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Let B be the unit ball and S the unit sphere in $\mathbb{C}^n$ $(n \geq 2)$. Let σ be the unique normalized rotation-invariant Borel measure on S and m the normalized area measure on $\mathbb{C}$.

We first prove that if Λ is a holomorphic homogeneous polynomial on $\mathbb{C}^n$ normalized so that Λ maps B onto the unit disk U in $\mathbb{C}$ and if $\mu = \sigma[(\Lambda|_S)^{-1}]$, then $\mu \ll m$ and the Radon-Nikodym derivative $d\mu/dm$ is radial and positive on U. Then we obtain the asymptotic behavior of $d\mu/dm$ for a certain, but not small, class of functions Λ. These results generalize two recent special cases of P. Ahern and P. Russo. As an immediate consequence we enlarge the class of functions for which Ahern-Rudin's Paley-type gap theorems hold.

1. Introduction. Let n be a positive integer. Write $B = B_n$ for the unit ball in $\mathbb{C}^n$ and let $S = S_n = \partial B_n$. When $n = 1$, we use the notation U and T in place of B_1 and S_1, respectively. We shall let $\sigma = \sigma_n$ denote the unique normalized rotation-invariant Borel measure on S and m the normalized area measure on $\mathbb{C}$. The symbol P_n stands for the class of holomorphic homogeneous polynomials Λ on $\mathbb{C}^n$ normalized so that $\Lambda(B) = U$. The maximum modulus set $\Lambda^{-1}(T) \cap S$ of $\Lambda \in P_n$ is denoted by $\operatorname{Max} \Lambda$. It is assumed $n \geq 2$ in the rest of the paper unless otherwise specified.

To begin with, let us look at some special cases which motivated the main results of this paper. We have the following "change-of-variables" formula for $\Lambda \in P_n$ of degree 1 [**Ru**, Section 1.4]:

$$\int_S \psi \circ \Lambda^* \, d\sigma = (n-1) \int_U \psi(\lambda)(1 - |\lambda|^2)^{n-2} \, dm(\lambda). \tag{1}$$

Here $\Lambda^* = \Lambda|_S$ and ψ denotes an arbitrary nonnegative Borel function on U. The similar integral formula for $\Lambda(z) = z_1^2 + \cdots + z_n^2$ has been recently proved by P. Russo [**Rus**]:

$$\int_S \psi \circ \Lambda^* \, d\sigma = (n-1)/2 \int_U \psi(\lambda)(1 - |\lambda|^2)^{(n-3)/2} \, dm(\lambda). \tag{2}$$

Also, P. Ahern [**A**] has shown that if $\Lambda(z) = n^{n/2} z_1 \cdots z_n$, then

$$\int_S \psi \circ \Lambda^* \, d\sigma = \int_U \psi(\lambda) w(\lambda) \, dm(\lambda) \tag{3}$$

where w is a *weight* such that

$$0 < \lim_{r\uparrow 1} w(r)/(1-r^2)^{(n-3)/2} < \infty. \tag{4}$$

Here and in what follows, a *weight* means a *positive* $[m]$ *a.e., radial, integrable Borel* function on U.

Formula (1) is an immediate consequence of Formula 1.4.4 in **[Ru]**. In **[A]** and **[Rus]**, while (2) and (3) are the important first steps to the main pullback results (see 3.4 below), their proofs strongly depend on the precise formula of Λ. In §2 we prove that every $\Lambda \in P_n$ induces a unique (up to a set of $[m]$ measure 0) weight w so that (3) holds. Note that the exponents in (2) and (4) are the same, while they are different from that in (1). *Why*? One possible way to explain this might be as follows. It is easily verified that if $\Lambda(z) = z_1^2 + \cdots + z_n^2$, then $\operatorname{Max}\Lambda$ is topologically equivalent to $S^{n-1} \times T$ and if $\Lambda(z) = n^{n/2} z_1 \cdots z_n$, then $\operatorname{Max}\Lambda$ is topologically equivalent to T^n. Also it is clear that if $\Lambda \in P_n$ is of degree 1, then $\operatorname{Max}\Lambda = T$, topologically. Hence a close look at the exponents in (1), (2) and (4) leads to the interesting fact that all these exponents can be written as a single expression $\delta(\Lambda)/2 - 1$. The notation $\delta(\Lambda)$ means the topological co-dimension of $\operatorname{Max}\Lambda$ in S, i.e., $\delta(\Lambda) = (2n-1) - \dim(\operatorname{Max}\Lambda)$ where $\dim(\operatorname{Max}\Lambda)$ denotes the topological dimension of $\operatorname{Max}\Lambda$. It is known that $\delta(\Lambda) \geq n-1$ (see **[Ru**, Section 11.4]). Although one cannot expect precise formulas of weights as in (1) and (2) in general, it is a pleasant fact that $\delta(\Lambda)/2-1$ is the *right* exponent for a certain class, say Ω_n, of functions $\Lambda \in P_n$. More precisely, in §3 it is shown that if w is the weight induced by $\Lambda \in \Omega_n$ in the sense of (3), then

$$(*) \qquad w(r) = c(\Lambda)(1-r^2)^{\delta(\Lambda)/2-1}[1 + O(\sqrt{1-r^2})] \quad \text{as } r \uparrow 1.$$

Even when $n = 1$, $\delta(\Lambda)/2 - 1$ is the right exponent at least in the weak sense. Of course $\delta(\Lambda) = 0$ for every $\Lambda \in P_1$ and it is not hard to verify that

$$\lim_{\alpha\downarrow 0} \alpha(1-|\lambda|^2)^{\alpha-1}\, dm(\lambda) = \frac{d\theta}{2\pi}$$

in the weak*-topology of the dual of $C(\overline{U})$. Hence

$$\begin{aligned}\int_0^{2\pi} \psi \circ \Lambda(e^{i\theta}) \frac{d\theta}{2\pi} &= \int_0^{2\pi} \psi(e^{i\theta}) \frac{d\theta}{2\pi} \\ &= \lim_{\alpha\downarrow 0} \alpha \int_{\overline{U}} \psi(\lambda)(1-|\lambda|^2)^{\alpha-1}\, dm(\lambda)\end{aligned}$$

for every $\psi \in C(\overline{U})$.

We do not know whether $(*)$ holds for every $\Lambda \in P_n$.

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2. Existence. In this section we shall prove that to each $\Lambda \in P_n$ corresponds a unique weight as mentioned above. This will be derived from a more general fact Theorem 2.4 below. We shall let $\langle\ ,\ \rangle$ denote the complex inner product on $\mathbb{C}^n$ and $|z|^2 = \langle z, z\rangle$ for $z \in \mathbb{C}^n$.

2.1. *Jacobian and co-area formula.* Consider a continuously differentiable function $\phi: S \to \mathbb{C}$. Fix $\xi \in S$ and $T_\xi(S)$ be the *real* tangent space to S at ξ. We regard $T_\xi(S)$ as a $(2n-1)$-dimensional real space sitting in $\mathbb{C}^n$. Also $\mathbb{C}$ is regarded as a 2-dimensional real space. Therefore, for $z \in \mathbb{C}^n$, $z \in T_\xi(S)$ if and only if $\operatorname{Re}\langle \xi, z\rangle = 0$. Let D be the *differential* of ϕ at ξ defined by

$$D(X) = \frac{d}{dt}\phi(\gamma(t))\bigg|_{t=0} \quad \text{for } X \in T_\xi(S)$$

where γ is any curve in S such that $\gamma(0) = \xi$ and $\gamma'(0) = X$. If $\operatorname{rank} D \leq 1$, then ξ is called a *critical* point of ϕ. The Jacobian of ϕ at ξ is defined by

$$J\phi(\xi) = \sqrt{\det D \circ D^*}.$$

By the Binet-Cauchy formula [**G**], one computes $J\phi(\xi)$ as the square root of the sum of squares of the 2 by 2 subdeterminants of the matrix realization of D with respect to orthonormal bases of $T_\xi(S)$ and $\mathbb{C}$. It follows that ξ is a critical point of ϕ if and only if $J\phi(\xi) = 0$.

Federer's co-area formula for manifolds [**F1**, Theorem 3.1] will be the key to proving the existence of weights. A simple version, which is enough for our purpose, is as follows:

$$\int_S g J\phi \, d\sigma = c(n) \int_{\mathbb{C}} \int_{\phi^{-1}(\lambda)} g \, dh_{2n-3} \, dm(\lambda)$$

for every nonnegative Borel function g on S. Here h_{2n-3} denotes the $(2n-3)$-dimensional Hausdorff measure on S.

2.2. LEMMA. *Let $\phi \in C^{2n-2}(S)$. Then*

$$\sigma[\phi^{-1}] \ll m \quad \textit{if and only if} \quad J\phi > 0 \ [\sigma] \ \textit{a.e.}$$

Proof. First assume $\sigma[\phi^{-1}] \ll m$. It suffices to show $m[\phi(K)] = 0$ where K is the set of all critical points of ϕ. But this follows directly from Sard's theorem (see [**F2**, p. 310]) because $\phi \in C^{2n-2}(S)$.

Conversely, assume $J\phi > 0$ [σ] a.e. By the co-area formula we have

$$\int_{\phi^{-1}(E)} J\phi \, d\sigma = c(n) \int_E h_{2n-3}[\phi^{-1}(\lambda)] \, dm(\lambda)$$

for every Borel set E in $\mathbb{C}$. In particular, if $m(E) = 0$, then $\sigma[\phi^{-1}(E)] = 0$. In other words, $\sigma[\phi^{-1}] \ll m$. The proof is complete. □

The following lemma was proved for $\Lambda(z) = z_1^2 + \cdots + z_n^2$ in **[Rus]**. The proof given below is a modification of P. Russo's proof found there.

2.3. LEMMA. *Suppose ϕ is a function on S which extends to a function holomorphic on some open set containing $\overline{B}$. Then*

$$J\phi = |\nabla\phi| \sqrt{|\nabla\phi|^2 - |R\phi|^2} \tag{1}$$

where $\nabla\phi$ is the complex gradient of ϕ and $R\phi$ is the radial derivative of ϕ, i.e., $R\phi(\xi) = \langle \nabla\phi(\xi), \overline{\xi} \rangle$.

Proof. Fix $\xi \in S$. Let D be the differential of ϕ at ξ and $T_\xi(S)$ the real tangent space to S at ξ. We may regard ϕ as a function holomorphic on some open set containing $\overline{B}$. Hence by the chain rule

$$D(W) = \langle W, \overline{\nabla\phi}(\xi) \rangle \tag{2}$$

for every $W \in T_\xi(S)$. Define $X = \overline{\nabla\phi}(\xi) - \overline{R\phi}(\xi)\xi$. Then (2) can be written

$$D(W) = \langle W, X \rangle + iR\phi(\xi) \operatorname{Re}\langle W, i\xi \rangle \tag{3}$$

because $\operatorname{Re}\langle W, \xi \rangle = 0$ for $W \in T_\xi(S)$. It is easily seen that $i\xi$, X and iX are orthogonal vectors in $T_\xi(S)$. Also it is easy to see $|X|^2 = |\nabla\phi(\xi)|^2 - |R\phi(\xi)|^2$ because $R\phi(\xi) = \langle \nabla\phi(\xi), \overline{\xi} \rangle$. Using (3), we easily obtain $D(i\xi) = iR\phi(\xi)$, $D(X) = |X|^2$ and $D(iX) = i|X|^2$. In addition, if $W \in T_\xi(S)$ is orthogonal to $i\xi$, X and iX, then $D(W) = 0$. Now assume that $X \neq 0$, extend $\{i\xi, X/|X|, iX/|X|\}$ to an ordered orthonormal basis for $T_\xi(S)$ and regard $\{1, i\}$ as an ordered orthonormal basis for $\mathbb{C}$. With respect to these orthonormal bases, D can therefore be realized as a 2 by $2n - 1$ matrix

$$\begin{pmatrix} -\operatorname{Im} R\phi(\xi) & |X| & 0 & 0 \cdots 0 \\ \operatorname{Re} R\phi(\xi) & 0 & |X| & 0 \cdots 0 \end{pmatrix}.$$

It is easily verified that this matrix realization of D is still valid when $X = 0$. A little computation now leads to (1). □

2.4. THEOREM. *Let $\phi \in C^{2n-2}(S)$ and $\mu = \sigma[\phi^{-1}]$. Suppose that ϕ is nonconstant and extends to a function holomorphic on B. Then*

(1) $\mu \ll m$,

(2) $d\mu/dm > 0$ $[m]$ *a.e. on* $\phi(S)$. *More explicitly,*

$$\frac{d\mu}{dm}(\lambda) = c(n) \int_{\phi^{-1}(\lambda)} \frac{dh_{2n-3}}{J\phi} \quad [m] \text{ a.e. } \lambda \in \mathbb{C}.$$

Proof. The proof of Theorem 1 in [S] shows that the set of critical points of ϕ has $[\sigma]$ measure 0. In other words, $J\phi > 0$ $[\sigma]$ a.e. Thus by Lemma 2.2, we obtain (1). Since $J\phi$ is also continuous, $\chi_{\phi^{-1}(E)}/J\phi$ is a Borel function on S whenever E is a Borel set in $\mathbb{C}$. Here $\chi_{\phi^{-1}(E)}$ denotes the characteristic function of $\phi^{-1}(E)$. Apply the co-area formula to this function and obtain

$$\mu(E) = c(n) \int_E \int_{\phi^{-1}(\lambda)} \frac{dh_{2n-3}}{J\phi}\, dm(\lambda).$$

From this it follows that

$$\frac{d\mu}{dm}(\lambda) = c(n) \int_{\phi^{-1}(\lambda)} \frac{dh_{2n-3}}{J\phi} \quad [m] \text{ a.e. } \lambda \in \mathbb{C}. \tag{3}$$

Now suppose that $\lambda \in \phi(S)$ is a *regular value.* This means that λ is not in the image of critical points under ϕ. Since λ is a regular value, $\phi^{-1}(\lambda)$ is a $(2n-3)$-manifold and thus the right side of (3) is strictly positive. Since $\phi \in C^{2n-2}(S)$, Sard's theorem says that the set of regular values of ϕ is of full $[m]$ measure in $\phi(S)$. This proves (2). □

We now come to the main result of this section.

2.5. THEOREM. *Let $\Lambda \in P_n$. Then there exists a unique (up to a set of $[m]$ measure 0) weight w on U such that*

$$\int_S \psi \circ \Lambda^*\, d\sigma = \int_U \psi w\, dm \tag{1}$$

for every nonnegative Borel function ψ on U.

Note. Since $\sigma(\operatorname{Max}\Lambda) = 0$, $\psi \circ \Lambda^*$ is a Borel function defined $[\sigma]$ a.e. on S. This w will be called *the weight induced by* Λ.

Proof. Let $\mu = \sigma[(\Lambda^*)^{-1}]$. By Theorem 2.4 $\mu \ll m$. Put $w = d\mu/dm$. Since $\Lambda(S) = \Lambda(\overline{B})$, $w > 0$ $[m]$ a.e. on U by Theorem 2.4. Recall that we have

$$w(\lambda) = c(n) \int_{(\Lambda^*)^{-1}(\lambda)} \frac{dh_{2n-3}}{J\Lambda^*} \quad \text{for } \lambda \in U$$

after redefining w on a set of $[m]$ measure 0. Since $\Lambda \in P_n$, $|\nabla\Lambda|$ is T-invariant, i.e., $|\nabla\Lambda(\xi)| = |\nabla\Lambda(\xi e^{i\theta})|$ for every $\xi \in S$ and real number θ. Since $R\Lambda = (\deg\Lambda)\Lambda$, $J\Lambda^*$ is T-invariant by Lemma 2.3. It follows from homogeneity that w is radial, because h_{2n-3} is also T-invariant. Thus w is a weight on U. Now, by definition of μ, (1) holds for every characteristic function ψ of a Borel set in U. The general case follows from a standard approximation. The proof is complete. □

2.6. REMARK Let w be the weight induced by $\Lambda \in P_n$. For every nonnegative Borel function ψ on $[0, 1)$ we have

$$\int_S \psi \circ |\Lambda^*|\, d\sigma = 2\int_0^1 r\psi(r)w(r)\, dr.$$

Thus w is completely determined by such ψ. This fact will be the basis for the asymptotic estimate of w near $r = 1$ in §3.

3. Estimation. We shall estimate the asymptotic behavior of the weight induced by a function belonging to a subclass of P_n.

We need some more notation and terminology. Suppose that ϕ_1 and ϕ_2 are $\mathbb{C}$-valued functions on $D_1 \subset \mathbb{C}^k$ and $D_2 \subset \mathbb{C}^l$, respectively. We shall let $\phi_1 \otimes \phi_2$ and $\phi_1 \oplus \phi_2$ denote the functions defined by $(\phi_1 \otimes \phi_2)(z, w) = \phi_1(z)\phi_2(w)$ and $(\phi_1 \oplus \phi_2)(z, w) = \phi_1(z) + \phi_2(w)$ for $(z, w) \in D_1 \times D_2 \subset \mathbb{C}^k \times \mathbb{C}^l = \mathbb{C}^{k+l}$. For simplicity $\Lambda \in P_n$ is called *good* if $\operatorname{Max}\Lambda$ is a disjoint union of finitely many compact manifolds in S and if the weight induced by Λ satisfies $(*)$. Also every $\Lambda \in P_1$ is called *good.*

3.1. THEOREM. *Suppose* $\phi_1 \in P_k$ $(k \geq 1)$ *and* $\phi_2 \in P_l$ $(l \geq 1)$. *Let* $\Lambda \in P_n$ $(n = k + l)$ *be one of the following:*

(a) $\Lambda = b\phi_1 \otimes \phi_2$, $b > 0$,

(b) $\Lambda = a\phi_1 \oplus \phi_2$, $\deg\Lambda \geq 3$, $0 \leq a \leq 1$,

(c) $\Lambda = a\lambda^2 \oplus \phi_2$, $\deg\Lambda = 2$, $0 \leq a < 1$.

In case (c), $k = 1$ *and* $l = n - 1$. *If* ϕ_1 *and* ϕ_2 *are good, then so is* Λ.

3.2. COROLLARY. *Suppose that* $\Lambda \in P_n$ *is a monomial or* $\Lambda(z) = a_1 z_1^d + \cdots + a_n z_n^d$. *Then* Λ *is good.* □

3.3. *Class* Ω_n. It is easily seen from Remark 2.6 that if Λ is good, then so is any power of Λ. Also, if Λ is good, then so is $\Lambda \circ \mathscr{U}$ for every unitary transformation $\mathscr{U}$ on $\mathbb{C}^n$ because Λ and $\Lambda \circ \mathscr{U}$ induce exactly the same weight and $\operatorname{Max}(\Lambda \circ \mathscr{U}) = \mathscr{U}^*(\operatorname{Max}\Lambda)$. Now choose any function as in Corollary 3.2, but of fewer variables, apply Theorem

3.1, (a) through (c), repeatedly to construct a new function in more variables, form any power of this new function, and compose it with any unitary transformation. Let Ω_n be the subclass of P_n obtained by any combination of this process. Then every $\Lambda \in \Omega_n$ is good. The class Ω_n is not small in the sense that it contains most of those functions which may occur in the actual examples or applications.

Before we prove Theorem 3.1, let us look at some consequences

3.4. *Pullback theorem.* Let $0 < p < \infty$. The Hardy space $H^p(B)$ consists of functions f holomorphic on B such that

$$\|f\|_p^p = \sup_{0<r<1} \int_S |f(r\xi)|^p \, d\sigma(\xi) < \infty.$$

For $\alpha > -1$, the weighted Bergman space $A_\alpha^p(U)$ consists of functions g holomorphic on U such that

$$\int_U |g(\lambda)|^p (1 - |\lambda|^2)^\alpha \, dm(\lambda) < \infty.$$

Now consider $\Lambda \in \Omega_n$ and let w be the weight induced by Λ. Then $g \in A_\alpha^p(U)$ where $\alpha = \delta(\Lambda)/2 - 1$ if and only if

$$\int_U |g(\lambda)|^p w(\lambda) \, dm(\lambda) < \infty.$$

Hence it follows that $g \circ \Lambda \in H^p(B)$ for every $g \in A_\alpha^p(U)$. This pullback property is sharp in the sense that the "weight" $\delta(\Lambda)/2 - 1$ cannot be improved.

It is known [**A**], [**Rus**] that if $\Lambda(z) = n^{n/2} z_1 \cdots z_n$ or $\Lambda(z) = z_1^2 + \cdots + z_n^2$, then $g \circ \Lambda^*$ is a function of bounded mean oscillation on S for every Bloch function g on U. P. Ahern and W. Rudin have recently found a new proof of this BMO-result, which does not seem to extend to the H^p-context mentioned above, and proved the same for every monomial $\Lambda \in P_n$. Their method has been used by the author [**C**] to prove the same BMO-result for a certain subclass of Ω_n containing all the previous functions.

3.5. *Gap theorem.* Let E be a set of positive integers. Following [**AR2**] we call E a Paley set if $|\{k \in E: N \le k \le 2N\}|$ stays bounded as $N \to \infty$. Also we recall that E is called a $\Lambda(q)$-set, for $q > 2$, if there is a constant C such that

$$\left(\int_0^{2\pi} \left| \sum_{k \in E} a_k e^{ik\theta} \right|^q \frac{d\theta}{2\pi} \right)^{1/q} \le C \left(\sum_{k \in E} |a_k|^2 \right)^{1/2}$$

for all $\{a_k\}$. It turns out that every Paley set is a $\Lambda(q)$-set for every $q > 2$. Fix $\Lambda \in P_n$. If $f \in L^1(\sigma)$ and if k is a nonnegative integer, then we shall write f_k for the projection of f into the one-dimensional space spanned by Λ^k. In their recent paper **[AR2]** P. Ahern and W. Rudin have also shown the following Paley-type gap theorems which have no analogue in the one variable case for a special class of functions $\Lambda \in P_n$, i.e., for Λ monomials and $\Lambda(z) = z_1^2 + \cdots + z_n^2$. We refer to **[AR2]** for further details:

THEOREM. *Let $1 < p < 2 < q < \infty$. Then the following are equivalent.*

(a) *E is a Paley set.*

(b) *$\sum_{k\in E} \|f_k\|_p^p \le C\|f\|_p^p$ for every $f \in H^p(B)$.*

(c) *There exists $t < 2$ so that $\sum_{k\in E} \|f_k\|_p^t \le C\|f\|_p^t$ for every $f \in H^p(B)$.*

(d) *$\|h\|_q^q \le C\sum_{k\in E} \|h_k\|_q^q$ for every $h \in H^q(B)$ of the form $h = \sum_{k\in E} a_k\Lambda^k$.*

(e) *There exists $s > 2$ so that $\|h\|_q^s \le C\sum_{k\in E} \|h_k\|_q^s$ for every $h \in H^q(B)$ of the form $h = \sum_{k\in E} a_k\Lambda^k$.*

THEOREM. *Let $1 < p < 2$ and q the conjugate exponent. If E is a $\Lambda(q)$-set, then the following two equivalent conditions hold.*

(a) *$\|h\|_q^2 \le C\sum_{k\in E} \|h_k\|_q^2$ for every $h \in H^q(B)$ of the form $h = \sum_{k\in E} a_k\Lambda^k$.*

(b) *$\sum_{k\in E} \|f_k\|_p^2 \le C\|f\|_p^2$ for every $f \in H^p(B)$.*

THEOREM. *If $2 \le p < \infty$, then $\|f\|_p^p \le C\sum_{j=0}^{\infty}(\sum_{2^j\le k<2^{j+1}} \|f_k\|_p)^p$ holds for every $f \in H^p(B)$ of the form $f = \sum_{k=1}^{\infty} a_k\Lambda^k$.*

The only property (except $\Lambda \in P_n$) used in their proof is the following:

$$\int_S |\Lambda|^{2m}\, d\sigma \approx \left(\frac{1}{\sqrt{m}}\right)^{\delta(\Lambda)}. \tag{1}$$

The notation $\approx$ means that the ratios of two terms are bounded above and below. The only importance of the exponent $\delta(\Lambda)$ is the positivity. It is not hard to verify via Stirling's formula that (1) holds for every $\Lambda \in \Omega_n$. It follows that Ahern-Rudin's Paley-type gap theorems mentioned above hold for every $\Lambda \in \Omega_n$. The above observation leads to a natural conjecture that (1) holds for every $\Lambda \in P_n$.

Now we return to the proof of Theorem 3.1. The following integral formula will be the main key to proving this. The normalized volume measure on $\mathbb{C}^k$ is denoted by ν_k.

3.6. PROPOSITION (*Integration by Averaging*). *If k and l are positive integers such that $k + l = n$, then for every $f \in L^1(\sigma_n)$*

$$\int_{S_n} f\,d\sigma_n = \binom{n-1}{k} \int_{B_k} \int_{S_l} f(z, \sqrt{1-|z|^2}\xi)\,d\sigma_l(\xi)(1-|z|^2)^{l-1}\,d\nu_k(z)$$

holds.

Proof. Let O be the group of all unitary transformations on $\mathbb{C}^l$ and fix $f \in L^1(\sigma_n)$. By Lemma 1.4.2 in **[Ru]** we have

$$\int_{S_n} f\,d\sigma_n = \int_{S_n} \int_O f(\zeta, \mathscr{U}\eta)\,d\mathscr{U}\,d\sigma_n(\zeta, \eta)$$

where $d\mathscr{U}$ is the Haar measure on O. In the right side of the above the inner integral is independent of η once ζ is chosen, and hence by Formula 1.4.4 in **[Ru]** the left side of the above is equal to

$$\binom{n-1}{k} \int_{B_k} \int_O f(z, \sqrt{1-|z|^2}\mathscr{U}\xi)\,d\mathscr{U}(1-|z|^2)^{l-1}\,d\nu_k(z)$$

where ξ is an arbitrary point in S_l. By Proposition 1.4.7 in **[Ru]** the inner integral of the above is in turn equal to

$$\int_{S_l} f(z, \sqrt{1-|z|^2}\xi)\,d\sigma_l(\xi).$$

The proof is complete. □

3.7. *Proof of Theorem* 3.1. We will assume $k \geq 2$ (in cases (a) and (b)) and $l \geq 2$. The proof when $l = 1$ or $k = 1$ is much simpler and in fact the proof given below can be repeated without any difficulty. ψ will denote an arbitrary nonnegative Borel function on $[0, 1)$. Also, v_i, $i = 1, 2$, denote the weights induced by ϕ_i and w the weight induced by Λ. We will handle the constants $c = c(k, l, \phi_1, \phi_2)$ in the usual manner; they are not necessarily the same at any two occurrences.

Case (a). In this case $\operatorname{Max}\Lambda = \operatorname{Max}\phi_1 \times \operatorname{Max}\phi_2$, topologically, and therefore $\delta(\Lambda) = \delta(\phi_1) + \delta(\phi_2) + 1$. It remains to verify that

$$(1) \qquad w(r) = c(1-r)^{(\delta(\phi_1)+\delta(\phi_2)-1)/2}[1 + O(\sqrt{1-r})] \quad \text{as } r \uparrow 1.$$

Put $d = \deg \phi_1$ and $e = \deg \phi_2$. Note that $d, e \geq 1$. Define

$$h(x) = b\sqrt{x^d(1-x)^e} \quad \text{for } 0 \leq x \leq 1.$$

Note that h is strictly increasing on $[0, x_0]$, strictly decreasing on $[x_0, 1]$, and $h(x_0) = 1$ where $x_0 = d/(d+e)$. Let f be the inverse function of h on $[0, x_0]$ and g the inverse function of h on $[x_0, 1]$.

Integration by averaging, integration in polar coordinates and change of variables show that

$$\begin{aligned}\int_{S_n} \psi \circ |\Lambda^*| \, d\sigma_n &= c \int_0^1 x^{k-1}(1-x)^{l-1} \\ &\quad \cdot \int_{S_k} \int_{S_l} \psi(h(x)|\phi_1(\xi)||\phi_2(\eta)|) \, d\sigma_l(\eta) \, d\sigma_k(\xi) \, dx \\ &= c \int_{S_k} \int_{S_l} \int_0^1 \theta(\rho)\psi(\rho|\phi_1(\xi)||\phi_2(\eta)|) \, d\rho \, d\sigma_l(\eta) \, d\sigma_k(\xi)\end{aligned}$$

where $\theta = f^{k-1}(1-f)^{l-1}f' - g^{k-1}(1-g)^{l-1}g'$. By definitions of v_1 and v_2, the latter integral is equal to

$$\begin{aligned}&c \int_0^1 \int_0^1 \int_0^1 \theta(\rho)\psi(\rho st) \, d(\rho st) v_1(s) v_2(t) \, ds \, dt \\ &\quad = c \int_0^1 \psi(r) \iint_{st \geq r} \theta_{s,t}(r) v_1(s) v_2(t) \, ds \, dt \, dr\end{aligned}$$

where $\theta_{s,t}(r) = \theta(r/st)$ for $st \geq r$. From this we conclude (see Remark 2.6)

$$\begin{aligned}w(r) &= c[1 + O(\sqrt{1-r})] \\ &\quad \times \iint_{st \geq r} \theta_{s,t}(r)(1-s)^{\delta(\phi_1)/2-1}(1-t)^{\delta(\phi_2)/2-1} \, ds \, dt.\end{aligned} \tag{2}$$

It remains to estimate $\theta_{s,t}(r)$ as $r \uparrow 1$. Since $h'(x_0) = 0$ and $h''(x_0) < 0$, we have

$$h(x) = 1 + \frac{h''(x_0)}{2}(x - x_0)^2[1 + O(|x - x_0|)] \quad \text{as } x \to x_0 \tag{3}$$

and

$$h'(x) = h''(x_0)(x - x_0)[1 + O(|x - x_0|)] \quad \text{as } x \to x_0. \tag{4}$$

From (3) and (4) it follows that

$$\frac{1}{h'(x)} = \frac{[1 + O(|x - x_0|)]}{\sqrt{-2h''(x_0)}\sqrt{1 - h(x)}} \quad \text{as } x \uparrow x_0. \tag{5}$$

Putting $x = f(r)$, we obtain from (3) and (5) that

$$\begin{cases} f(r) = x_0[1 + O(\sqrt{1-r})] \\ f'(r) = \dfrac{[1 + O(\sqrt{1-r})]}{\sqrt{-2h''(x_0)}\sqrt{1-r}} \end{cases} \quad \text{as } r \uparrow 1.$$

Similarly,

$$\begin{cases} g(r) = x_0[1 + O(\sqrt{1-r})] \\ g'(r) = -\dfrac{1 + O(\sqrt{1-r})]}{\sqrt{-2h''(x_0)}\sqrt{1-r}} \end{cases} \quad \text{as } r \uparrow 1.$$

Hence

$$\theta(r) = \frac{c[1 + O(\sqrt{1-r})]}{\sqrt{1-r}} \quad \text{as } r \uparrow 1$$

and therefore

$$\theta_{s,t}(r) = \frac{c[1 + O(\sqrt{1-r})]}{\sqrt{st-r}} \quad \text{uniformly in } st \geq r \text{ as } r \uparrow 1. \tag{6}$$

Insert (6) into (2) and estimate the double integral to obtain (1). □

Cases (b) *and* (c). Let $d = \deg \Lambda$. As above, we have

$$\begin{aligned} &\int_{S_n} \psi \circ |\Lambda^*| \, d\sigma_n \\ &\quad = c \int_0^1 x^{k-1}(1-x)^{l-1} \\ &\qquad \cdot \int_{S_k} \int_{S_l} \psi(|ax^{d/2}\phi_1(\xi) + (1-x)^{d/2}\phi_2(\eta)|) \, d\sigma_l(\eta) \, d\sigma_k(\xi) \, dx \\ &\quad = c \int_0^1 x^{k-1}(1-x)^{l-1} \\ &\qquad \cdot \int_U \int_U \psi(|ax^{d/2}\lambda + (1-x)^{d/2}z|) v_2(z) \, dm(z) v_1(\lambda) \, dm(\lambda) \, dx \\ &\quad = c \int_0^1 x^{k-1}(1-x)^{l-1} \\ &\qquad \cdot \int_0^1 \int_{\operatorname{Im} z \geq 0} \psi(|ax^{d/2}s + (1-x)^{d/2}z|) v_2(z) \, dm(z) s v_1(s) \, ds \, dx. \end{aligned}$$

Compute the innermost integral in polar coordinates, make change of variables, and get

$$\int_0^1 \int_{-1}^1 \frac{\psi(\sqrt{a^2x^ds^2 + (1-x)^dt^2 + 2astx^{d/2}(1-x)^{d/2}y}) \, dy t v_2(t) \, dt}{\sqrt{1-y^2}}.$$

In the inner integral of the above we make another substitution

$$r = \sqrt{a^2x^ds^2 + (1-x)^dt^2 + 2astx^{d/2}(1-x)^{d/2}y},$$

simplify the integral obtained, and arrive at

$$\int_{g_{s,t}(x)}^{h_{s,t}(x)} \frac{2r\psi(r)\,dr}{\sqrt{r^2 - g_{s,t}^2(x)}\sqrt{h_{s,t}^2(x) - r^2}}$$

where

$$h_{s,t}(x) = asx^{d/2} + t(1-x)^{d/2} \quad \text{and}$$
$$g_{s,t}(x) = |asx^{d/2} - t(1-x)^{d/2}|.$$

So far,

$$\int_{S_n} \psi \circ |\Lambda^*|\, d\sigma_n$$
$$= c\int_0^1\int_0^1 \left\{ \int_0^1 \int_{g_{s,t}(x)}^{h_{s,t}(x)} \frac{x^{k-1}(1-x)^{l-1} r\psi(r)\,dr\,dx}{\sqrt{r^2 - g_{s,t}^2(x)}\sqrt{h_{s,t}^2(x) - r^2}} \right\} \cdot sv_1(s)tv_2(t)\,ds\,dt.$$

For each fixed s and t, the integral in the bracket is taken over the region $\{(x,r)\colon g_{s,t}(x) \le r \le h_{s,t}(x)\}$ and therefore the interchange of the order of integration yields $\theta_{s,t}$ such that

$$\int_{S_n} \psi \circ |\Lambda^*|\, d\sigma_n = c\int_0^1 r\psi(r) \int_0^1\int_0^1 \theta_{s,t}(r) sv_1(s)tv_2(t)\,ds\,dt\,dr.$$

Accordingly,

$$w(r) = c\int_0^1\int_0^1 \theta_{s,t}(r) sv_1(s)tv_2(t)\,ds\,dt. \tag{7}$$

Now we subdivide into two cases (b) and (c). Before going further, we assume $a > 0$ (if $a = 0$, then $\operatorname{Max}\Lambda = \operatorname{Max}\phi_2$, topologically, and the proof in case (a) can be easily adapted).

First, consider case (b). Since $d \ge 3$, $h_{s,t}$ is strictly convex for every s and t. By elementary calculus if $t \ge (1/2)^{d/2-1}$, then $h_{s,t}$ and $g_{s,t}$ are strictly decreasing on $[0, 1/3]$ for every s. For such t, let $\alpha_{s,t}$ and $\beta_{s,t}$ be the inverse functions of $h_{s,t}$ and $g_{s,t}$ on $[0, 1/3]$, respectively. We first prove two sublemmas.

SUBLEMMA 1. *Define for* $t \geq r$ *and* $r \approx 1$

$$E_{s,t}(r) = \int_{\beta_{s,t}(r)}^{\alpha_{s,t}(r)} \frac{x^{k-1}(1-x)^{l-1}\,dx}{\sqrt{r^2 - g_{s,t}^2(x)}\sqrt{h_{s,t}^2(x) - r^2}}.$$

Then $E_{s,t}(r) = c(t-r)^{k-1}[1 + O(\sqrt{1-r})]$ *uniformly in* $t \geq r$ *and* s *as* $r \uparrow 1$.

Proof of Sublemma 1. We easily obtain

$$\begin{cases} d\alpha_{s,t}(r) = 2(t-r)[1 + O(\sqrt{1-r})], \\ d\beta_{s,t}(r) = 2(t-r)[1 + O(\sqrt{1-r})], \\ \alpha_{s,t}(r) = \beta_{s,t}(r)[1 + O(\sqrt{1-r})], \end{cases} \tag{8}$$

where $O(\sqrt{1-r})$ is uniform in $t \geq r$ and s as $r \uparrow 1$.

Define $H_{s,t} = h_{s,t}^2$ and $G_{s,t} = g_{s,t}^2$. Since $d \geq 3$, $H'_{s,t}(x) = -dt^2[1 + O(\sqrt{x})]$ uniformly in $t \approx 1$ and s as $x \downarrow 0$. Thus by the mean value theorem and (8), if $\beta_{s,t}(r) < x < \alpha_{s,t}(r)$, then

$$H_{s,t}(x) - r^2 = d(\alpha_{s,t}(r) - x)[1 + O(\sqrt{1-r})]$$

uniformly in $t \geq r$, s and $x \in (\beta_{s,t}(r), \alpha_{s,t}(r))$ as $r \uparrow 1$. Similarly,

$$r^2 - G_{s,t}(x) = d(x - \beta_{s,t}(r))[1 + O(\sqrt{1-r})]$$

uniformly in $t \geq r$, s and $x \in (\beta_{s,t}(r), \alpha_{s,t}(r))$ as $r \uparrow 1$. It follows that

$$E_{s,t}(r) = c[1 + O(\sqrt{1-r})] \int_{\beta_{s,t}(r)}^{\alpha_{s,t}(r)} \frac{x^{k-1}(1-x)^{l-1}\,dx}{\sqrt{x - \beta_{s,t}(r)}\sqrt{\alpha_{s,t}(r) - x}}$$

uniformly in $t \geq r$ and s as $r \uparrow 1$. In this integral, make the substitution

$$y = \frac{x - \beta_{s,t}(r)}{\alpha_{s,t}(r) - \beta_{s,t}(r)}$$

and get

$$\int_0^1 ([\alpha_{s,t}(r) - \beta_{s,t}(r)]y + \beta_{s,t}(r))^{k-1} \cdot (1 - \beta_{s,t}(r) + [\beta_{s,t}(r) - \alpha_{s,t}(r)]y)^{l-1} \frac{dy}{\sqrt{y(1-y)}}.$$

This shows the sublemma by (8). □

If $a = 1$ and if $s \geq (1/2)^{d/2-1}$, then $h_{s,t}$ and $g_{s,t}$ are strictly increasing on $[2/3, 1]$ for every t. For such s, let $\gamma_{s,t}$ and $\delta_{s,t}$ be the inverse functions of $h_{s,t}$ and $g_{s,t}$ (with $a = 1$) on $[2/3, 1]$, respectively. The proof of the following sublemma is "symmetric" and is omitted.

SUBLEMMA 2. *Define for* $s \geq r$ *and* $r \approx 1$

$$F_{s,t}(r) = \int_{\gamma_{s,t}(r)}^{\delta_{s,t}(r)} \frac{x^{k-1}(1-x)^{l-1}\,dx}{\sqrt{r^2 - g_{s,t}^2(x)}\sqrt{h_{s,t}^2(x) - r^2}}.$$

Then $F_{s,t}(r) = c(s-r)^{l-1}[1 + O(\sqrt{1-r})]$ *uniformly in* $s \geq r$ *and* t *as* $r \uparrow 1$. □

Assume $a = 1$. Then $\operatorname{Max}\Lambda$ is the disjoint union of $\operatorname{Max}\phi_1$ and $\operatorname{Max}\phi_2$. Hence it is enough to prove that

$$w(r) = [c(1-r)^{\delta(\phi_1)/2+l-1} + c(1-r)^{\delta(\phi_2)/2+k-1}] \cdot [1 + O(\sqrt{1-r})] \quad \text{as } r \uparrow 1. \tag{9}$$

By a geometric consideration, for r sufficiently close to 1,

$$\theta_{s,t}(r) = \begin{cases} 0 & \text{if } t \leq r \text{ and } s \leq r, \\ E_{s,t}(r) & \text{if } s \leq r \leq t, \\ F_{s,t}(r) & \text{if } t \leq r \leq s, \\ E_{s,t}(r) + F_{s,t}(r) & \text{if } r \leq t \text{ and } r \leq s. \end{cases}$$

Hence by (7), up to a constant factor,

$$\begin{aligned} w(r) &= \int_r^1 \int_0^r E_{s,t}(r) s v_1(s)\,ds t v_2(t)\,dt \\ &\quad + \int_0^r \int_r^1 F_{s,t}(r) s v_1(s)\,ds t v_2(t)\,dt \\ &\quad + \int_r^1 \int_r^1 (E_{s,t}(r) + F_{s,t}(r)) s v_1(s)\,ds t v_2(t)\,dt \\ &= \int_r^1 \int_0^1 E_{s,t}(r) s v_1(s)\,ds t v_2(t)\,dt \\ &\quad + \int_0^1 \int_r^1 F_{s,t}(r) s v_1(s)\,ds t v_2(t)\,dt = \mathrm{I} + \mathrm{II}. \end{aligned}$$

By Sublemma 1

$$\begin{aligned} \mathrm{I} &= c[1 + O(\sqrt{1-r})] \int_r^1 (t-r)^{k-1}(1-t)^{\delta(\phi_2)/2-1}\,dt \\ &= c(1-r)^{\delta(\phi_2)/2+k-1}[1 + O(\sqrt{1-r})] \quad \text{as } r \uparrow 1. \end{aligned}$$

Similarly, by Sublemma 2

$$\mathrm{II} = c(1-r)^{\delta(\phi_1)/2+l-1}[1 + O(\sqrt{(1-r)})] \quad \text{as } r \uparrow 1.$$

This proves (9).

Now assume $a < 1$. Then $\operatorname{Max}\Lambda = \operatorname{Max}\phi_2$, topologically. Thus we need show

$$w(r) = c(1-r)^{\delta(\phi_2)/2+k-1}[1 + O(\sqrt{1-r})] \quad \text{as } r \uparrow 1. \tag{10}$$

As above, for r sufficiently close to 1,

$$\theta_{s,t}(r) = \begin{cases} 0 & \text{if } t \le r, \\ E_{s,t}(r) & \text{if } r \le t. \end{cases}$$

Thus by (7) and Sublemma 1

$$\begin{aligned} w(r) &= c\int_r^1 \int_0^1 E_{s,t}(r) s v_1(s)\, ds\, t v_2(t)\, dt \\ &= c[1 + O(\sqrt{1-r})] \int_r^1 (t-r)^{k-1}(1-t)^{\delta(\phi_2)/2-1}\, dt \quad \text{as } r \uparrow 1. \end{aligned}$$

After a little calculation we obtain (10).

Finally, we consider case (c). In this case $k = 1$ and $l = n-1$. Also, $\operatorname{Max}\Lambda = \operatorname{Max}\phi_2$, topologically. Thus we need verify

$$w(r) = c(1-r)^{\delta(\phi_2)/2}[1 + O(\sqrt{1-r})] \quad \text{as } r \uparrow 1.$$

Note that (7) is still available with $s = 1$, i.e.,

$$w(r) = c\int_0^1 \theta_{1,t}(r) t v_2(t)\, dt. \tag{11}$$

For $r > a$,

$$\theta_{1,t}(r) = \begin{cases} 0 & \text{if } t \le r, \\ E_t(r) & \text{if } r \le t, \end{cases} \tag{12}$$

where $E_t(r)$ is the function as in Sublemma 1 with appropriately defined functions. More precisely, for $t \ge r > a$,

$$\begin{aligned} h_t(x) &= ax + t(1-x), & g_t(x) &= |ax - t(1-x)|, \\ \alpha_t(r) &= \frac{t-r}{t-a}, & \beta_t(r) &= \frac{t-r}{t+a} \end{aligned}$$

and

$$E_t(r) = \int_{\beta_t(r)}^{\alpha_t(r)} \frac{(1-x)^{n-2}\, dx}{\sqrt{r^2 - g_t^2(x)}\sqrt{h_t^2(x) - r^2}}. \tag{13}$$

By (11) and (12), for $r > a$,

$$w(r) = c[1 + O(\sqrt{1-r})] \int_r^1 E_t(r)(1-t)^{\delta(\phi_2)/2-1}\, dt \quad \text{as } r \uparrow 1.$$

Thus it suffices to show that

$$E_t(r) = c[1 + O(\sqrt{1-r})] \quad \text{uniformly in } t \geq r \text{ as } r \uparrow 1. \tag{14}$$

Clearly $\alpha_t(r) = O(1-r)$ and $\beta_t(r) = O(1-r)$ uniformly in $t \geq r$ as $r \uparrow 1$. Hence it is not hard to see that

$$h_t^2(x) - r^2 = 2(1-a)(\alpha_t(r) - x)[1 + O(1-r)]$$

and

$$r^2 - g_t^2(x) = 2(1+a)(x - \beta_t(r))[1 + O(1-r)]$$

uniformly in $t \geq r$ and $x \in (\beta_t(r), \alpha_t(r))$ as $r \uparrow 1$. Substitute these into (13) and obtain (14) as before. □

References

[A] P. Ahern, *On the behavior near torus of functions holomorphic in the ball*, Pacific J. Math., **107** (1983), 267–278.

[AR1] P. Ahern and W. Rudin, *Bloch functions,* BMO *and boundary zeros*, Indiana Univ. Math. J., **36** (1987), 131–148.

[AR2] ———, *Paley-type gap theorems for H^p-functions on the ball*, Indiana Univ. Math. J., to appear.

[C] B. R. Choe, *Cauchy integral equalities and applications*, Trans. Amer. Math. Soc., to appear.

[F1] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc., **93** (1959), 418–491.

[F2] ———, *Geometric Measure Theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1969.

[G] F. R. Gantmacher, *The Theory of Matrices*, Vol. I, Chelsea, 1959.

[Ru] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[Rus] P. Russo, *Boundary behavior of $BMO(B_n)$*, Trans. Amer. Math. Soc., **292**, (1985), 733–740.

[S] N. Sibony, *Valeurs au bord de fonctions holomorphes et ensembles polynomialement convexes*, Lecture Notes in Mathematics, No. 578, pp. 300–313, Springer-Verlag, Heidelberg, 1977.

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