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The following boundary value result is obtained: If ϕ is a quasiregular function on a plane domain U with non-polar complement and ϕ satisfies a growth condition analogue to the classical H^p-condition for analytic functions, then there exists a uniformly elliptic diffusion X_t such that for a.a. $\eta \in \partial U$ with respect to its elliptic-harmonic measure the limit of ϕ along the η-conditional X_t-paths exists a.s.

It is proved that if U is the unit disc then convergence along the η-conditional X_t-paths implies the classical non-tangential convergence. Therefore the result above is a generalization of the classical Fatou theorem. As an application, using known properties of elliptic-harmonic measure we obtain that there exists $\alpha > 0$ (depending on ϕ) such that for every interval $J \subset \partial D$ there is a subset $F \subset J$ of positive α-dimensional Hausdorff measure such that the non-tangential limit of ϕ exists at every point of F.

1. Introduction. The classical Fatou theorem states that if f is an analytic function on the unit disc $D = \{z; |z| < 1\}$ in the complex plane $\mathbf{C}$ and there exists $p > 0$ such that

$$\sup_{r<1}\left(\frac{1}{2\pi}\int_{|z|=r}|f(re^{i\theta})|^p\,d\theta\right) < \infty \tag{1.1}$$

then f has radial limits a.e. on $T = \{z; |z| = 1\}$, i.e.

$$\lim_{r\to 1} f(re^{i\theta}) \text{ exists} \tag{1.2}$$

for a.a. $\theta \in [0, 2\pi)$ w.r.t. Lebesgue measure. In fact, the limit exists non-tangentially, for a.a. θ. (See for example Garnett [**10**].)

The purpose of this article is to generalize this result in two directions:

First, the analytic function f is replaced by a *quasiregular* function ϕ. In this case it is known that the Fatou theorem in the strong form above stating radial convergence almost everywhere (with respect to Lebesgue measure) is false (see [**16**, p. 119]), so we are looking for an appropriate modification of "almost everywhere".

Second, the domain D is replaced by any open subset U of $\mathbf{C}$ with non-polar complement, i.e. such that

$$(1.3) \qquad C_0(\mathbf{C}\backslash U) > 0,$$

where C_0 denotes logarithmic capacity. Of course, by considering the second generalization we must find an appropriate replacement for "radial" or "non-tangential" convergence. This is obtained by considering convergence along the *conditional paths* X_t^η of a suitable uniformly elliptic diffusion X_t (depending on ϕ) for a.a. $\eta \in \partial D$ w.r.t. the *elliptic-harmonic* measure $\mu^X = \mu_{x,U}^X$ of U for X_t. More precisely, the process X_t^η in U has the property that

$$(1.4) \qquad \lim_{t\to\zeta} X_t^\eta = \eta \text{ a.s. } P^{x,\eta}, \text{ for a.a. } \eta \in \partial U$$
$$\text{w.r.t. } \mu^X \text{ and all } x \in U,$$

where ζ is the life time of X_t^η and $P^{x,\eta}$ is the probability law of X_t^η starting at x. And we prove:

THEOREM 3.2 (*Stochastic Fatou Theorem*). *Suppose* $\phi \in H_{QR}^p(U)$ *for some* $p > 0$, *i.e.* ϕ *is a quasiregular function on* U *satisfying a growth condition similar to the* H^p*-condition* (1.1) (*e.g. it suffices to have* $\text{Area}(\phi(U)) < \infty$). *Then for all* $x \in U$

$$(1.5) \qquad \lim_{t\to\zeta} \phi(X_t^\eta)$$

exists a.s. $P^{x,\eta}$ *for a.a.* $\eta \in \partial U$ *w.r.t.* μ^x.

In the special case when $U = D$ we show that the a.s. convergence (1.5) of ϕ at a point $\eta \in \partial D$ implies the non-tangential convergence of ϕ at η (Theorem 4.1). Thus Theorem 3.2 is indeed a generalization of the Fatou theorem. As an application, combining Theorems 3.2 and 4.1 with metric properties of elliptic-harmonic measure we obtain the following:

COROLLARY 5.2. *Suppose* $\phi \in H_{QR}^p(D)$ *for some* $p > 0$. *Then there exists* $\alpha > 0$ (*depending only on* ϕ) *such that in every interval* $J \subset \partial D$ *there is a subset* $F \subset J$ *of positive* α*-dimensional Hausdorff measure such that the non-tangential limits of* ϕ *exist at every point of* F.

Results like this corollary have been known to experts for some time, but it seems to be difficult to find them stated explicitly in the literature.

The results of this paper are related to those in the paper by Caffarelli, Fabes, Mortola & Salsa [**5**]. There it is proved that a *positive* solution u in a Lipschitz domain G in $\mathbf{R}^n$ of the equation $Lu = 0$ in G (where L is a uniformly elliptic second order partial differential operator) has non-tangential limits a.e. on ∂G with respect to the elliptic harmonic measure corresponding to L. So their result implies in particular that the same holds for a quasiregular function ϕ on a Lipschitz domain in the plane ($n = 2$) provided that the real and imaginary parts of ϕ are both positive (or bounded). The purpose of this paper is to show that for a quasiregular function ϕ the same conclusion can be obtained under much weaker conditions on ϕ if we use a different approach: The idea is to consider ϕ directly (not its real and imaginary parts separately) and apply a stochastic method. The key to this method is the fact (see [**17**]) that there exists a uniformly elliptic diffusion X_t (depending on ϕ) which is mapped into a time change of (2-dimensional) Brownian motion by ϕ. Thereby we also obtain the generalized stochastic Fatou theorem above, valid without any conditions on the boundary of the domain.

2. Conditional uniformly elliptic diffusions. Let $(X_t(\omega), \Omega, P^x)$ (where $t \geq 0$, $\omega \in \Omega$, $x \in U$) be a *uniformly elliptic diffusion* in an open set $U \subset \mathbf{R}^2$ with generator

$$Af = \operatorname{div}(a\nabla f). \tag{2.1}$$

Here $a = [a_{ij}]$ is a symmetric 2×2 matrix where each element $a_{ij} = a_{ij}(x)$ is a bounded measurable function and there exists $M < \infty$ such that

$$\frac{1}{M}|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \leq M|\xi|^2 \quad \text{for all } x \in U,\ \xi \in \mathbf{R}^2. \tag{2.2}$$

The constant M is called the *ellipticity constant* of the diffusion. For example, X_t may be obtained as the Hunt process associated to the Dirichlet form

$$\mathscr{E}(u,v) = \int_{\mathbf{R}^2} \nabla u^T a \nabla v\, dx; \qquad u, v \in C_0^\infty(U) \tag{2.3}$$

where dx denotes Lebesgue measure (Fukushima [**9**]).

Assume that U has a nonpolar complement. Then

$$\tau_U = \tau_U^X = \inf\{t > 0; X_t \notin U\} \tag{2.4}$$

(the first exit time from U) is finite a.s. P^x, and we can define the *harmonic measure* $\mu_x = \mu_{x,U}$ for X as follows:

$$\mu_{x,U}(F) = P^x[X_{\tau_U} \in F], \qquad F \subset \partial U. \tag{2.5}$$

It is well known that the Harnack principle holds for such operators A, i.e. for all x there exists a neighbourhood $W \ni x$ and $C < \infty$ such that

$$\frac{1}{C} \le \frac{d\mu_y}{d\mu_x} \le C \quad \text{for all } y \in W. \tag{2.6}$$

Fix $x_0 \in U$ and put

$$K(x,\eta) = \frac{d\mu_x}{d\mu_{x_0}}(\eta); \qquad x \in U,\ \eta \in \partial U. \tag{2.7}$$

Let $H = L^2(U, dx)$ and let $T_t\colon H \to H$ be the transition operators of X_t killed when it exits from U, i.e. $(T_t f)(x) = E^x[f(X_t)]$ $(\overset{\text{def}}{=} E^x[f(X_t) \cdot \chi_{\{t<\tau_U\}}];\ t \ge 0,\ f \in H)$. Fix $\eta \in \partial U$ such that $k(x) \overset{\text{def}}{=} K(x,\eta) > 0$ for all $x \in U$ and define $\tilde{H} = L^2(U; k^2(x)\,dx)$. Let $T_t^\eta\colon \tilde{H} \to \tilde{H}$ be given by

$$T_t^\eta(g) = \frac{T_t(kg)}{k}; \qquad t \ge 0,\ g \in \tilde{H}. \tag{2.8}$$

Then $\{T_t^\eta\}$ is a symmetric, strongly continuous contraction semigroup on $\tilde{H}$ (since $\{T_t\}$ is on H), with generator $\tilde{A}f = A[kf]/k$ and corresponding Dirichlet form

$$\begin{aligned}\mathscr{E}^\eta(u,v) &= -(\tilde{A}u, v)_{\tilde{H}} = -\left(\frac{A(ku)}{k}, v\right)_{\tilde{H}} \\ &= -(A(ku), kv)_H = \mathscr{E}(ku, kv)\end{aligned} \tag{2.9}$$

for $u, v \in \mathscr{D}(\mathscr{E}^\eta) = \{f/k; f \in \mathscr{D}(\mathscr{E})\}$.

This form is regular and

$$\mathscr{E}^\eta(u,v) = -\int A(ku)kv\,dx = 0 \tag{2.10}$$

if u is constant in a neighbourhood of supp$[v]$, for a.a. $\eta \in \partial U$ w.r.t. μ_{x_0}.

The property (2.10) can be proved as follows: For all $g \in C_0^\infty(U)$, $f \in C_0(\partial U)$ we have

$$\int_{\partial U} f(\eta)\mathscr{E}(g(x), K(x,\eta))\, d\mu_{x_0}(\eta)$$
$$= \int_{\partial U} f(\eta)\left(\int_U \nabla g^T(x)a(x)\nabla_x K(x,\eta)\right) d\mu_{x_0}(\eta),$$
$$\int_{\partial U} \nabla g^T(x)a(x)\nabla_x\left(\int_{\partial U} f(\eta)K(x,\eta)\, d\mu_{x_0}(\eta)\right) dx$$
$$= -(A\tilde{f}, g)_H = 0,$$

since $\tilde{f}(x) = \int_{\partial U} f(\eta)\, d\mu_x(\eta)$ is the A-harmonic extension of f to U. So $\mathscr{E}(g, K(\cdot,\eta)) = -(Ak, g) = 0$ a.e. (μ_{x_0}), as claimed. It also follows that $Ak = 0$. Therefore k is Hölder continuous ([7], [15]). We conclude that for a.a. $\eta \in \partial U$ w.r.t. μ_{x_0} there exists a Hunt process $(X_t^\eta(\omega), \Omega, P^{x,\eta})_{t\geq 0, \omega\in\Omega}$, whose generator is $\tilde{A}$. Moreover, from the property (2.10) of $\mathscr{E}^\eta$ we know that X_t^η is t-continuous and no killing of X_t^η occurs inside U (see [9]). We let $\zeta = \zeta_U$ denote the life time of X_t^η. The process X_t^η will be called the conditioning of the process X_t with respect to η (or, more precisely, with respect to the A-kernel function $k(x)$).

The next result justifies the name "conditional" for the process X_t^η: ($E^{x,\eta}$ and E^x denotes expectation w.r.t. the measures $P^{x,\eta}$ and P^x, respectively). We refer the reader to [2, Lemma 4] for a proof.

LEMMA 2.1. *Let $g_1, \dots, g_k$ be bounded Borel functions on U. Then*

$$E^{x,\eta}[g_1(X_{t_1}^\eta)\cdots g_k(X_{t_k}^\eta)] = E^x[g_1(X_{t_1})\cdots g_k(X_{t_k})|X_{\tau_U} = \eta].$$

3. A stochastic Fatou theorem. Let ϕ be a *quasiregular* function in U, i.e. $\phi \in ACL^2$ (ϕ is absolutely continuous on a.e. straight line and with partial derivatives in L^2_{loc}) and

$$|\phi'(x)|^2 \leq K \cdot J_\phi(x) \quad \text{for a.a. } x \in U \tag{3.1}$$

for some constant K, where $\phi' = [\partial\phi_i/\partial x_j]$; $1 \leq i, j \leq 2$ and $J_\phi = \det(\phi')$ is the Jacobian. See [13] for information about quasiregular functions. In [17] it is proved that there exists a uniformly elliptic diffusion X_t (depending on ϕ) such that the process $\phi(X_t)$; $t < \tau$ is a time change of Brownian motion in $\mathbf{C}$. More precisely, define

$$\beta_t = \int_0^t J_\phi(X_s)\, ds, \quad \alpha_t = \inf\{s; \beta_s > t\}, \tag{3.2}$$

and let $(\hat{B}_t, \hat{P}^y)$ be a Brownian motion in $\mathbf{C}$. Then

$$B_t = \begin{cases} \phi(X_{\alpha_t}); & t < \beta_\tau, \\ \phi^* + \hat{B}_{t-\beta_\tau}; & t \geq \beta_\tau \end{cases} \tag{3.3}$$

is again a Brownian motion in $\mathbf{C}$. Here $\phi^* = \lim_{t\to\tau} \phi(X_t)$, which exists a.s. on $\{\beta_\tau < \infty\}$.

From now on we let X_t denote this special process associated to ϕ and as before we let X_t^η denote its conditioned process, defined for a.a. $\eta \in \partial U$. We will assume that ϕ satisfies one of the following two growth conditions (3.4), (3.5):

LEMMA 3.1. *Let $0 < p < \infty$. The following are equivalent:*

$$\sup_{\sigma<\tau_U} E^x[|\phi(X_\sigma)|^p] < \infty \quad \textit{for each } x \in U, \tag{3.4}$$

the sup *being taken over all X_t-stopping times $\sigma < \tau_U$.*

$$E^x[\beta_{\tau_U}^{p/2}] < \infty \quad \textit{for each } x \in U. \tag{3.5}$$

Conditions (3.4), (3.5) are satisfied if

$$E^y[\hat{\tau}_{\phi(U)}^{p/2}] < \infty \quad \text{for each } y \in \phi(U), \tag{3.6}$$

where $\hat{\tau}_{\phi(U)}$ is the first exit time from $\phi(U)$ of Brownian motion in $\mathbf{C}$.

Condition (3.6) holds if

$$\operatorname{Area} \phi(U) < \infty. \tag{3.7}$$

REMARK. Note that condition (3.4) coincides with the classical H^p-condition (1.1) in the special case when ϕ is analytic and $U = D$. We therefore *define* $H^p_{QR}(U)$ as the set of quasiregular functions ϕ on U satisfying (3.4).

Proof of Lemma 3.1. (3.7) $\Rightarrow$ (3.6). This follows from the estimates of Aizenman and Simon [**1**] of the moments of the exit time for Brownian motion.

(3.6) $\Rightarrow$ (3.5). Since the process B_t in (3.3) is a Brownian motion and obviously $B_t \in \phi(U)$ for $t < \beta_\tau$ it is clear that

$$\beta_\tau \leq \hat{\tau}_{\phi(U)} \tag{3.8}$$

and therefore (3.6) $\Rightarrow$ (3.5).

(3.5) $\Rightarrow$ (3.4). First note that by (3.3) we have

$$E^x[|\phi(X_\sigma)|^p] = E^{\phi(x)}[|B_{\beta_\sigma}|^p] \leq E^{\phi(x)}\left[\sup_{t\leq\beta_\sigma} |B_t|^p\right]. \tag{3.9}$$

The Burkholder-Gundy inequalities state that

$$E^y\left[\sup_{t\le T}|B_t|^p\right] \sim E^y[T^{p/2}], \tag{3.10}$$

for all stopping times T, where $\sim$ means that the ratio is bounded and bounded below by constants (only depending on p and y and the dimension (here 2)). See for example [**4**]. It was pointed out by B. Davis that these inequalities also hold for the so-called quasistopping or Markov times [**6**, p. 304], which include the random times β_σ above. By (3.9) and (3.10) for $T = \beta_\sigma$ we see that (3.5) $\Rightarrow$ (3.4).

Conversely, by Doob's martingale inequality we have that

$$E^y\left[\sup_{t\le \beta_\sigma}|B_t|^p\right] \sim E^y[|B_{\beta_\sigma}|^p] \tag{3.11}$$

for all $p > 1$. To obtain this relation for all $p > 0$ for Brownian motion in the plane we proceed as in [**8**, p. 156–157]:

Let $p > 0$ and assume for simplicity that $y = 1$. Then since the probability that B_t hits 0 is 0, we may define a pathwise logarithm $G_t = \log B_t$ such that $G_0 = 1$ a.s. Then G_t is a martingale and so is

$$H_t = e^{pG_t/2},$$

since $z \to e^z$ is analytic. So by Doob's martingale inequality we have

$$E^y\left[\sup_{t\le\beta_\sigma}|B_t|^p\right] = E^y\left[\sup_{t\le\beta_\sigma}|H_t|^2\right] \sim E^y[|H_{\beta_\sigma}|^2] = E^y[|B_{\beta_\sigma}|^p],$$

which proves (3.11) for all $p > 0$. Thus we have obtained that

$$E^x[|\phi(X_\sigma)|^p] \sim E^{\phi(x)}[\beta_\sigma^{p/2}] \quad \text{for all } p > 0, \tag{3.12}$$

and the equivalence of (3.4) and (3.5) follows.

Now assume that ϕ satisfies (3.4). Let $U_k \subset\subset U$ be an increasing sequence of open, relatively compact subsets of U such that $U = \bigcup_{k=1}^\infty U_k$ and put $\tau_k = \tau_{U_k}^X$. Then by (3.2), (3.3) and (3.12) we have, for $k < m$

$$E^x[|\phi(X_{\tau_k}) - \phi(X_{\tau_m})|^p] \sim E^{\phi(x)}\left[\left(\int_{\tau_k}^{\tau_m} J_\phi(X_t)\,dt\right)^{p/2}\right] \to 0$$

as $k, m \to \infty$. Thus $\{\phi(X_{\tau_k})\}_k$ constitute a Cauchy sequence in $L^p(P^x)$. Let ϕ^* be the limit of this sequence. With the convention that $\phi(X_{t\wedge\tau})$ means ϕ^* if $t \ge \tau$ we have that $\phi(X_{t\wedge\tau})$ is a martingale in $\mathbf{C}$, so with

$G_t = \log \phi(X_t)$, $H_t = \exp((p/2) \cdot G_t)$ as above we get by Doob's martingale inequality

$$(3.13)\quad P^x\left[\sup_{\tau_k<t<\tau} |\phi(X_t) - \phi(X_{\tau_k})|^p > \lambda^p\right]$$
$$= P^x\left[\sup_{\tau_k<t<\tau} |H_t - H_{\tau_k}|^2 > \lambda^p\right]$$
$$\leq \frac{c}{\lambda^p} E^x[|H_\tau - H_{\tau_k}|^2] = \frac{c}{\lambda^p} E^x[|\phi^* - \phi(X_{\tau_k})|^p] \to 0$$

as $k \to \infty$, for all $\lambda > 0$, where c is an absolute constant.

We are now ready to prove the main result of this paper:

THEOREM 3.2. *Let $U \subset \mathbf{C}$ be open with $C_0(\mathbf{C}\backslash U) > 0$ and let $\phi \in H^p_{QR}(U)$ for some $p > 0$. Then*

$$\lim_{t\to\zeta} \phi(X^\eta_t)$$

exists a.s. $P^{x,\eta}$, for a.a. $\eta \in \partial U$ w.r.t. μ_{x_0}.

Proof. With U_k, τ_k as above and $\lambda > 0$ consider

$$\int P^{x,\eta}\left[\sup_{\tau_k<t<\tau} |\phi(X^\eta_t) - \phi(X^\eta_{\tau_k})| > \lambda\right] d\mu_x(\eta)$$
$$= E^x\left[\left(P^{x,\eta}\left[\sup_{\tau_k<t<\tau} |\phi(X^\eta_t) - \phi(X^\eta_{\tau_k})| > \lambda\right]\right)_{\eta=X_\tau}\right]$$
$$= E^x\left[\left(P^x\left[\sup_{\tau_k<t<\tau} |\phi(X_t) - \phi(X_{\tau_k})| > \lambda | X_\tau\right]\right)\right] \quad \text{(by Lemma 2.1)}$$
$$= P^x\left[\sup_{\tau_k<t<\tau} |\phi(X_t) - \phi(X_{\tau_k})| > \lambda\right]$$
$$\to 0 \quad \text{as } k \to \infty \text{ by (3.13).}$$

So by bounded convergence

$$\int \lim_{k\to\infty} P^{x,\eta}\left[\sup_{\tau_k<t<\tau} |\phi(X^\eta_t) - \phi(X^\eta_{\tau_k})| > \lambda\right] d\mu_x(\eta) = 0.$$

Hence

$$P^{x,\eta}\left[\lim_{k\to\infty} \sup_{\tau_k<t<\tau} |\phi(X^\eta_t) - \phi(X^\eta_{\tau_k})| > \lambda\right] = 0 \quad \text{for a.a. } \eta.$$

Since this holds for all $\lambda > 0$ we obtain the theorem.

4. Conditional convergence implies non-tangential convergence. It is natural to ask if the convergence of ϕ along the conditional paths X_t^η implies non-tangential convergence in the case when U is the open unit disc D. We will prove that this is indeed the case. Thus the situation is analogous to that for a harmonic function converging along the conditional paths of Brownian motion, in which case the equivalence to non-tangential convergence was first established by Brelot and Doob. The proof in our case will adopt basic ideas of the proof of Brossard in the Brownian motion case. See Durrett **[8]** for further references and an exposition of Brossard's proof. We say that a real function u on U is called *A-harmonic* (or X_t-harmonic) if

$$Au = 0 \quad \text{in } U$$

in the sense of distribution. This is equivalent to the mean value property

$$u(x) = E^x[u(X_{\tau_w})]$$

for all stopping times τ_W, where $W \subset\subset U$.

The main result of this section can now be stated as follows:

THEOREM 4.1. *Let u be an A-harmonic function in the open unit disc $D \subset \mathbf{C}$. Suppose*

$$\lim_{t\to\zeta} u(X_t^\eta(\omega)) \tag{4.1}$$

exists for a.a. $\omega \in \Omega$ w.r.t. $P^{x,\eta}$, for some $\eta \in \partial D$, $x \in D$. Then this limit is the same for a.a. ω and it coincides with the non-tangential limit of u at η.

We split the proof into several lemmas. If τ is a stopping time for X_t and $z \in U$ we say that $X_t(\omega)$ *makes a loop around* z for $0 \le t \le \tau$ if z does not belong to the unbounded component of $\mathbf{C}\backslash\{X_t(\omega); 0 \le t \le \tau\}$. A similar terminology is used for X_t^η.

LEMMA 4.2. *Let $W \subset\subset U$ and let K be a compact subset of W. Then there exists $\varepsilon > 0$ such that*

$$P^x[X_t(\omega) \text{ makes a loop around } z;\ 0 \le t \le \tau_W] \ge \varepsilon \quad \text{for all } x, z \in K. \tag{4.2}$$

Proof. We use the notation $D_r(y) = \{x;\ |x-y| < r\}$. Let $x, z \in K$. Then if $r > 0$ is small enough the z-component V of $\phi^{-1}(D_r(\phi(z)))$

is a *normal* neighbourhood of z and $\phi^{-1}(\phi(z)) \cap V = \{z\}$. See Martio, Rickman and Väisälä **[13]**. Since K is compact we can choose r independent of x and z. Since $\phi(\partial V) = \partial(\phi(V))$ we have

$$\beta_{\tau_V} = \tau_{\phi(V)}$$

and therefore

$$\{\phi(X_{\alpha_t});\ 0 \le t \le \beta_{\tau_V}\}$$

coincides with the path of a Brownian motion

$$\{B_t;\ 0 \le t \le \tau_{\phi(V)}\}.$$

It is well known that B_t winds around $\phi(z)$ with positive probability (**[8]**). It follows that X_t, when starting from V, winds around z with positive probability before exiting from V. Since the probability that X_t hits any neighbourhood of z before exiting from W is positive, by "the communication property" of uniformly elliptic diffusions, the lemma follows.

LEMMA 4.3. *The same conclusion as in Lemma* 4.2 *holds for the conditioned process* X_t^η.

Proof. First note that by induction it follows from (2.8) that

$$(4.3)\qquad E^{x,\eta}[g_1(X_{t_1}^\eta)\cdots g_k(X_{t_k}^\eta)] = \frac{1}{k(x)}\cdot E^x[g_1(X_{t_1})\cdots g_k(X_{t_k})k(X_{t_k})]$$

for all $0 \le t_1 \le t_2 \le \cdots \le t_k$. Therefore, if $W \subset\subset U$ then the law $P^{x,\eta}$ of $X_t^{x,\eta}$ for $t \le \tau_W$ is absolutely continuous with respect to the law P^x of X_t for $t \le \tau_W$ with Radon-Nikodym derivative

$$(4.4)\qquad \frac{dP^{x,\eta}}{dP^x} = \frac{k(X_{\tau_W})}{k(x)}.$$

Since k is bounded away from 0 on W we conclude that Lemma 4.3 is a consequence of Lemma 4.2.

For Brownian motion B_t starting at the point x it is well known that the scaled process $\hat{B}_t = x + r(B_t - x)$, $t \ge 0$, (where $r > 0$ is fixed) is again a Brownian motion except for a time change. A uniformly elliptic diffusion is not scaling invariant in the same strong sense. However, scaling a uniformly elliptic diffusion always gives us another uniformly elliptic diffusion (with a time change) with the same ellipticity constant. Moreover, the conditioned process X_t^η behaves similarly under scaling:

LEMMA 4.4 (*Scaling lemma*). *Let* $0 < r \leq 1$, $\eta \in \partial D$ *and define*

$$\xi(x) = rx + (1-r)\eta \quad \textit{for } x \in D \tag{4.5}$$

and

$$Z_t = \xi(X_t), Z_t^\eta = \xi(X_t^\eta); \qquad t \geq 0.$$

Then Z_t *is a uniformly elliptic diffusion with generator* A_ξ *which satisfies*

$$\begin{aligned}(A_\xi f)(\xi(x)) &= A(f \circ \xi)(x) \\ &= r^2[\operatorname{div}(a \circ \xi^{-1} \nabla f)](\xi(x)).\end{aligned} \tag{4.6}$$

Therefore

$$Z_t \simeq \tilde{X}_{r^2 t} \quad (\textit{where } \simeq \textit{ means "identical in law"}) \tag{4.7}$$

where $\tilde{X}_t$ *is the uniformly elliptic diffusion with generator*

$$\tilde{A} f = \operatorname{div}(a \circ \xi^{-1} \cdot \nabla f). \tag{4.8}$$

Moreover,

$$Z_t^\eta \simeq \tilde{X}_{r^2 t}^\eta, \tag{4.9}$$

where $\tilde{X}^\eta$ *is the process obtained by conditioning* $\tilde{X}_t$ *with respect to the* $\tilde{A}$*-kernel* $k \circ \xi^{-1}$.

Proof. By definition of Z_t we have for $f \in C_0^2(D)$

$$\begin{aligned}(A_\xi f)(\xi(x)) &= \lim_{t \to 0} \frac{E^x[f(\xi(X_t)) - f(\xi(x))]}{t} = A[f \circ \xi](x) \\ &= \operatorname{div}[a \nabla (f \circ \xi)](x) = \operatorname{div}[ra \cdot ((\nabla f) \circ (\xi))](x) \\ &= r^2 \operatorname{div}[(a \circ \xi^{-1}) \cdot \nabla f](\xi(x)),\end{aligned}$$

which proves (4.6) and (4.7).

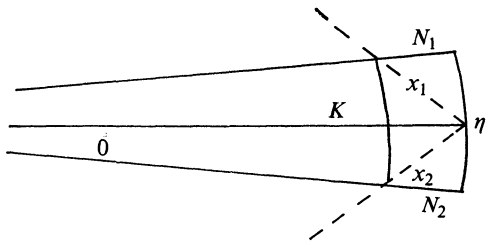
Similarly, if A_ξ^η denotes the generator of Z_t^η we get

$$(A_\xi^\eta f)(\xi(x)) = r^2 \left(\frac{\operatorname{div}((a \circ \xi^{-1}) \cdot \nabla (f \cdot k \circ \xi^{-1}))}{k \circ \xi^{-1}} \right) (\xi(x)),$$

which shows (4.9).

Before stating the next lemma we need some notation: For $\eta \in \partial D$ and $0 < \rho < 1$ let $S = S_\rho(\eta)$ denote the *Stoltz domain* associated to η

and ρ, i.e. $S_\rho(\eta)$ is the interior of the convex hull of the circle $|z| = \rho$ and the point η.



For $\rho < r < 1$ put

$$K = K_r = \{z \in S;\ |z| = r\}.$$

Let L_1, L_2 be the two lines connecting the point $-\eta$ with the points $\eta e^{\pm(1-r)i}$ and let N_1, N_2 be the segments of these lines which connect K to ∂D.

LEMMA 4.5. *There exists a constant $\delta > 0$ depending only on ρ and the ellipticity constant M of X_t such that*

$$P^{x,\eta}[X_t^\eta \text{ hits } K \text{ for some } t < \zeta] \geq \delta \quad \text{for all } x \in N_1 \cup N_2.$$

Proof. Since the transition semigroup T_t^η on $\tilde{H}$ given by (2.8) is symmetric, the corresponding resolvent $\{U_\alpha^\eta\}_{\alpha \geq 0}$ trivially satisfies the duality condition in Theorem VI.1.4 in [3] (relative to the measure $d\xi = k^2\,dx$).

Therefore, by Proposition VI. 4.3 in [3] we can write

$$P^{x,\eta}[X_t^\eta \text{ hits } K \text{ for some } t < \zeta] = \int_K G^\eta(x,y)\,d\lambda(y), \tag{4.10}$$

where G^η is the Green function of X_t^η and $\lambda \geq 0$ is the unique measure on K with the property that

$$\lambda(K) = \sup\left\{\nu(K);\ \nu \geq 0 \text{ measure on } K,\ \int G^\eta(x,y)\,d\nu(y) \leq 1 \text{ for all } x \notin K\right\}.$$

From (2.8) it follows that

$$G^\eta(x,y) = \frac{G(x,y)k(y)}{k(x)}, \tag{4.11}$$

where G is the Green function of X_t in D. Therefore

$$\text{(4.12)}\quad P^{x,\eta}[X_t^\eta \text{ hits } K \text{ for some } t < \zeta] = \int \frac{G(x,y)k(y)}{k(x)}\,d\lambda(y)$$
$$\geq \frac{a}{k(x)}\cdot\int_K G(x,y)\,d\lambda(y), \quad \text{where } a = \inf_K k.$$

The two positive functions $u(x) = \int_K G(x,y)\,d\lambda(y)$ and $k(x)$ are A-harmonic in $D\backslash K$ and they vanish on $\partial D\backslash\{\eta\}$, so by the Comparison Theorem (Theorem 1.4) of Caffarelli, Fabes, Mortola and Salsa **[5]** combined with the Scaling Lemma 4.4 above and a conformal map from D onto the half plane there exists a constant C_1 depending only on ρ and the ellipticity constant such that

$$\text{(4.13)}\qquad \frac{u(x)}{k(x)} \geq C_1\frac{u(x_i)}{k(x_i)} \quad \text{for all } x \in N_i,$$

where x_i is the midpoint of N_i; $i = 1, 2$. Combining (4.10)–(4.13) we get that, with $b = \sup_K k$,

$$\text{(4.14)}\quad P^{x,\eta}[X_t^\eta \text{ hits } K \text{ for some } t < \zeta] \geq \frac{a}{k(x_i)}\int_K G(x_i,y)\,d\lambda(y)$$
$$\geq \frac{C_1 a}{b}\cdot\int_K G^\eta(x_i,y)\,d\lambda(y)$$
$$= \frac{C_1 a}{b}\cdot P^{x_i,\eta}[X_t^\eta \text{ hits } K \text{ for some } t < \zeta]$$

for all $x \in N_i$.

By the Scaling Lemma 4.4 the last hitting probability in (4.14) is bounded below by a positive constant only depending on ρ and the ellipticity constant M. Moreover, if we use the interpretation of $k(z)/k(x)$ as the Radon-Nikodym derivative $d\mu_z/d\mu_x$ of the two exit distributions of X_t starting from z and x, we see that it follows from the Scaling Lemma 4.4 that

$$\frac{a}{b} = \frac{\inf_K k}{\sup_K k} \geq C_2 > 0,$$

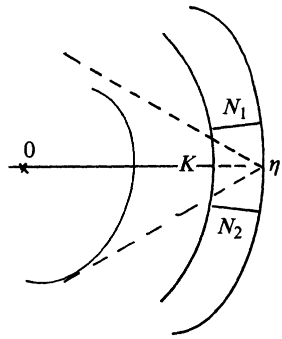
where C_2 only depends on ρ and M. That completes the proof of Lemma 4.5.

LEMMA 4.6. *There exists $\varepsilon > 0$ only depending on ρ, x and M such that*

$$P^{x,\eta}[X_t^\eta \text{ makes a loop around } z;\ 0 \leq t < \zeta] \geq \varepsilon$$

for all $z \in S_\rho$.

Proof. Put $L = L(z) = \{\omega;\ X_t^\eta \text{ makes a loop around } z; 0 \leq t < \zeta\}$.



By the strong Markov property of X_t^η we have

$$P^{x,\eta}[L] = \int_{K \cup N_1 \cup N_2} P^{y,\eta}[L] P^{x,\eta}[X_{\tau_W}^\eta \in dy],$$

where $W = D \backslash V$, V is the closed set bounded by $K = K(r)$, N_1, N_2 and the arc of ∂D between $\eta e^{-(1-r)i}$ and $\eta e^{(1-r)i}$ with $r = 2(1 - |z|)$.

Since $P^{x,\eta}[X_{\tau_W}^\eta \in K \cup N_1 \cup N_2] = 1$ we get from Lemma 4.5

$$\begin{aligned} P^{x,\eta}[L] \geq \delta \cdot P^{x,\eta}[X_{\tau_W}^\eta \in N_1 \cup N_2] \\ + \inf_{y \in K} P^{y,\eta}[L](1 - P^{x,\eta}[X_{\tau_W}^\eta \in N_1 \cup N_2]). \end{aligned}$$

By Lemma 4.3 and Scaling Lemma 4.4 we have

$$\inf_{y \in K} P^{y,\eta}[L] \geq \varepsilon.$$

Since ε and δ only depend on ρ and M, Lemma 4.6 follows.

The proof of Theorem 4.1 is now completed by following the main idea of Brossard, as described in [**8**, p. 114–115]:

First we note that it suffices to prove the following:

(4.15) Suppose $z_n \in S_\rho(\eta)$ for all $n = 1, 2, \ldots$ and $u(z_n) \to \alpha \in [-\infty, \infty]$ as $n \to \infty$. Then $u(X_t^\eta) \to \alpha$ as $t \to \zeta$ a.s. $P^{x,\eta}$.

Put

$$G_n = L(z_n) = \{\omega;\ X_t^\eta \text{ makes a loop around } z_n;\ 0 \leq t \leq \zeta\}$$

and

$$G = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n} G_k \right) = \{\omega;\ \omega \text{ belongs to infinitely many } G_n\text{'s}\}.$$

By the 0-1 law we have that $P^{x,\eta}(G)$ is either 0 or 1. Since by Lemma 4.6

$$P^{x,\eta}(G) = \lim_{n\to\infty} P^{x,\eta}\left(\bigcup_{k=n}^{\infty} G_k\right) \geq \varepsilon,$$

we conclude that $P^{x,\eta}(G) = 1$.

Hence $X_t^{x,\eta}(\omega)$ winds around infinitely many z_n's for a.a. ω. For each such ω with the additional property that

$$\lim_{t\to\zeta} u(X_t^\eta(\omega))$$

exists, we get by the mean value property of u applied to the region inside each loop around z_n that

$$\lim_{t\to\zeta} u(X_t^\eta(\omega)) = \lim_{n\to\infty} u(z_n) = \alpha.$$

Hence (4.15) holds and the proof is complete.

5. Applications. Combining Theorem 3.2 and 4.1 we get

THEOREM 5.1. *Suppose $\phi \in H_{QR}^p(D)$ for some $p > 0$. For $x \in D$ let $\mu = \mu_x$ be the elliptic-harmonic measure of the uniformly elliptic diffusion X_t associated with ϕ. Then*

$$\lim_{z\to\eta} \phi(z)$$

exists non-tangentially for a.a. $\eta \in \partial D$ with respect to μ.

Finally we point out how Theorem 5.1 can be combined with known properties of elliptic-harmonic measures to obtain new results about the boundary behaviour of quasiregular functions, even if we restrict ourselves to results involving only non-stochastic concepts. For $\alpha > 0$ let Λ_α denote the α-dimensional Hausdorff measure.

COROLLARY 5.2 (*Non-stochastic Fatou theorem*). *Suppose $\phi \in H_{QR}^p(D)$ for some $p > 0$. Then there exists $\alpha > 0$ (depending only on ϕ) such that in every non-empty, open interval $J \subset \partial D$ there is a subset $F \subset J$ with $\Lambda_\alpha(F) > 0$ such that ϕ has non-tangential boundary limits at every point of F.*

Proof. As before we let X_t be the uniformly elliptic diffusion associated with ϕ and we let μ be its elliptic-harmonic measure. Then by the doubling property of μ [5], it follows that

$$\mu(J) > 0 \tag{5.1}$$

for every non-empty open interval J in ∂D. On the other hand, it follows from Lemma 3.4 in [12] that there exists $\alpha > 0$ (depending on the ellipticity constant) such that

$$\mu << \Lambda_\alpha. \tag{5.2}$$

By Theorem 5.1 the non-tangential limit of ϕ exists a.e. μ on J, hence by (5.1) and (5.2) on a subset F of J with $\Lambda_\alpha(F) > 0$. That completes the proof.

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