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This article studies uniqueness, in the class of distributions, of solutions of the Cauchy problem for a class of degenerate hyperbolic second order equations, when the initial curve contains a doubly characteristic point. The techniques employed are Carleman estimates and the concatenation method.

1. Introduction. This work is concerned with the uniqueness in the characteristic Cauchy problem for operators with double characteristics at a point of the initial curve.

We obtain an extension of results in **[T2]** and **[BP]**: these authors study uniqueness across $y = 0$ for the operator

$$(D_x - xD_y)(D_x + xD_y) - cD_y;$$

throughout the article we use the notation $D_x = \partial/\partial x$, $D_y = \partial/\partial y$.

Our work has an overlapping with **[N]** and **[K]**. In **[N]**, the study is made in the context of hyperfunction theory and results are proven for operators, e.g., of the type: $(D_x - x^k D_y)(D_x + x^k D_y) - cx^{k-1}D_y$, k a natural number; our method of proof uses only the theory of distributions. In **[K]**, operators, e.g., like

$$P(a, b) = [D_x - a(x)D_y][D_x + a(x)D_y] + b(x)D_y,$$

where a has a zero of order one at zero, are dealt with; in our work, if, say, b is non-negative then a is allowed to vanish to an arbitrary odd order k at zero.

Section 2 contains the proof of the Carleman estimates which yield the uniqueness across $y = 0$, in the class C^2, for the operator $P(a, b)$, under suitable assumptions. The results of this section are more general than what we needed in our applications. In the beginning of §3, we specialize our operator $P(a, b)$ to the case $a(x) = -ax^k$, $b(x) = -cx^{k-1}$, k odd, and, by using the concatenations in **[GT]**, we prove uniqueness, in the class C^m, where m depends on c and c avoids a certain sequence of real numbers. When c takes on such values, it is possible to prove that there is non-uniqueness, even in the class C^∞

(see [N]). The result in Theorem 3.2 refers to uniqueness when the initial curve lies midway between the pair of characteristics through the origin.

Section 3 also contains a result of uniqueness—not covered by [N]—for operators of the type

$$(D_x + ax^k D_y)(D_x + bx^l D_y) - (cx^{k-1} + dx^{l-1})D_y$$

where k, l are odd natural numbers and a, b, c, d are real numbers satisfying certain conditions (see Theorem 3.3).

In §4, we extend the method used in **[BP]** and we show how to obtain uniqueness in the class of distributions from uniqueness in the class C^m for some m, in the case of a certain type of operators among which are all the above mentioned operators. It is interesting to compare Theorem 4.1 with Theorem 4.4.8 in **[H]**.

The authors hope to prove, in a forthcoming publication, the uniqueness in the Cauchy problem for operators such as

$$[D_x + a(x,y)D_y][D_x + b(x,y)D_y] - c(x,y)D_x - d(x,y)D_y - e(x,y)$$

where both $a(x,0)$ and $b(x,0)$ have zeros of orders greater than one at $x = 0$. This will be accomplished by means of certain approximate concatenations since we haven't been able to find exact ones in this more general set up.

2. Carleman estimates and uniqueness in the class C^2 for the operator $P(a,b) = (D_x - a(x)D_y)(D_x + a(x)D_y) + b(x)D_y$; $a \in C^1(\mathbf{R},\mathbf{R}), b \in C^0(\mathbf{R},\mathbf{R})$.

PROPOSITION 2.1. *Assume that the following condition is satisfied:*
(H) *There exist $M > 0$ and $r > 0$ such that*

$$g(x) = M[a(x)]^2 + a'(x) + b(x) \geq 0, \qquad x \in [-r, r]$$

where $a'(x)$ is the derivative of $a(x)$.

Then the following estimates hold for all $v \in C_c^2(X, \mathbf{C})$, where $X \subset \{(x,y) \in \mathbf{R}^2 \colon |x| \leq r\}$ is a nonempty bounded open subset of $\mathbf{R}^2$:

(T1) *If $P^\#(a,b,t) = \exp(ty)P(a,b)\exp(-ty)$, then for $t \geq M$*

$$\operatorname{Re}\langle P^\#(a,b,t)v,\ D_y v\rangle \geq C \iint_X g|v|^2\,dx\,dy$$

where $\langle\ ,\ \rangle$ is the usual inner product of $L^2(X,\mathbf{C})$ and where the constant $C > 0$ is independent of v.

(T2) *For the same constant $C > 0$ and for $t \geq M$,*

$$C \iint_X g(x) \exp(2ty)|v|^2 \, dx \, dy \leq \iint_X \exp(2ty)[|P(a,b)v \cdot \overline{D_y v}| + t|P(a,b)v \cdot \bar{v}|] \, dx \, dy.$$

Proof. First we use (T1) to prove (T2). We have

$$\begin{aligned} C \iint_X g(x) \exp(2ty)|v|^2 \, dx \, dy &\leq \operatorname{Re}\langle P^{\#}(a,b,t)(v \exp(ty)), D_y(v \exp(ty))\rangle \\ &= \operatorname{Re}\langle \exp(ty) P(a,b)(v), \exp(ty)(tv + D_y v)\rangle \\ &\leq \left| \iint_X \exp(2ty)[P(a,b)(v) \cdot \overline{D_y v} + tP(a,b)(v) \cdot \bar{v}] \, dx \, dy \right| \\ &\leq \iint_X \exp(2ty)[|P(a,b)(v) \cdot \overline{D_y v}| + t|P(a,b)(v) \cdot \bar{v}|] \, dx \, dy. \end{aligned}$$

We shall now prove (T1). We first get

$$\int_{-\infty}^{+\infty} |v|^2 \, dy \leq C^{-1} \int_{-\infty}^{+\infty} |D_y v|^2 \, dy$$

where the constant $C > 0$ is independent of $v \in C_c^2(X, \mathbf{C})$.

Then

$$Cg(x) \int_{-\infty}^{+\infty} |v|^2 \, dy \leq g(x) \int_{-\infty}^{+\infty} |D_y v|^2 \, dy$$

and

$$\begin{aligned} C \iint_X g|v|^2 \, dx \, dy &\leq \iint_X g|D_y v|^2 \, dx \, dy \\ &= \iint_X [Ma^2 + a' + b]|D_y v|^2 \, dx \, dy \\ &\leq \iint_X [2ta^2 + a' + b]|D_y v|^2 \, dx \, dy \\ &= \operatorname{Re}\langle P^{\#}(a,b,t)(v), D_y v\rangle \end{aligned}$$

since $t \geq M$.

THEOREM 2.1. *Assume that the following conditions are satisfied:*
(H1) *There exist* $M > 0$ *and* $r > 0$ *such that*

$$g(x) = M[a(x)]^2 + a'(x) + b(x) \geq 0, \qquad x \in [-r, r]$$

and g *is not identically zero on any subinterval of* $[-r, r]$.
(H2) $X \subset \{(x, y) \in \mathbf{R}^2 : |x| \leq r\}$ *is a nonempty bounded open subset of* $\mathbf{R}^2$ *and* F *is a relatively closed subset of* X *whose intersection with* $X_+ = \{(x, y) \in X : y \geq 0\}$ *is a compact set* K.
Then the following holds
(T) *There exists an open neighborhood* U *of* K *such that every function* $u \in C^2(X, \mathbf{C})$ *satisfying*

$$(1) \quad P(a, b)u = 0 \quad \text{in } X; \qquad (2) \quad \operatorname{supp} u \subset F$$

also satisfies $u = 0$ *in* U.

Proof. The proof uses the estimates proved in the previous section following the lines of Theorem 2.3 of **[T1]**.

3. Concatenations and uniqueness in the class C^m for the operator $P(a, b, c, k) = (D_x + ax^k D_y)(D_x + bx^k D_y) - cx^{k-1} D_y$. We are going to use

LEMMA 3.1. *Assume that*
(H1) *For* $a, \mu \in \mathbf{C}$ *and* $k \in N$, *let*

$$Q(a, k, \mu) = x(D_x + ax^k D_y) + \mu,$$
$$R(a, k) = D_x + ax^k D_y.$$

Then the following holds for $a, b, c \in \mathbf{C}$ *and* $(a - b)b \neq 0$.
(T1) $Q(a, k, \mu)P(a, b, c, k) = P(a, b, c', k)Q(a, k, \mu - 2)$ *if* $c' = c + (k + 1)(a - b)$ *and* $\mu - 2 = k + c/(a - b)$.
(T2) $xP(a, -a, c, k) - R(-a, k)Q(a, k, \mu - 2) = (-c'/2a)R(a, k)$, *if* $a \neq 0$.

Proof. For (T1) see **[GT]**. A simple computation shows that (T2) also holds.

LEMMA 3.2. *For* $m \geq 2$ *and* $c \in \mathbf{C}$, *consider the statement:*
$S(m, c)$: *every function* $u \in C^m(X, \mathbf{C})$ *satisfying*
(1) $P(a, b, c, k)u = 0$ *in* X *with* $a = -b$,
(2) $\operatorname{supp} u \subset F$
vanishes identically in U (*here,* X, F *and* U *are as in Theorem* 2.1).

If $c' = c + (k+1)(a-b) \neq 0$ *and* $\mu - 2 = k + c/(a-b) \neq 0$ *then* $S(m, c')$ *implies* $S(m+1, c)$.

Proof. Let $u \in C^{m+1}(X, \mathbf{C})$ be such that
(1) $P(a, b, c, k)u = 0$ in U with $a = -b$,
(2) $\operatorname{supp} u \subset F$.
From $Q(a, k, \mu)P(a, b, c, k)u = P(a, b, c', k)Q(a, k, \mu - 2)u$ it follows that

$$P(a, b, c', k)Q(a, k, \mu - 2)u = 0 \quad \text{in } U$$

and therefore by hypothesis

$$Q(a, k, \mu - 2)u = 0 \quad \text{in } U.$$

Thus, if $a \neq 0$,

$$\begin{aligned} 0 &= xP(a, -a, c, k)u - R(-a, k)Q(a, k, \mu - 2)u \\ &= \frac{-c'}{2a}R(a, k)u \quad \text{in } U. \end{aligned}$$

Since $c' \neq 0$,

$$0 = Q(a, k, \mu - 2)u = xR(a, k)u + (\mu - 2)u \quad \text{in } U$$

and so $(\mu - 2)u = 0$ in U; since $\mu - 2 \neq 0$, we get $u = 0$ in U. The lemma is proved.

THEOREM 3.1. *Assume that the following condition is satisfied:*
(H) *X is a nonempty bounded open subset of $\mathbf{R}^2$ and F is a relatively closed subset of X whose intersection with $X_+ = \{(x, y) \in X : y \geq 0\}$ is a compact set K.*
Then the following holds
(T) *There exist $m \geq 2$, depending on c, and an open neighborhood U of K such that every function $u \in C^m(X, \mathbf{C})$ satisfying:*

$$(1) \quad P(a, -a, c, k)u = 0 \quad \textit{in } X; \qquad (2) \quad \operatorname{supp} u \subset F$$

also satisfies $u = 0$ in U, when one of the following holds:
(i) $0 \neq a$, $c \in \mathbf{R}$; $ka + c = 0$
(ii) $a, c \in \mathbf{R}$; *k odd and* $ka + c < 0$
(iii) $0 > a$, $c \in \mathbf{R}$; *k odd and for all* $j = 0, 1, 2, \ldots$ $c \neq -2aj(k+1)$, $c \neq -2a[k + j(k+1)]$.

Proof. The first two cases are consequences of Theorem 2.1; in these cases, $m = 2$. In the third case, if c is non-positive, we are in the preceding case and $m = 2$. If $c > 0$, let j_0 be the smallest natural number such that $c_1 = c + 2a(k+1)j_0 < 0$. As in the preceding case, $S(2, c_1)$ holds. We apply Lemma 3.2 j_0 times and conclude that $S(2 + j_0, c)$ holds; in this case, $m = 2 + j_0$. The proof is complete.

THEOREM 3.2. *Assume that the following condition is satisfied:*

(H) *X is a nonempty bounded open subset of $\mathbf{R}^2$ and F is a relatively closed subset of X whose intersection with*

$$X_{a,b,k} = \left\{ (x, y) \in X : y \geq \left(\frac{a+b}{2} \right) \frac{x^{k+1}}{k+1} \right\}$$

is a compact set K.

Then the following holds:

(T) *There exist $m \geq 2$ depending on c, and an open neighborhood U of K such that every function $u \in C^m(X, \mathbf{C})$ satisfying*

$$(1) \quad P(a, b, c, k)u = 0 \quad \text{in } X; \qquad (2) \quad \operatorname{supp} u \subset F$$

also satisfies $u = 0$ in U, when one of the following holds:

(i) *$a, b, c \in \mathbf{R}$; $a \neq b$; $k((a-b)/2) + c = 0$*

(ii) *$a, b, c \in \mathbf{R}$; k odd and $k((a-b)/2) + c < 0$*

(iii) *$a, b, c \in \mathbf{R}$; $a - b < 0$; k odd and for all $j = 0, 1, 2, \ldots$ $c \neq j(b-a)(k+1)$, $c \neq (b-a)[k + j(k+1)]$.*

THEOREM 3.3. *Let $P = (D_x + p(x)D_y)(D_x + q(x)D_y) - s(x)D_y$, where p, q and s are smooth functions vanishing at $x = 0$. Assume that the following condition is satisfied:*

(H) *There exist $M > 0$ and $r > 0$ such that*

$$g(x) = M \left(\frac{p(x) - q(x)}{2} \right)^2 - \frac{p'(x) - q'(x)}{2} - s(x) \geq 0, \qquad x \in [-r, r].$$

Then the following holds:

(T) *If $X \subset \{(x, y) \in \mathbf{R}^2 : |x| \leq r\}$ is a nonempty bounded open subset of $\mathbf{R}^2$ and F is a relatively closed subset of X whose intersection with*

$$X_{p,q} = \{(x, y) \in X : 2y \geq p_1(x) + q_1(x);$$
$$p_1'(x) = p(x), p_1(0) = 0, q_1'(x) = q_(x), q_1(0) = 0\}$$

is a compact set K then: "there exists an open neighborhood U of K

such that every function $u \in C^2(X;\mathbf{C})$ *satisfying*:

$$(1)\quad Pu = 0 \quad in\ X; \qquad (2)\quad \operatorname{supp} u \subset F$$

also satisfies $u = 0$ *in* U".

In particular, if $p(x) = ax^k$, $q(x) = bx^l$, $s(x) = cx^{k-1} + dx^{l-1}$, where k and l are odd with $k < l$, then the conditions $a \neq 0$, $ka+2c < 0$ imply uniqueness in the class C^2 across

$$y = \frac{a}{2}\frac{x^{k+1}}{k+1} + \frac{b}{2}\frac{x^{l+1}}{l+1}.$$

Theorems 3.2 and 3.3 follow, after a change of variables, from Theorem 3.1.

4. Uniqueness in the class of distributions for the operator $P = D_{xx}^2 - A(x)D_{xy}^2 - B(x)D_{yy}^2 - C(x)D_x - D(x)D_y - E(x)$ **where** A, B, C, D, E **are smooth functions of the variable** x. In this section we have three results: the first one, Theorem 4.1, shows how to obtain uniqueness in the class of distributions from uniqueness in the class C^m, for some $m \geq 2$. The other two (given without proof) are applications of it to the operator $P(a,b,c,k)$ of the preceding section.

THEOREM 4.1. *Assume that the following conditions are satisfied*:

(H1) X *is a nonempty bounded open subset of* R^2 *and* F *is a relatively closed subset of* X *whose intersection with* $X_+ = \{(x,y) \in X : y \geq 0\}$ *is a compact set* K.

(H2) *There exist a natural number* $m \geq 2$ *and an open neighborhood* V *of* K *such that every* $v \in C^m(X,\mathbf{C})$ *satisfying*:

(1) $Pv = 0$ *in* X,

(2) $\operatorname{supp} v \subset F$

vanishes identically on V.

Then the following holds

(T) *There exists an open neighborhood* U *of* K *such that every distribution* $u \in D'(U)$ *satisfying*:

(1) $Pu = 0$ *in* X,

(2) $\operatorname{supp} u \subset F$

vanishes identically on U.

Proof. We shall follow closely the article **[BP]**.

Preliminaries. Let $u \in D'(X)$ be such that

(1) $Pu = 0$ in X,

(2) $\operatorname{supp} u \subset F$.

Pick $r > 0$ such that

$$K_{3r} = \{(x, y) \in R^2 : d((x, y), K) \leq 3r\} \subset X.$$

The closure of the set

$$(K_{3r} \backslash K_r) \cap \{(x, y) \in R^2 : y = 0\}$$

is compact and disjoint from F, and therefore its distance to F is strictly positive.

Now choose $p < 0$ with the following properties:

(i) $0 < |p| < r$,

(ii) $K_r \cap F \cap X_p = K_{3r} \cap F \cap X_p$, where

$$X_p = \{(x, y) \in X : y > p\}.$$

By using a partition of unity, it is possible to express $u \in D'(X)$ as $u = u_1 + u_2$ with $\operatorname{supp} u_1 \subset K_{2r} \cap F$ and $\operatorname{supp} u_2 \subset K_r^c \cap F$. (It suffices to take $g \in C_c^\infty(X, \mathbf{R})$ with $g \equiv 0$ in K_{2r}^c and $g \equiv 1$ in an open neighborhood of K_r and then take $u_1 = gu$ and $u_2 = (1 - g)u$.) We have

(iii) $(\operatorname{supp} u_2) \cap K_{3r} \cap X_p = \varnothing$.

(Indeed, by using (ii), we get

$$\begin{aligned} \phi &= K_r \cap X_p \cap \operatorname{supp} u_2 = K_r \cap X_p \cap F \cap \operatorname{supp} u_2 \\ &= K_{3r} \cap X_p \cap F \cap \operatorname{supp} u_2 = K_{3r} \cap X_p \cap \operatorname{supp} u_2.) \end{aligned}$$

Notice $Pu = 0$ in X implies $Pu_1 = -Pu_2$ in X and $\operatorname{supp} Pu_1 \subset \operatorname{supp} u_1 \cap \operatorname{supp} u_2$. Also, by (iii)

(iv) $X_p \cap \operatorname{supp} Pu_1 = \varnothing$.

Extend u_1 to a distribution on R^2 by setting it equal to zero outside K_{3r}. By (iv), $Pu_1 = 0$ in X_p. By L. Schwartz's theorem on the structure of distributions with compact support, there exist a non-negative integer n and continuous functions f_{jk} such that $\operatorname{supp} f_{jk} \subset K_{3r}$ and

(v) $u_1 = \sum_{j+k \leq n} D_x^j D_y^k f_{jk}$

in the sense of distributions.

In order to simplify our notation we pick $b \in R$ such that

$$K_{4r} \subset Q = \{(x, y) \in R^2 : x > b, \ y > b\}.$$

For each continuous function $f : R^2 \to C$ with $\operatorname{supp} f \subset Q$, we define

$$[D_x^{-1} f](x, y) = \int_b^x f(t, y)\, dt,$$

$$[D_x^{-1} f](x, y) = \int_b^y f(x, t)\, dt.$$

It is clear that $[D_x^{-1}f](x,y)=0$ if $x \le b$ and that $[D_y^{-1}f](x,y)=0$ if $y \le b$. It is also obvious that D_x, D_y, D_x^{-1} and D_y^{-1} "commute" (e.g., if f is C^1, we have $D_xD_x^{-1}f = D_x^{-1}D_xf = f = D_yD_y^{-1}f = D_y^{-1}D_yf$). We define, for $n = 1, 2, \ldots$, $D_x^{-n} = [D_x^{-1}]^n$ and $D_y^{-n} = [D_y^{-1}]^n$. Also, $D_x^0 f = D_y^0 f = f$. With these notations, we may write

(v) $u_1 = D_x^n D_y^n f$

where $f = \sum_{j+k\le n} D_x^{j-n} D_y^{k-n} f_{jk}$. We have

$$\operatorname{supp} f \subset \{(x,y) \in R^2 \colon x > b + r,\ y > b + r\}.$$

Regularization of u_1 in the variable y. Let $\psi \in C_c^\infty(R,R)$ with $\int_{-\infty}^{+\infty}\psi(t)\,dt = 1$ and $\psi(t) = 0$ for $|t| > 1/2$. For each $\varepsilon > 0$, we set $\psi_\varepsilon(t) = \varepsilon^{-1}\psi(\varepsilon^{-1}t)$, $t \in R$ and $v_\varepsilon = u_1 *' \psi_\varepsilon$ where $*'$ denotes convolution in variable y only.

The fact that the coefficients of P are independent of y implies $Pv_\varepsilon = Pu_1 *' \psi_\varepsilon$ for $\varepsilon > 0$. If $0 < \varepsilon < |p| < r$ then $\operatorname{supp} v_\varepsilon \subset K_{4r} \subset Q$. Also

$$v_\varepsilon = u_1 *' \psi_\varepsilon = (D_x^n D_y^n f) *' \psi_\varepsilon = D_x^n[f *' D_y^n \psi_\varepsilon] = D_x^n g$$

where the function $g = f *' D_y^n \psi_\varepsilon$ is continuous, has partial derivatives of any order in the variable y and satisfies $\operatorname{supp} g \subset Q$.

Substitution of $v_\varepsilon = D_x^n g$ in $Pv_\varepsilon = 0$ on $X_{p+\varepsilon/2}$.

$$\begin{aligned} D_{xx}^2[D_x^n g] &= AD_{xy}^2[D_x^n g] + BD_{yy}^2[D_x^n g] \\ &\quad + CD_x[D_x^n g] + DD_y[D_x^n g] + ED_x^n g. \end{aligned}$$

First Case. $n = 1, 2, 3, \ldots$.

$$\begin{aligned} AD_{xy}^2[D_x^n g] &= AD_y[D_x^{n+1} g] \\ &= D_x^{n+1}[AD_y g] - \binom{n+1}{1} A'(x) D_x^n[D_y g] \\ &\quad - \sum_{j=2}^{n+1}\binom{n+1}{j} D_x^j A D_x^{n+1-j}[D_y g] \\ &= D_x^{n+1}[AD_y g] - \binom{n+1}{1}\left[D_x^n[A'D_y g] - \sum_{j=1}^{n}\binom{n}{j} D_x^j A' D_x^{n-j}[D_y g]\right] \\ &\quad - \sum_{j=2}^{n+1}\binom{n+1}{j} D_x^j A D_x^{n+1-j}[D_y g] \\ &= D_x^{n+1}[AD_y g + A_1 D_x^{-1}[A'D_y g] + \cdots + A_{n+1} D_x^{-n-1}[A^{(n+1)} D_y g]], \end{aligned}$$

$$
\begin{aligned}
&BD_y^2[D_x^n g]D_x^n[BD_y^2 g] - \sum_{j=1}^{n}\binom{n}{j} D_x^j B D_x^{n-j}[D_y^2 g] \\
&\quad = D_x^{n+1}[D_x^{-1}[BD_y^2 g] + B_1 D_x^{-2}[B' D_y^2 g] + \cdots + B_n D_x^{-n-1}[B^{(n)} D_y^2 g]], \\
&CD_x^{n+1}[g] = D_x^{n+1}[Cg] - \sum_{j=1}^{n+1}\binom{n+1}{j} D_x^j C D_x^{n+1-j} g \\
&\qquad\qquad = D_x^{n+1}[Cg + C_1 D_x^{-1}[C'g] + \cdots + C_{n+1} D_x^{-n-1}[C^{(n+1)} g]], \\
&DD_y[D_x^n g] = D_x^n[DD_y g] - \sum_{j=1}^{n}\binom{n}{j} D_x^j D D_x^{n-j}[D_y g] \\
&\quad = D_x^{n+1}[D_x^{-1}[DD_y g] + D_1 D_x^{-2}[D' D_y g] + \cdots + D_n D_x^{-n-1}[D^{(n)} D_y g]], \\
&ED_x^n[g] = D_x^n[Eg] - \sum_{j=1}^{n}\binom{n}{j} D_x^j E D_x^{n-j} g \\
&\qquad = D_x^{n+1}[D_x^{-1}[Eg] + E_1 D_x^{-2}[E'g] + \cdots + E_n D_x^{-n}[E^{(n)} g]].
\end{aligned}
$$

We get $D_x^{n+2} g = D_x^{n+1} h$ or $D_x g = h$. Therefore, $v_\varepsilon = D_x^{n-1} h$ where h has the same properties of g.

Second case. $n = 0$.

$$
\begin{aligned}
D_x^2 g &= AD_x[D_y g] + BD_y^2 g + CD_x g + DD_y g + Eg \\
&= D_x[AD_y g - D_x^{-1}[A' D_y g] + D_x^{-1}[BD_y^2 g] \\
&\qquad + Cg - D_x^{-1}[C'g] + D_x^{-1}[DD_y g] + D_x^{-1}[Eg]].
\end{aligned}
$$

We get $D_x^2 g = D_x h$ or $D_x g = h$. Therefore, $v_\varepsilon = D_x^{-1} h$ where h has the same properties of g.

Third case. $-n = +1, +2, \ldots$.

We shall use the following "Leibniz Formula":

$$
D_x^{-n}[FG] = FD_x^{-n}[G] - \sum_{j=1}^{n} D_x^{-j}[D_x[F] D_x^{-n+j-1}[G]],
$$

$$
\begin{aligned}
AD_{xy}^2[D_x^{-n} g] &= AD_x^{-n+1}[D_y g] \\
&= \sum_{j=1}^{n-1} D_x^{-j}[A' D_x^{-n+j}[D_y g]] + D_x^{-n+1}[AD_y g].
\end{aligned}
$$

By applying Leibniz formula to $A'D_x^{-n+1}[D_y g]$, $A'D_x^{-n+2}[D_y g], \ldots,$ $A'D_x^{-1}[D_y g]$ and to the other terms of the equation $Pv_\varepsilon = 0$ we arrive at $D_x^{-n+2} g = D_x^{-n+1} h$ or $D_x g = h$. Therefore, $v_\varepsilon = D_x^{-n-1} h$ where h has the same properties as g.

Epilogue. The analysis above furnishes the inductive step in proving that for each integer n there exists g_n, continuous, with $\operatorname{supp} g_n \subset Q$ and

$$v_\varepsilon = D_x^n g_n \quad \text{in } X_{p+\varepsilon/2}$$

g being smooth in y.

We reach the conclusion that v_ε restricted to X is in $C^\infty(X_{p+\varepsilon/2}, C)$.

Now assumption (H2) implies that $v_\varepsilon = 0$ in $X_{p+\varepsilon/2}$. Therefore $u_1 = 0$ in X_p since v_ε converges to u_1 in X_p when ε decreases to zero. Now (iii) implies that $u_1 = u$ in $\mathring{K}_{3r} \cap X_p$. We may take $U = \mathring{K}_{2r} \cap X_p$. The proof is complete.

COROLLARY 4.1. *Assume that the following condition is satisfied:*

(H) *X is a nonempty bounded open subset of $\mathbf{R}^2$ and F is a relatively closed subset of X whose intersection with $X_+ = \{(x, y) \in X : y \geq 0\}$ is a compact set K.*

Then the following holds

(T) *There exists an open neighborhood U of K such that every distribution $u \in D'(U)$ satisfying*

(1) *$P(a, -a, c, k) = [(D_x + ax^k D_y)(D_x - ax^k D_y) - cx^{k-1} D_y]u = 0$ in X,*

(2) $\operatorname{supp} u \subset F$

also satisfies $u = 0$ in U, when one of the following holds:

(i) *$0 \neq a$, $c \in \mathbf{R}$; $ka + c = 0$,*

(ii) *$a, c \in \mathbf{R}$, k odd and $ka + c < 0$,*

(iii) *$0 > a$, $c \in \mathbf{R}$; k odd and for all $j = 0, 1, 2, \ldots$ $c \neq -2aj(k+1)$, $c \neq -2a[k + j(k+1)]$.*

COROLLARY 4.2. *Assume that the following condition is satisfied:*

(H) *X is a nonempty bounded open subset of $\mathbf{R}^2$ and F is a relatively closed subset of X whose intersection with*

$$X_{a,b,k} = \left\{ (x, y) \in X : y \geq \left(\frac{a+b}{2}\right) \frac{x^{k+1}}{k+1} \right\}$$

is a compact set K.

Then the following holds:

(T) *There exists an open neighborhood U of K such that every distribution $u \in D'(U)$ satisfying*

$$(1)\quad P(a,b,c,k)u = 0 \quad \text{in } X; \qquad (2)\quad \operatorname{supp} u \subset F$$

also satisfies $u = 0$ in U, when one of the following holds:

(i) $a, b, c \in \mathbf{R}$; $a \neq b$; $k((a-b)/2) + c = 0$

(ii) $a, b, c \in \mathbf{R}$; k *odd and* $k((a-b)/2) + c < 0$

(iii) $a, b, c \in \mathbf{R}$; $a - b < 0$; k *odd and for all* $j = 0, 1, 2, \ldots$ $c \neq j(b-a)(k+1)$, $c \neq (b-a)[k + j(k+1)]$.

References

[BP] B. Birkeland and J. Persson, *The local Cauchy problem in R^2 at a point where two characteristic curves have a common tangent*, J. Differential Equations, **30** (1978), 64–88.

[GT] A. Gilioli and F. Treves, *An example in the solvability theory of linear PDE's*, Amer. J. Math., **92** no. 2 (1974), 366–384.

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators* I: *Distribution Theory and Fourier Analysis*, Springer-Verlag (1983).

[N] S. Nakane, *Propagation of singularities and uniqueness in the Cauchy problem at a class of doubly characteristic points*, Comm. in PDE, **6**(8) (1981), 917–927.

[K] L. Korbly, *Uniqueness of solutions of partial differential equations when the initial surface is characteristic at a point*, Proc. Amer. Math. Soc., **86** no. 4 (1982), 617–624.

[T1] F. Treves, *Linear Partial Differential Equations With Constant Coefficients*: *Existence, Approximation and Regularity of Solutions*, Math. and its Applications, **6** Gordon and Breach, New York (1966).

[T2] ———, *Discrete phenomena in uniqueness in the Cauchy problem*, Proc. Amer. Math. Soc., **46** (1974), 229–233.

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