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Let $\{e_i; i = 0, \pm 1, \pm 2, \dots\}$ be a family of projections with the property; (a) $e_i e_{i\pm 1} e_i = \lambda e_i$ for some $\lambda \leq 1$, (b) $e_i e_j = e_j e_i$ for $|i - j| \geq 2$, (c) the von Neumann algebra M generated by the family $\{e_i; i = 0, \pm 1, \pm 2, \dots\}$ is a hyperfinite II_1-factor with the trace tr and (d) $\mathrm{tr}(we_i) = \lambda\,\mathrm{tr}(w)$ if w is a word on 1 and e_j $(j \leq i-1)$. Let N be a von Neumann algebra generated by $\{e_i; i = \pm 1, \pm 2, \dots\}$. Then N is a subfactor of M. If $\lambda = (1/4)\sec^2(\pi/m)$ for some integer m $(m \geq 3)$, then $N' \cap M = \mathbf{C}1$ and the index $[M:N] = (m/4)\mathrm{cosec}^2(\pi/m)$.

1. Introduction. The index theory for finite factors was introduced by Jones in [3]. In that paper, the following sequence $\{e_i; i = 1, 2, \dots\}$ of projections plays an important role:

(a) $e_i e_{i\pm 1} e_i = \lambda e_i$ for some $\lambda \leq 1$,

(b) $e_i e_j = e_j e_i$ for $|i - j| \geq 2$,

(c) the von Neumann algebra P generated by $\{e_i; i = 1, 2, \dots\}$ is a hyperfinite II_1-factor,

(d) $\mathrm{tr}(we_i) = \lambda\,\mathrm{tr}(w)$ if w is a word on $1, e_1, e_2, \dots, e_{i-1}$, where tr is the canonical trace of P and 1 is the identity operator.

If Q is the subfactor of P generated by $\{e_i; i = 2, 3, \dots\}$, then the index $[P:Q]$ of Q in P is $1/\lambda$. In the case $\lambda > 1/4$, Q has trivial relative commutant in P and $[P:Q] = 4\cos^2(\pi/m)$ for some $m = 3, 4, \dots$. Hence by his basic construction, we have the family $\{e_i; i = \dots, -2, -1, 0-, 1, 2, \dots\}$ of projections with the properties (a), (b), (c′) and (d′);

(c′) $\{e_i; i = 0, \pm 1, \pm 2, \dots\}$ generates a hyperfinite II_1 factor M,

(d′) $\mathrm{tr}(we_i) = \lambda\mathrm{tr}(w)$ for the trace tr of M if w is a word on 1 and $\{e_j; j < i\}$ (cf. [5]).

We shall call this family $\{e_i; i = 0, \pm 1, \pm 2, \dots\}$ the *Jones' two sided sequence of projections for* λ. The main purpose of this note is to show the following theorem.

THEOREM. *Let $\{e_i; i = 0, \pm 1, \pm 2, \dots\}$ be the Jones' two sided sequence of projections for $\lambda = (1/4)\sec^2(\pi/m)$ for some m $(m = 3, 4, \dots)$. If M (resp. N) is the von Neumann algebra generated by $\{e_i; i = 0, \pm 1, \pm 2, \dots\}$* (resp. $\{e_i; i = \pm 1, \pm 2, \dots\}$), *then N is a subfactor of M*

with the index

$$[M:N] = (m/4)\operatorname{cosec}^2(\pi/m),$$

and the relative commutant of N in M is trivial, that is, $N' \cap M = \mathbf{C}1$.

The author learned from the referee that A. Ocneanu obtained the same formula independently. She would like to express her hearty thanks to the referee for many valuable comments.

2. Notations and preliminaries. Let B be a subfactor of a II_1-factor A. Then Jones defined in **[3]** the index $[A:B]$ of B in A using the coupling constants of A and B due to Murray and von Neumann (**[4]**) and he (and also, Pimsner-Popa in **[5]**) gives some methods to get the number $[A:B]$. In **[6]**, Wenzl gets another method to compute $[A:B]$ in the case where those factors are σ-weak closures of the union of increasing sequences of finite dimensional algebras, which satisfy some good conditions.

In this note, we shall use the results in **[6]** to give a proof of Theorem.

(2.1) Let A be a finite dimensional von Neumann algebra. Then A is decomposed into a direct sum $\sum_{i=1}^m \oplus A_i$ of $a(i)$ by $a(i)$ matrix algebra A_i. The vector $a = (a(i))$ is called the *dimension vector* of A, following Wenzl **[6]**. Each trace ϕ on the algebra A is determined by a column vector $w = (w(i))$ which satisfies $\phi(x) = \sum_{i=1}^m w(i)\mathrm{Tr}(x_i)$ for $x \in A$, where $x = \sum \oplus x_i$ $(x_i \in A_i)$ and Tr is the usual nonnormalized trace on the matrix algebra. The row vector w is called the *weight vector* of the trace ϕ. Let B be a von Neumann subalgebra of A with direct summand $B = \sum_{i=1}^n \oplus B_i$ of $b(i)$ by $b(i)$ matrix algebras B_i. The inclusion of B in A is specified up to conjugacy by an n by m matrix $[g_{i,j}]$, where $g_{i,j}$ is the number of simple components of a simple A_j module viewed as a B_i module. The matrix $[g_{i,j}]$ is called the *inclusion matrix* of B in A which we denote by $[B \to A]$. Let $b = (b(i))$ be the dimension vector of B and v the weight vector of the restriction of ϕ of B, then

(e) $b[B \to A] = a$ and $[B \to A]w = v$.

(2.2) Let $\{e_i; i = 0, \pm 1, \pm 2, \dots\}$ be Jones' two sided sequence of projections for λ $(\lambda \leq 1)$. A reduced word is a word on e_i's of minimal length for the rules (a), (b) and $e_i^2 \leftrightarrow e_i$. As a trivial consequence of Jones' method in **[3]**, we have that the von Neumann algebra N generated by $\{e_i; i = \pm 1, \pm 2, \dots\}$ is a subfactor of the hyperfinite II_1 factor M generated by $\{e_i; i = 0, \pm 1, \pm 2, \dots\}$.

(2.3) The factor M is the σ-weak closure of the union of the increasing sequence of the following von Neumann algebras $\{M_k; k = 1, 2, \dots\}$:

$$M_1 = \mathbf{C}1, \quad M_{2m} = \{e_j; |j| \leq m-1\}'', \quad M_{2m+1} = \{M_{2m}, e_m\}''.$$

The subfactor N of M is generated by the following increasing sequence of $\{N_k; k = 1, 2, \dots\}$:

$$N_1 = N_2 = \mathbf{C}_1, \quad N_{2m} = \{e_j; 0 \neq |j| \leq m-1\}'', \quad N_{2m+1} = \{N_{2m}, e_m\}''.$$

The algebras M_k and N_k are all finite dimensional ([**3**]). We denote by a_k (resp. b_k) the dimension vector of M_k (resp. N_k). In the case where M_k is the direct sum of d_k matrix algebras, we say d_k is the *length of the dimension vector* a_k.

(2.4) Every N_k is a subalgebra of M_k. Let $E(B)$ be the conditional expectation of M onto the von Neumann subalgebra B of M conditioned by $\mathrm{tr}(xE(B)(y)) = \mathrm{tr}(xy)$ for $x \in B$ and $y \in M$.

LEMMA 1. *$E(N_{k+1})E(M_k) = E(N_k)$ and $E(N)E(M_k) = E(N_k)$ for all k.*

Proof. Since $E(N_{k+1})E(M_k) = E(N_k)$ if and only if $E(N_{k+1})E(M_K) = E(N_{k+1})E(N_k)E(M_k)$, it is sufficient to prove that $\mathrm{tr}(yE(N_{k+1})(x)) = \mathrm{tr}(yE(N_k)(x))$, for $x \in M_k$, $y \in N_{k+1}$. Every reduced word $y \in N_{2m+1}$ has a form $y = vw_1e_mw_2$, where v is a reduced form on $\{e_i; i = -m+1, -m+2, \dots, -1\}$ and w_i $(i = 1, 2)$ is a reduced word on $\{e_i; i = 1, 2, \dots, m-1\}$. Let w be a reduced word in M_{2m}; then

$$\begin{aligned}\mathrm{tr}(yE(N_{2m+1})(w)) = \mathrm{tr}(yw) = \lambda\mathrm{tr}(w_2wvw_1) &= \lambda\mathrm{tr}(E(N_{2m})(w)vw_1w_2) \\ = \mathrm{tr}(w_2E(N_{2m})(w)vw_1e_m) &= \mathrm{tr}(yE(N_{2m})(w)).\end{aligned}$$

Since each algebra is generated by reduced words, $E(N_{2m+1})E(M_{2m}) = E(N_{2m})$. Similarly $E(N_{2m})E(M_{2m+1}) = E(N_{2m-1})$. Since $E(N_{k+1})E(M_k) = E(N_{k+i})E(M_{k+i-1})E(M_k) = E(N_{k+i-1})E(M_k) = \dots = E(M_k)$,

$$E(N)E(M_k) = E(M_k) \quad \text{for all } k.$$

(2.5) Let (A_k) and (B_k) be sequences of finite dimensional von Neumann algebras such that $B_k \subset A_k$ for all k. Following after [**6**], we write $(A_k)_k \supset (B_k)_k$ if $(A_k)_k$ (resp. $(B_k)_k$) generates a II_1-factor A (resp. a subfactor B of A) and satisfies the property of Lemma 1. So, by (c′), (2.2) and Lemma 2, we have $(N_k)_k \subset (M_k)_k$. Such a sequence (M_k) is said to be *periodic* with period r if there is a number m such that $[M_{n+r} \to M_{n+r+i}] = [M_n \to M_{n+i}]$ for $n \geq m$ $(i = 1, 2, \ldots)$ and the matrix $[M_n \to M_{n+r}]$ is primitive for $n \geq m$. The sequences $(M_k)_k \supset (N_k)_k$ is *periodic* if both (M_k) and (N_k) are periodic with same period r and $[N_{n+r} \to M_{n+r}] = [N_n \to M_n]$ for a large enough n ([**6**]). In Section 6, we show the periodicity of $(N_k)_k \subset (M_k)_k$.

3. Bratteli diagram for $(M_k)_k$ and path maps. For convenience' sake, throughout we put

(3.1) for a positive integer k, $p = [k/2]$ and $q = k - p$.

In this section, we shall get, for the sequence (M_k) in (2.3), the components of the inclusion matrix $[M_q \to M_k]$, which we need to obtain the inclusion matrix $[N_k \to M_k]$. Let $A_k = \{1, e_1, \ldots, e_k\}''$. Then M_k is $*$-isomorphic to A_{k-1} for $k \geq 2$. On the other hand there is a unitary u in M_{2m} which satisfies $ue_iu^* = e_{-i}$ and $ue_{-i}u^* = e_i$ for all $i = 0, 1, \ldots, m-1$ ([**3**]). Hence $[M_k \to M_{k+1}] = [A_{k-1} \to A_k]$ for all $k \geq 2$. It is clear that $[M_1 \to M_2]$ is the 1 by 2 matrix $[1, 1]$. In [**3**], Jones gets the Bratteli diagram ([**1**]) for the sequence (A_k) and so we get the Bratteli diagram for (M_k). The dimension vector a_k of M_k, the length d_k of a_k and the weight vector w_k of the restriction of tr on M_k are as follows:

(3.2) If $\lambda \leq 1/4$, then

$$d_k = p + 1,$$

$$a_k(i) = \begin{cases} \binom{k}{p+1-i} - \binom{k}{p-i} & \text{if } i = 1, 2, \ldots, d_k - 1, \\ 1 & \text{if } i = d_k, \end{cases}$$

$$w_k(i) = \lambda^{p+1-i} P_{k-1-2p+2i}(\lambda),$$

where P_j is the polynomial defined in [**3**] by $P_1(x) = P_2(x) = 1$ and $P_{n+1}(x) = P_n(x) - xP_{n-1}(x)$.

$$[M_k \to M_{k+1}] = [\delta_{i,j} + \delta_{i+1,j}]_{i,j}, \quad \text{for Kronecker's } \delta_{i,j},$$

where $i = 1, 2, \ldots, [(k+1)/2] + 1$ and

$$j = \begin{cases} 1, 2, \ldots, [(k+1)/2] + 1 & \text{if } k \text{ is even} \\ 1, 2, \ldots, (k+3)/2 & \text{if } k \text{ is odd.} \end{cases}$$

(3.3) If $\lambda > 1/4$, then $\lambda = (1/4)\sec^2(\pi/n+2)$ for some $n = 1, 2, \ldots$. The Blatteri diagram for $M_1 \subset M_2 \subset \cdots \subset M_n$ has the same form as in the case of $\lambda \leq 1/4$ and the diagram for $M_{n+2i-1} \subset M_{n+2i}$ (resp. $M_{n+2i} \subset M_{n+2i-1}$) is the same as the one for $M_{n-1} \subset M_n$ (resp. the reverse form of one for $M_{n-1} \subset M_n$), for all $i = 0, 1, 2, \ldots$.

Now we consider the Bratteli diagram for $(M_k)_k$ as a graph Λ, the set of vertices of which is the set of points where $a_k(i)$ ($k = 1, 2, \ldots, i = 1, 2, \ldots, d_k$) stand. We denote the vertex in Λ corresponding to $a_k(i)$ by the same notation $a_k(i)$. We denote by $[a_k(i) \to a_{k+1}(j)]$ the edge from $a_k(i)$ to $a_{k+1}(j)$. A *path* on Λ is a sequence $\xi = (\xi_r)$ of edges such that $\xi_r = [a_{k(r)}(i_r) \to a_{k(r)+1}(j_r)]$ for some i_r, j_r and $k(r)$ such that $k(r+1) = k(r) + 1$. The set of all paths in Λ with the starting point $a_k(i)$ and the ending point $a_r(j)$ is called a *polygon from the vertex* $a_k(i)$ *to the vertex* $a_r(j)$ and denoted by $[a_k(i) \to a_r(j)]$. Also the set of all paths in Λ with $a_k(i)$ as the starting point and for some j $a_r(j)$ as the ending point is called a *path map from the vertex* $a_k(i)$ *to the floor* a_r and denoted by $[a_k(i) \to a_r]$. Let Ξ_m be the set of paths on Λ consisting of m edges. For a ξ in Ξ_1 and y in Ξ_m let $\xi \circ y = \{\xi \circ \eta; \eta \in y\}$. Let $x \in \Xi_m$ be a polygon. If there are polygons y and z in Ξ_{m-1} such that as sets of paths x is either the union of $\xi \circ y$ and $\eta \circ z$ or the union of $y \circ \xi$ and $z \circ \eta$ for some ξ and η in Ξ_1, we say x is the *direct sum of* y *and* z and we write $x = y \oplus z$ for $y = x \ominus z$.

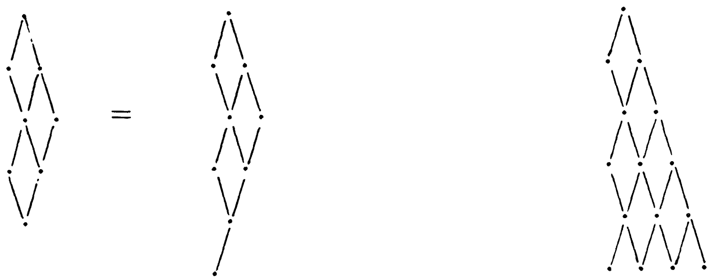
REMARK 2. The ith coordinate $a_k(i)$ of the dimension vector a_k represents a cardinal number of different paths in the polygon $[a_1(1) \to a_k(i)]$. Below, we consider $a_k(i)$ as the polygon $[a_1(1) \to a_k(i)]$ and the dimension vector a_k as the path map $[a_1(1) \to a_k]$. Also, for path map $x = (x(1), \ldots, x(m))$, we denote by the same notation x the path map $(x(1), \ldots, x(m), 0, \ldots, 0)$. We shall identify two polygons or path maps if they are same as figures.

Under such identification, we define the direct sum of path maps. Let $x = (x(1), \ldots, x(h))$, $y = (y(1), \ldots, y(m))$ and $z = (z(1), \ldots, z(n))$ be path maps. If $h = \max\{h, m, n\}$ and $x(i) = y(i) + z(i)$ for every polygon $\{x(i), y(i), z(i)\}$, we say x is the *direct sum* of y and z, and we write $x = y \oplus z$.

REMARK 3. If we use the method of path model in [4], a polygon corresponds to a matrix algebra and a path map corresponds to a multi-matrix algebra.

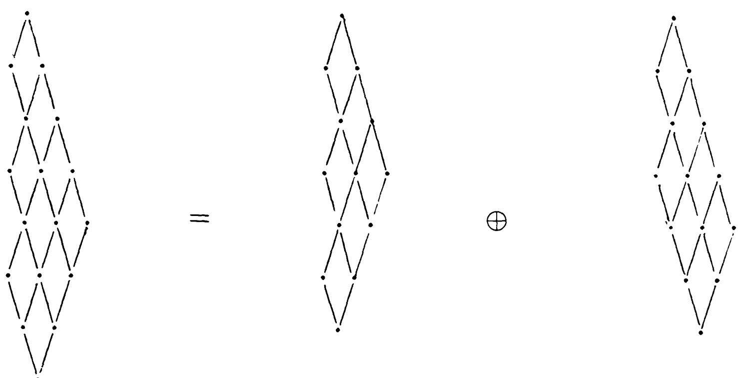
EXAMPLE. (1) The polygon $a_6(1) = (a_1(1) \to a_6(1))$ and the path map $a_6 = (a_1(1) \to a_6)$ are as follows in the case of either $\lambda \leq 1/4$ or $n \geq 6$:

$a_6(1)$ a_6

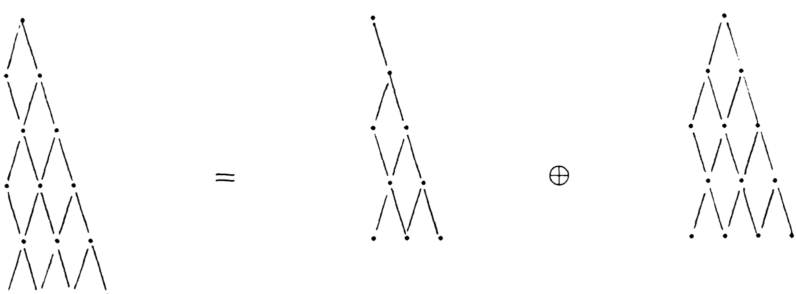


(2) Let $x \in \Xi_7$, $y \in \Xi_6$ and $Z \in \Xi_6$ be polygons, then $x = y \oplus z$ are as follows:

x y z



(3) Direct sum of path maps.



Now we discuss the inclusion matrix $[M_q \to M_k]$. It is obvious that the (i,j)-component of $[M_q \to M_k]$ means the cardinal number of $[a_q(i) \to a_k(j)]$. Hence the ith row vector x_i of $[M_q \to M_k]$ is considered as the path map $[a_q(i) \to a_k]$.

Under the identification of vectors and path maps, we define the polynomials $f_i(m)$ of path maps on Λ by

$$f_i(0) = a_i, \quad f_i(1) = a_{i+1} \quad \text{and} \quad f_i(m+1) = f_{i+1}(m) \ominus f_i(m-1).$$

Then for all positive integers i and m, $f_i(2m)$ (resp. $f_i(2m+1)$) is a polynomial on path maps $\{a_{i+2j};\ j = 0,1,2,\dots,m\}$ (resp. $\{a_{i+2j+1};\ j = 0,1,2,\dots,m\}$ with positive integers as coefficients.

LEMMA 4. *Let x_i be the ith row vector of the inclusion matrix $[M_q \to M_k]$, for a triplet $\{k,p,q\}$ in (3.1). Then, the path map x_i is as follows for all i $(i = 1,2,\dots,d_q)$;*

$$x_i = \begin{cases} f_p(2i-2) & \textit{if } q \textit{ is even}, \\ f_p(2i-1) & \textit{if } q \textit{ is odd}, \end{cases}$$

under the identification for vectors that $(y(1),\dots,y(m),0,\dots,0) = (y(1),\dots,y(m))$ for $y(j) \neq 0$ $(j = 1,\dots,m)$.

Proof. Since the path map x_1 is $(a_q(1) \to a_k)$, it is clear by the shape of graph Λ that

$$x_1 = \begin{cases} a_{p+1} = f_p(1) & \text{if } q \text{ is odd}, \\ a_p = f_p(0) & \text{if } q \text{ is even}. \end{cases}$$

Suppose the statements are true for all $j \leq i$. As a path map, we have

$$x_{i+1} = [a_q(i+1) \to a_k] = \begin{cases} [a_{2i}(i+1) \to a_{p+2i}] & \text{if } q \text{ is even}, \\ [a_{2i+1}(i+1) \to a_{p+2i+1}] & \text{if } q \text{ is odd}, \end{cases}$$

by sliding up the line combining $a_q(1)$ and $a_q(i+1)$ as possible. Then the assumptions of the induction means that

$$[a_{2(i-1)}(i) \to a_{p+2i-2}] = f_p(2i-2)$$

and

$$[a_{2(i-1)+1}(i) \to a_{p+2(i-1)+1}] = f_p(2i-1).$$

Since

$$[a_{2i}(i) \to a_{p+2i}] \oplus [a_{2i}(i+1) \to a_{p+2i}] = [a_{2i-1}(i) \to a_{p+2i}],$$

we have

$$\begin{aligned}[a_{2i}(i+1) \to a_{p+2i}] &= [a_{2i-1}(i) \to a_{p+2i}] \ominus [a_{2i}(i) \to a_{p+2i}] \\ &= [a_{2(i-1)+1}(i) \to a_{p+1+2(i-1)}] \ominus [a_{2(i-1)}(i) \to a_{p+2(i-1)}] \\ &= f_{p+1}(2i-1) \ominus f_p(2i-2) = f_p(2i).\end{aligned}$$

On the other hand,

$$\begin{aligned}&[a_{2i+1}(i) \to a_{p+2i+1}] \oplus [a_{2i+1}(i+1) \to a_{p+2i+1}] \\ &\quad = [a_{2i}(i+1) \to a_{p+2i+1}].\end{aligned}$$

Hence

$$\begin{aligned}&[a_{2i+1}(i+1) \to a_{p+2i+1}] \\ &\quad = [a_{2i}(i+1) \to a_{p+1+2i}] \ominus [a_{2(i-1)+1}(i) \to a_{p+2(i-1)+1}] \\ &\quad = f_{p+1}(2i) \ominus f_P(2i-1) = f_p(2i+1).\end{aligned}$$

Thus $x_{i+1} = f_p(2i)$ if q is even and $x_{i+1} = f_p(2(i+1)-1)$ if q is odd.

4. Bratteli diagram for $(N_k)_k$. Let (N_k) be the sequence in (2.3). Let $N_k(+) = \{e_i \in N_k; j \geq 1\}''$ and $N_k(-) = \{e_j \in N_k; j \leq -1\}''$. Then N_k is generated by the commuting pair $N_k(+)$ and $N_K(-)$. For a triplet $\{k, p, q\}$ in (3.1), $N_k(+)$ is isomorphic to M_q and $N_k(-)$ is isomorphic to M_p. Two dimension vectors and weight vectors of a finite dimensional von Neumann algebra are respectively conjugate by an inner automorphism. We may take a dimension vector b_k of N_k and the weight vector u_k for the restriction of the trace tr of M to N_k as

$$b_k = (a_p(1)a_q, a_p(2)a_q, \ldots, a_p(d_p)a_q) \tag{4.1}$$

and

$$^t u_k = (t_p(1)^t w_q, t_p(2)^t w_q, \ldots, t_p(d_p)^t w_q), \tag{4.2}$$

where $^t y$ denotes the transposed vector of the vector y. Since we obtained the inclusion matrices for (M_k) in (3.1),

$$[N_k \to N_{k+1}] = \begin{cases} I_p \otimes [M_p \to M_{p+1}] & \text{if } k \text{ is odd,} \\ [M_p \to M_{p+1}] \otimes I_q & \text{if } k \text{ is even,} \end{cases} \tag{4.3}$$

where I_k denotes the d_k by d_k identity matrix. It is easy to check that $[N_k \to N_{k+1}]$ satisfies the property (e) for b_k and u_k. The Bratteli diagram for (N_k) comes from the diagram for (M_k) following after the above information.

In the case $\lambda = (1/4)\sec^2(\pi/n+2)$ for some n ($n = 1, 2, \ldots$), the diagram for $N_1 = N_2 \subset N_3 \subset \cdots \subset N_{2n}$ has the same form as in the case $\lambda \leq 1/4$, the diagram for $N_{2n+4i-2} \subset N_{2n+4i-1}$ (resp. $N_{2n+4i-1} \subset N_{2n+4i}$) is similar to one for $N_{2n-2} \subset N_{2n-1}$ (resp. $N_{2n-1} \subset N_{2n}$) and the diagram for $N_{2n+4i} \subset N_{2n+4i+1}$ (resp. $N_{2n+4i+1} \subset N_{2n+4i+2}$) has the reverse form of order changed one for $N_{2n-1} \subset N_{2n}$ (resp. $N_{2n-2} \subset N_{2n}$).

5. Inclusion matrix of N_k in M_k. Let $\{k, p, q\}$ be a triplet in (3.1). Let $x_i(j)$ be the (i, j)-component of $[M_q \to M_k]$ and x_i the ith column vector of $[M_q \to M_k]$. Here we consider $x(i, j)$ and x_i as a polygon and a path map in Ξ_p. By Lemma 4, the polygon $x_i(j)$ can be decomposed into the direct sum of polygons $\{a_{p+j}(i); j = 0, 1, \ldots, i = 1, 2, \ldots, d_p\}$. Then we define the matrix $[a_p \to x_i] = [h(j, k)]$ such that $h(j, k)$ is the number that $a_p(j)$ is contained in $x_i(k)$. We call the matrix $[a_p \to x_i]$ the *inclusion matrix of the path map a_p in the path map x_i.*

REMARK 5. Let x, y and z be path maps on Λ such that $[x \to y]$ and $[x \to z]$ are defined. Then, by the definition of the direct sum of path maps and the inclusion matrix for path maps, the matrix $[x \to (y \oplus z)]$ is defined and

$$[x \to (y \oplus z)] = [x \to y] + [x \to z].$$

By this property and Lemma 4, the inclusion matrix $[a_p \to x_i]$ of the path map a_p in the path map x_i is defined from the inclusion matrices $[M_p \to M_r]$ $(r \geq p)$ by the natural method.

LEMMA 6. *Let $\lambda = (1/4)\sec^2(\pi/n + 2)$ and $p \geq n - 1$.*

(1) *If n is odd and p is even, then*

$$[a_p \to f_p(m)](i, j) = \begin{cases} 1, & -\left[\frac{m}{2}\right] \leq i - j \leq \left[\frac{m+1}{2}\right],\ \left[\frac{m}{2}\right] + 2 \leq i + j \leq 2\left[\frac{n}{2}\right] - \left[\frac{m-1}{2}\right], \\ 0, & \textit{otherwise.} \end{cases}$$

If n is odd and p is odd, then

$$[a_p \to f_p(m)](i, j) = \begin{cases} 1, & -\left[\frac{m+1}{2}\right] \leq i - j \leq \left[\frac{m}{2}\right],\ 1 + \left[\frac{m-1}{2}\right] \leq i + j \leq 2\left[\frac{n}{2}\right] - \left[\frac{m}{2}\right], \\ 0, & \textit{otherwise.} \end{cases}$$

(2) *If n is even and p is odd, then*

$$[a_p \to f_p(m)](i, j) = \begin{cases} 1, & -\left[\frac{m+1}{2}\right] \leq i - j \leq \left[\frac{m}{2}\right], 1 + \left[\frac{m+1}{2}\right] \leq i + j \leq 2\left[\frac{n}{2}\right] - \left[\frac{m}{2}\right], \\ 0, & \textit{otherwise.} \end{cases}$$

If n is even and p is even, then

$$[a_p \to f_p(m)](i,j) = \begin{cases} 1, & -\left[\frac{m}{2}\right] \le i-j \le \left[\frac{m+1}{2}\right],\ \left[\frac{m}{2}\right]+2 \le i+j \le 2\left[\frac{n}{2}\right]-\left[\frac{m+1}{2}\right], \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. It is sufficient to prove the statement for $p = n-1$ and $p = n$, because $f_p(m)$ is the polynomial on $\{a_{p+j}; j = [m/2], j$ is odd (resp. even)$\}$ if m is odd (resp. even) and $[a_p \to a_{p+j}] = [a_{p+2} \to a_{p+2+j}]$ for all $p \ge n-1$ and j. Since $f_p(1) = a_{p+1}$, it is clear that $[a_p \to f_p(1)]$ satisfies the conditions for all n and p. For a given n, assume that the statements hold for $p = n-1$, n and $m = 1, 2, \ldots, k$. Then we can give a proof of the statements for $p = n-1$, n and $m = k+1$ by the relation;

$$[a_p \to f_p(k+1)] = [a_p \to a_{p+1}][a_{p+1} \to f_{p+1}(k)] - [a_p \to f_p(k-1)]$$

and

$$[a_{n+1} \to f_{n+1}(k)] = [a_{n-1} \to f_{n-1}(k)].$$

LEMMA 7. *Let* $\lambda = (1/4)\sec^2(\pi/n+2)$ *and* x_i *the ith column vector of* $[M_q \to M_k]$. *Assume* $q \ge n$.

(1) *If n is odd, then* $[a_p \to x_i]$ *is a* $(1+[n/2])$ *square matrix with the following form*:

(5.1) *If* $p = q$ *is an odd number, then*

$$[a_p \to x_i](j,r) = \begin{cases} 1, & 1-i \le r-j \le i < j+r \le n+2-i, \\ 0, & \textit{otherwise.} \end{cases}$$

(5.2) *If* $p+1 = q$ *is even, then*

$$[a_p \to x_i](j,r) = \begin{cases} 1, & |r-j| < i \le j+r \le n+2-i, \\ 0, & \textit{otherwise.} \end{cases}$$

(5.3) *If* $p = q$ *is even, then*

$$[a_p \to x_i](j,r) = \begin{cases} 1, & |r-j| < i < j+r \le n+3-i, \\ 0, & \textit{otherwise.} \end{cases}$$

(5.4) *If* $p+1 = q$ *is odd, then*

$$[a_p \to x_i](j,r) = \begin{cases} 1, & -i \le r-j < i < j+r \le n+2-i, \\ 0, & \textit{otherwise.} \end{cases}$$

(2) *Let n be even.*

(6.1) *If $p = q$ is odd, then $[a_p \to x_i]$ is an $n/2$ by $1 + (n/2)$ matrix with*

$$[a_p \to x_i](j,r) = \begin{cases} 1, & 1 - i \leq r - j \leq i < j + r \leq n + 2 - i, \\ 0, & \textit{otherwise.} \end{cases}$$

(6.2) *If $p + 1 = q$ is even, then $[a_p \to x_i]$ is an $n/2$ square matrix with*

$$[a_p \to x_i](j,r) = \begin{cases} 1, & |r - j| < i \leq j + r \leq n + 2 - i, \\ 0, & \textit{otherwise.} \end{cases}$$

(6.3) *If $p = q$ is even, then $[a_p \to x_i]$ is a $1 + (n/2)$ square matrix with*

$$[a_p \to x_i](j,r) = \begin{cases} 1, & |r - j| < i < j + r \leq n + 3 - i, \\ 0, & \textit{otherwise} \end{cases}$$

(6.4) *If $p + 1 = q$ is odd, then $[a_p \to x_i]$ is a $1 + (n/2)$ by $n/2$ matrix with*

$$[a_p \to x_i](j,r) = \begin{cases} 1, & -i \leq r - j < i < j + r \leq n + 2 - i, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let n be odd. Then $d_j = d_{n-1}$ for all $j \geq n - 1$. Since $d_{n-1} = [n/2] + 1$, $[M_q \to M_k]$ is a $1 + [n/2]$ square matrix. It means that a_j $(j \geq n - 1)$ and each x_i are path maps consisting of $1 + [n/2]$ polygons in Ξ_{p+1}. Similarly, if n is even, then a_j is a path map with $[n/2]$ (resp. $[n/2] + 1$) polygons for odd (resp. even) $j \geq n - 1$. Hence x_i is a path map with $[n/2]$ (resp. $[n/2] + 1$) polygons if k is odd (resp. even). Therefore by Lemma 5 and Lemma 7, the statements hold.

LEMMA 8. *For the weight vector w_k of the restriction of* tr *to M_k, we have*

$$[a_p \to x_i] w_k = w_q(i) w_p \qquad (i = 1, 2, \ldots, d_q).$$

Proof. We denote the matrix $[[a_p \to a_{p+i}], 0, \ldots, 0]$ by the same notation $[a_p \to a_{p+i}]$, where 0 is the column vector with all components 0. Then by the Bratteli diagram for (M_k), we have for all i $(i = 0, 1, \ldots)$

$$[a_p \to a_{p+i}] w_k = \lambda^{n(i)} w_p \quad \text{for } n(i) = \left[\tfrac{q}{2}\right] - \left[\tfrac{i}{2}\right].$$

Since x_i is given by the polynomials on $\{a_{p+j}; j = 0, 1, \ldots\}$ by Lemma 4, we have the statement by Lemma 5, (3.2) and the relation between the polynomial f_j's and P_j's, because

$$w_k(i) = \lambda^{p+1-i} P_{k-1-2p+2i}(\lambda),$$

where P_j is the polynomial defined in [3] by $P_1(x) = P_2(x) = 1$ and $P_{n+1}(x) = P_n(x) - xP_{n-1}(x)$.

Let G_k be the d_pd_q by d_k matrix, the $(d_q(j-1)+i)$th row vector of which is the jth row vector of the matrix $[a_p \to x_i]$, where $i = 1,2,\dots,d_q$, $j = 1,2,\dots,d_p$. That is, the transposed matrix ${}^tG_k$ of G_k is as follows;

$$ {}^tG_k = [G[1]_1, G[2]_1, \dots, G[d_q]_1, G[1]_2, \dots, G[d_q]_2, \dots, G[1]_{d_p}, \dots, G[d_q]_{d_p}], $$

where $G[i]_j$ is the transposed vector of the jth row vector of $[a_p \to x_i]$.

LEMMA 9. *The matrix G_k satisfies the following:*

$$ b_kG_k = a_k, \quad G_kw_k = u_k \quad \& \quad G_k[M_k \to M_{k+1}] = [N_k \to N_{k+1}]G_{k+1}, $$

where a_k, b_k are dimension vectors of M_k, N_k and W_k, u_k are weight vectors of M_k, N_k.

Proof. Since $a_q[M_q \to M_k] = a_k$, we have, by the relation (4.1),

$$ b_kG_k = \sum_i a_q(i)a_p[a_p \to x_i] = \sum_i a_q(i)x_i = a_k, $$

where i runs over $\{1,2,\dots,d_q\}$.

Lemma 6 implies that $G_kw_k = u_k$, combining the definition of G_k and (4.2).

If $\lambda > 1/4$ and $k \geq 2n$, by Lemma 7, we have $G_k[M_k \to M_{k+1}] = [N_k \to N_{k+1}]G_{k+1}$. For another case, we need a similar lemma as Lemma 7. Below we do not need such cases. Hence we omit the proof of such cases.

Thus we can get a method of inclusion of N_k in M_k. Hence we denote G_k by $[N_k \to M_k]$.

6. Periodicity of $(N_k)_k \subset (M_k)_k$ in the case $\lambda > 1/4$. In this section, we assume that $\lambda = (1/4)\sec^2\pi/(n+2)$ for some n $(n = 1,2,\dots)$.

LEMMA 10. *The sequence (M_k) is periodic with period 2 and the sequence (N_k) is periodic with period 4.*

Proof. Combining the discussions in (2.5) in §3 with results in [2], we have that the sequence (M_k) is periodic with period 2. The fact implies that (N_k) is periodic with period 4, by the Bratteli diagram for (N_k).

LEMMA 11. *Let x_i (resp. y_i) be the ith row vector of* $[M_q \to M_k]$ (*resp.* $[M_{q+2} \to M_{k+4}]$). *If* $q \geq n$, *then*

$$[a_p \to x_i] = [a_{p+2} \to y_i] \qquad (i = 1, 2, \ldots, d_q).$$

Proof. First we remark that both $[M_q \to M_k]$ and $[M_{q+2} \to M_{k+4}]$ are d_q by d_k matrices, because (M_k) is periodic with period 2 and $[M_{q+2} \to M_{k+4}] = [M_q \to M_k][M_k \to M_{k+2}]$. Since $p = [k/2]$ and $q = k - p$, we have $p + 2 = [(k+4)/2]$ and $q + 2 = k + 4 - (p + 2)$, that is, $\{k+4, p+2, q+2\}$ satisfies (3.1). Hence $x_i = f_p(2i-2)$ (resp. $x_i = f_p(2i-1)$) if and only if $y_i = f_{p+2}(2i-2)$ (resp. $f_{p+2}(2i-1)$). By the definition, $f_j(2m)$ (resp. $f_j(2m+1)$) is a linear combination on $\{a_j, a_{j+2}, \ldots, a_{j+2m}\}$ (resp. $\{a_{j+1}, a_{j+3}, \ldots, a_{j+2m+1}\}$) with integer coefficients. Therefore, by Remark 5, we have $[a_p \to x_i] = [a_{p+2} \to y_i]$, because (M_k) is periodic with period 2.

LEMMA 12. *The sequence $(N_k) \subset (M_k)$ is periodic.*

Proof. We already proved that both (M_k) and (N_k) are periodic with same period 4. Hence it is sufficient to prove that

$$[N_k \to M_k] = [N_{k+4} \to M_{k+4}] \quad \text{for } k \geq 2n.$$

By the form of the matrix $[N_k \to M_k] = G_k$, it is nothing else but Lemma 11. Thus $(N_k) \subset (M_k)$ is periodic.

7. Proof of Theorem.

LEMMA 13. *If* $\lambda = (1/4)\sec^2(\pi/m)$ *for some* m $(m = 3, 4, \ldots)$, *then*

$$[M\colon N] = (m/4)\operatorname{cosec}^2(\pi/m).$$

Proof. The factors M and N are generated by the periodic sequences $(N_k) \subset (M_k)$ of finite dimensional algebras. Hence, by [**6**, Theorem 1.5], for the weight vectors w_k and u_k of the restriction tr to M_k and N_k, we have that $[M\colon N] = \|u_k\|_2^2/\|w_k\|_2^2$ for a large enough k. By (4.2),

$$\|u_k\|_2^2 = \|w_p\|_2^2\|w_q\|_2^2 \quad \text{for a } \{k, p, q\} \text{ in (3.1)}.$$

Put $n = m - 2$. Then we have

$$[M\colon N] = \|u_k\|_2^2/\|w_k\|_2^2 \quad \text{for all } k \geq n - 1.$$

Since $\|w_k\|_2^2/\|w_{k+2}\|_2^2 = 1/\lambda$ for all $k \geq n - 1$,

$$[M\colon N] = \|w_{n-1}\|_2^4/\|w_{2(n-1)}\|_2^2 = \|w_{n-1}\|_2^2/\lambda^{n-1}.$$

By the discussion in 3,

$$\|w_{n-1}\|_2^2 = \sum_j \lambda^{2j} P_{n-2j}(\lambda)^2,$$

where j runs over $\left\{0, 1, \ldots, \left[\frac{n-1}{2}\right]\right\}$.

On the other hand, by [3],

$$P_k((1/4)\sec^2\theta) = \sin k\theta / 2^{k-1}\cos^{k-1}\theta \sin\theta \quad \text{for all } k \text{ and } \theta.$$

Hence

$$\begin{aligned}[M{:}N] &= \frac{\sum_j \sin^2(n-2j)\pi/(n+2)}{\sin^2(\pi/(n+2))} \\ &= \frac{\sum_j\{2 - \exp(2(n-2j)/(n+2))\pi i - \exp(2(2j-n)/(n+2))\pi i\}}{4\sin^2(\pi/(n+2))} \\ &= ((n+2)/4)\operatorname{cosec}^2(\pi/(n+2)) = (m/4)\operatorname{cosec}^2(\pi/m),\end{aligned}$$

because $\sum_{j=1}^k \exp((j/k)2\pi i) = 0$, for all integer k.

REMARK. 14 (1) If $m = 3$ or 4, then $[M{:}N] = [P{:}Q]$ for the subfactor $Q = \{e_i; i = 2, 3, \ldots\}''$ of the factor $P = \{e_i; i = 1, 2, \ldots\}''$. That is, $[M{:}N] = 1$ if $m = 3$ and $[M{:}N] = 2$ if $m = 4$.

(2) If $m \geq 5$, then $[M{:}N] \neq [P{:}Q]$. If $m = 5$, then $[M{:}N] < 4$. Hence there is an integer k ($k \geq 3$) such that $[M{:}N] = 4\cos^2(\pi/k)$. H. Choda gets the number k, that is $k = 10$. (Here the author thanks H. Choda for helping her by computing a lot of indices $[M{:}N]$.) On the other hand, by the proof of Lemma 14,

$$[M{:}N] = 4\cos^2(\pi/3) + 4\cos^2(\pi/5).$$

This implies the following equation (the equation is proved by an elementary method, which M. Fujii tells us);

$$\cos^2(\pi/3) + \cos^2(\pi/5) = \cos^2(\pi/10).$$

The following lemma is an easy consequence of Skau's theorem ([7]). Here we shall denote another proof of it as an application of Lemma 7.

LEMMA 15. *The relative commutant $N' \cap M$ of N in M is trivial.*

Proof. Since $[M{:}N]$ is finite, $N' \cap M$ is finite dimensional. Let c be the dimension vector of $N' \cap M$. Since $(M_k) \supset (N_k)$ is periodic, by [6, Theorem 1.7],

$$\|c\|_1 \leq \alpha = \min\{\|G[i]_j\|_1;\ k \geq 2n,\ i = 1, 2, \ldots, d_q, j = 1, 2, \ldots, d_p\},$$

where $G[i]_j$ is the vector in §5. By Lemma 8, there are many $\{i,j\}$'s such that ${}^tG[i]_j = (1,0,\dots,0)$. It implies $\alpha = 1$. Hence $N' \cap M$ is 1-dimensional, so that $N' \cap M = \mathbf{C}1$.

8. A generalization. Fix a positive integer n. Let

$$L = \{\dots, e_{-n-1}, e_{-n}, e_1, e_2, e_3, \dots\}''.$$

In the case $n = 1$, $L = N$. It is clear that L is a subfactor of M, for all n. Also, L is a subfactor of N and $[N{:}L] = 4\cos^2(\pi/m)$. Hence

$$[M{:}L] = (m/4)\operatorname{cosec}^2(\pi/m)\{4\cos^2(\pi/m)\}^{n-1}.$$

Let

$$L_1 = L_2 = \mathbf{C}_1, \quad L_{2i-1} = L_{2i} = \{e_i; i = 1,2,\dots,n-1\}'' \quad \text{if } i \le n$$

and

$$L_{2i+1} = \{L_{2i}, e_i\}'', \quad L_{2i+2} = \{e_{-i}, L_{2i+1}\}'' \quad \text{if } i \ge n.$$

The sequence (L_k) is periodic with period 4 and generates L. By a similar method as for $(N_k) \subset (M_k)$, we get the inclusion matrix $[L_k \to M_k]$. For a triplet $\{k,p,q\}$ in (3.1), we consider the matrix $[a_{p-(n-1)} \to x_i]$ for a large k, where x_i is the same as in §3, that is the ith row vector of $[M_q \to M_k]$. Then $(N_k) \subset (M_k)$ is periodic. Let h be the dimension vector of $L' \cap M$.

If q is even, then $x_1 = a_p$; hence $[a_{p-(n-1)} \to x_1] = [a_{p-(n-1)} \to a_p]$.

If $n = 2$, we have $N' \cap M = \mathbf{C}1$, by the form of $[a_k \to a_{k+1}]$ for an odd k.

If $n \ge 3$, $\{e_{-n+2}, e_{-n+3}, \dots, e_{-1}\}''$ is contained in $L' \cap M$ and isomorphic to M_{n-1}. Hence we have

$$L' \cap M = \{e_{-n+2}, e_{-n+3}, \dots, e_{-1}\}''.$$

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