive step, k is at least four.   □

Before leaving this section, it is worth noting that for the generating function in Lemma 3.1, m_r is the number of mountains of height at least r. For the generating function in Lemma 3.2, m_r is the number of ranges of height at least r.

4. Proof of Proposition 1.4 for equations (1.5) **and** (1.6). To interpret the left hand side of equation (1.5) we first observe that by the argument at the end of §2 together with Lemma 3.1, we have that the function

$$\frac{q^{m_i^2+\cdots+m_r^2-m_t}(1-q^{m_t})}{(q)_{m_t-m_{t+1}}\cdots(q)_{m_{r-1}-m_r}(q)_{m_r}(q;q^2)_{m_r}},$$

where $t = k - i$, is the generating function for lattice paths starting at $(0, 1)$ which have no valleys above height $2(r - (k - i) + 1) + 1 - 3 = 2i - k - 1$ and m_j, $k - i \le j \le r$, counts the number of relative peaks for which the difference between the first and second coordinate is at least $j + i - k + 1$ and whose second coordinate is at most $k - 2j - 1$. This establishes the case $h = k - i$ of the following lemma which we can then prove by induction.

LEMMA 4.1. *For* $1 \le h \le k - i \le (k - 1)/2 = r$, $t = k - i$, *the function*

$$\frac{q^{m_h^2+\cdots+m_r^2-m_h-\cdots-m_t}(1-q^{m_t})}{(q)_{m_h-m_{h+1}}\cdots(q)_{m_{r-1}-m_r}(q)_{m_r}(q;q^2)_{m_r}}$$

is the generating function for lattice paths starting at $(0, k - i - h + 1)$, *which have no valleys above height* $k - 2h - 1$ *and for which* m_j *counts the number of relative peaks for which the difference between the first and second coordinate is at least* $j - h + 1$ *and the second coordinate is at most* $k - 2j - 1$.

Proof. We assume the lemma is true for $h + 1$ and prove that it also holds for h. The factor

$$\frac{q^{m_h^2-m_h}}{(q)_{m_h-m_{h+1}}}$$

is encoded as follows. Each pre-existing peak is increased in height by one, increasing the weight of the path by m_{h+1}^2. We then insert at the front of the path of $m_h - m_{h+1}$ peaks connecting the vertices:

$$(0, k - i - h + 1), \quad (1, k - i - h), \quad (2, k - i - h + 1),$$
$$(3, k - i - h), \dots, (2(m_h - m_{h+1}) - 1, k - i - h).$$

This further increases the weight of the path by

$$(m_h - m_{h+1}) * (m_h - m_{h+1} - 1) + (2^*(m_h - m_{h+1}) - 1) * m_{h+1}$$
$$= m_h^2 - m_{h+1}^2 - m_h,$$

accounting for the factor of $q^{m_h^2 - m_h}$.

The factor $(q)^{-1}_{m_h - m_{h+1}}$ is used to move the new peaks to the right in exactly the same manner as in §3.   □

A similar lemma, established in exactly the same way, proves Proposition 1.4 for equation (1.6).

LEMMA 4.2. *For* $1 \le h \le k - i \le k/2 = r$, $t = k - i$, *the function*

$$\frac{q^{m_h^2 + \cdots + m_r^2 - m_h - \cdots - m_t + m_t(m_t - 1)/2}(1 - q^{m_t})}{(q)_{m_h - m_{h+1}} \cdots (q)_{m_{r-1} - m_r}(q)_{m_r}(q; q^2)_{m_r}}$$

is the generating function for lattice paths starting at $(k - i - h + 1, 0)$, *which have no valleys above height* $k - 2h - 1$ *and for which* m_j *counts the number of relative peaks for which the difference between the first and second coordinate is at least* $j - h + 1$ *and whose second coordinate is at most* $k - 2j - 1$.

5. Proof of Proposition 1.4 for equations (1.3) **and** (1.4). The next two lemmas will conclude the proof of Proposition 1.4.

LEMMA 5.1. *For* $1 \le i \le r + 1 = (k + 1)/2$, *the function*

$$\frac{q^{m_1^2 + \cdots + m_r^2 + m_1 + \cdots + m_{i-1} + 2m_i + \cdots + 2m_r}}{(q)_{m_1 - m_2} \cdots (q)_{m_{r-1} - m_r}(q)_{m_r}(q; q^2)_{m_r + 1}}$$

is the generating function for lattice paths starting at height $k - i$, *with no valleys above height* $k - 3$ *and such that for* $1 \le j \le r$ *there are exactly* $m_j + 1$ *relative peaks whose difference between first and second coordinates is at least* j *and whose second coordinate is at most* $k - 2j - 1$. *The number of mountains of height at least* r *is* $m_r + 1$.

Proof. When $r = 1$ this is the result proved in §2. We proceed by decreasing induction on r. Assume the lemma is true for $r - 1$. Let $k' = k - 2$, $i' = i - 1$ if i is at least two, $i' = 1$ if $i = 1$. Then $1 \le i' \le (k' + 1)/2$. Our induction hypothesis is that

$$\frac{q^{m_2^2 + \cdots + m_r^2 + m_2 + \cdots + m_{i'} + 2m_{i'+1} + \cdots + 2m_r}}{(q)_{m_2 - m_3} \cdots (q)_{m_{r-1} - m_r}(q)_{m_r}(q; q^2)_{m_r + 1}}$$

is the generating function for lattice paths starting at $(0, k' - i')$ with no valleys above height $k' - 3 = k - 5$ and such that for $2 \leq j \leq r$ there are exactly $m_j + 1$ relative peaks for which the difference in coordinates is at least $j - 1$ and whose second coordinate is at most $k' - 2(j - 1) - 1 = k - 2j - 1$.

If i is larger than one, then the new factor is

$$\frac{q^{m_1^2 + m_1}}{(q)_{m_1 - m_2}}.$$

As before, increasing the height of each of the $m_2 + 1$ existing peaks by one and then inserting the lattice path of $m_1 - m_2$ peaks starting at $(0, k - i)$ and ending at $(2(m_1 - m_2) - 1, k - i - 1)$ accounts for the increase in weight of $m_1^2 + m_1$. The factor $(q)_{m_1 - m_2}^{-1}$ is used to move the new peaks to the right.

If i equals one, then the pre-existing path started at $(0, k - 3)$. We increase each of the pre-existing peaks by one and then insert the lattice path of $m_1 - m_2$ peaks starting at $(0, k - 1)$, passing through $(2, k - 3)$ and ending at $(2(m_1 - m_2), k - 3)$. This increases the weight of the path by $m_1^2 + 2m_1$. The factor $(q)_{m_1 - m_2}^{-1}$ is used to move the new peaks to the right.   □

LEMMA 5.2. *For* $1 \leq i \leq r + 1 = (k + 2)/2$, *the function*

$$\frac{q^{m_1^2 + \cdots + m_r^2 + m_1 + \cdots + m_{i-1} + 2m_i + \cdots + 2m_r + m_r(m_r - 1)/2}}{(q)_{m_1 - m_2} \cdots (q)_{m_{r-1} m_r}(q)_{m_r}(q; q^2)_{m_r}}$$

is the generating function for lattice paths starting at height $k - i$, with no valleys above height $k - 3$ and such that for $1 \leq j < r$ there are exactly $m_j + 1$ relative peaks whose difference between first and second coordinates is at least j and whose second coordinate is at most $k - 2j - 1$. The parameter m_r counts the number of ranges of height at least r which follow the first plain.

Proof. Again, this is proved exactly as Lemma 5.1 except that we start with at least one horizontal step in front of each of the initial m_r mountains.   □

6. **Partitions with "$n + t$ copies of n".** If we specify the constants k and i, $1 \leq i \leq k$, and are then given a lattice path which begins at height $k - i$ and has no valleys above height $k - 3$, we can encode this path as the sequence of the weights of the peaks with each weight

subscripted by the height of the respective peak. Thus the path
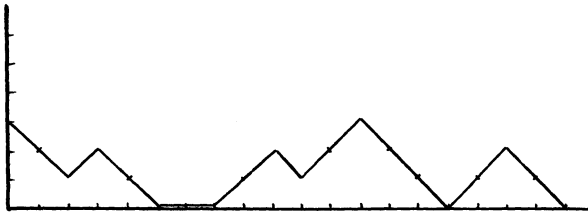
FIGURE 17

is encoded as $(0_3, 3_2, 9_2, 12_3, 17_2)$. Given this encoding and the parameters k and i, the path is uniquely reconstructible.

It should be obvious that not every finite increasing sequence of subscripted non-negative integers corresponds to one of our lattice paths. First of all, the subscripts are always at least one and the subscript for an integer n cannot be larger than $n + k - i$. If we denote the constant $k - i$ by t, this means that we have $n + t$ possible subscripts for any given non-negative integer n. Agarwal and Andrews [A&A] have described the sequences which correspond to our lattice paths as restricted partitions with $n + t$ copies of n. This is somewhat misleading as no weight can be repeated. The next proposition provides the link between the partitions with $n + t$ copies of n as defined by Agarwal and Andrews and our lattice paths.

PROPOSITION 6.1. *Given k and i, $1 \leq i \leq k$, let $t = k - i$. A finite increasing sequence of subscripted non-negative integers corresponds to a lattice path starting at $(0, t)$ and with no valleys above height $k - 3$ if and only if the sequence satisfies the following properties*:

(1) *The subscript on each integer n lies in the range $[1, n + t]$.*

(2) *If $t > 0$, then the subscript on the first term of the sequence is exactly t more than the value of that entry.*

The remaining conditions apply to every pair a_x, b_y of consecutive terms in the sequence.

(3) $b - a \geq |x - y| + 2$.

(4) *If $b - a \leq x + y$, then $a + x$ and $b + y$ have the same parity.*

(5) *If $k \geq 3$, then $b - a \geq x + y - 2k + 6$.*

(6) *If $k = 2$, then $b - a \geq x + y + 1$.*

Proof. The first property was explained above. For property 2, if the path starts at $(0, t)$, $t > 0$, then the first southeast step defines the first peak and the height of this peak will be t more than its weight.

Property 3 says that the weights of two successive peaks differ by at least two more than the absolute difference in heights. This follows from the fact that between these peaks one must descend to a height at or below the minimum of $x - 1$ and $y - 1$.

For property 4, if the weights of two successive peaks differ by at most the sum of the heights, then these two peaks must lie in the same range. All vertices on a given range have the same parity for the sum of height and weight.

Since no valley lies above height $k - 3$, if $k \geq 3$ then to get from one peak to the next one must descend at least to height $k - 3$ (a descent of at least $x - (k - 3)$) and then ascend from at most height $k - 3$ (an ascent of at least $y - (k - 3)$). Thus the difference in weights is at least

$$x - (k - 3) + y - (k - 3) = x + y - 2k + 6.$$

If $k = 2$, then to get from one peak to the next one must descend to height 0, make at least one horizontal step and then ascend to the next peak so that the difference in weights of the peaks is at least $x + 1 + y$.

We leave it to the readers to convince themselves that properties 1–4 describe all sequences corresponding to lattice paths which start at $(0, t)$ and have no other restrictions. Properties 5 and 6 are precisely what guarantees that all valleys descend to height $k - 3$ or below.   □

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