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We outline a new geometric theory of orientations under the action of a group G and formulate bordism theories for G-oriented manifolds. These theories extend the classical G-bordism theories (graded on $\mathbb{Z}$), as well as the $\mathrm{RO}(G)$-graded oriented G-bordism theories which describe bordism of G-manifolds with restricted local representation structure. The theories we obtain account for oriented and unoriented bordism of G-manifolds with and without restricted local representation structure. We further obtain spectral sequences converging to these theories through adjacent family constructions.

1. Introduction and statement of results. The literature contains divergent views both on the notion of equivariant dimension and on equivariant orientability for smooth actions of transformation groups.

As to the former, one has firstly the classical G-bordism groups **[D2, S2]** graded on $\mathbb{Z}$, where one regards the dimension of a smooth G-manifold as an integer in the usual sense. Secondly, one has the $\mathrm{RO}(G)$-graded G-bordism theories of Pulikowski, Kosniowski and the authors **[P2, K1, C1, W1]**, where one considers G-manifolds modelled on a fixed virtual representation of G, and regards the dimension of a G-manifold as the element of $\mathrm{RO}(G)$ determined by this virtual representation.

As regards equivariant orientability, there are several notions in the literature. Stong **[S3]** considered equivariant oriented bordism of oriented manifolds in which the group action preserves the orientation. This idea, which goes back to work of Conner and Floyd **[CF]** who studied oriented manifolds with actions by maps of odd period, appears in much of the subsequent work on oriented G-bordism. (See for example **[R1]**, **[R2]** and **[W4]**.) One also has the folklore requirement that the fixed-sets, together with their normal data, be "coherently" oriented, a weak version of this having appeared in work of Sebastiani and Rothenberg-Sondow **[S1]**, **[RS]**, in order to define equivariant connected sums. (See also **[B2; §VI.8]**.) The point here is that one wishes to avoid having to restrict to orientation-preserving actions.

In RO(G)-graded bordism of G-manifolds modelled on a single representation, the folklore requirement amounts to a reduction of the structure group of the normal bundle **[W1-2]**; one requires that the change-of-coordinate maps have equivariant degree one in a suitable sense.

In **[CMW]** May and the authors of this article present a comprehensive theory of orientations of G-manifolds in terms of categorical constructions which essentially rigorize the folklore view above. The authors in **[CMW]** regard the "dimension" of a G-manifold as an object called a "groupoid representation", discussed in §2 below; this is a categorical construction that captures the interlocking structure of the local representations of subgroups of G.

By way of recent applications, this theory has been shown, in **[CWW]**, to provide a setting in which oriented equivariant cutting and pasting constructions are well-defined and in which oriented equivariant vector field bordism can be studied. It turns out that one of the determining invariants is the "equivariant Euler characteristic" which is defined in terms of the categorical constructions used in the formulation of orientation theory.

It is our purpose here to apply equivariant orientation theory to the study of G-bordism and thereby provide a link between $\mathbb{Z}$-graded G-bordism and G-bordism modelled on a single (virtual) representation. Specifically, we show that the RO(G)-graded G-bordism theories (in both the oriented and unoriented cases) extend to theories graded on a ring of "virtual groupoid representations", with classical unoriented $\mathbb{Z}$-graded G-bordism as a special case. We also show that the existence of "universal" groupoid representations in each dimension leads to a theory of $\mathbb{Z}$-graded *oriented* G-bordism which is new. Further, we provide in §6 a generalization of Kosniowski's **[K2]** inductive exact couple and show the existence of a spectral sequence converging from various forms of free G-bordism to the theories we construct.

This paper is organized as follows. In §2 we summarize equivariant orientation theory so as to render this paper somewhat self-contained. The theory of oriented G-manifolds and their fixed-set data is discussed in §3, and our G-bordism theories constructed in §4. In §5 we construct explicit representing Thom spectra for oriented G-bordism theories, taking the view that these are an important calculational tool, and in §6, we describe a spectral sequence associated with ascending chains of families of subgroups. In §7 we construct the $\mathbb{Z}$-graded theories, and specialize the spectral sequence results to these.

2. Groupoid representations. We summarize the geometric theory of G-orientations given in **[CMW]**, and refer the reader there for more detailed arguments. At first we allow the transformation group G to be an arbitrary compact Lie group, but we specialize to finite groups in §5 when we use transversality results which fail for general compact Lie group actions.

Let G be a compact Lie group; subgroups will be assumed to be closed. Let $\mathcal{G}$ be the category of orbit spaces G/H and G-maps between them, and let $h\mathcal{G}$ be its homotopy category. Note that $\mathcal{G}$ and $h\mathcal{G}$ coincide when G is finite.

Recall that a groupoid is a small category each of whose maps is an isomorphism. A groupoid is *skeletal* if each of its isomorphism classes consists of a single object.

DEFINITION 2.1. A *groupoid over* $h\mathcal{G}$ is a small category $\mathcal{C}$ together with a functor $\varphi: \mathcal{C} \to h\mathcal{G}$ which satisfies the following properties. For each object $a \in h\mathcal{G}$, let $\mathcal{C}(a)$ denote the subcategory of $\mathcal{C}$ consisting of those objects and morphisms which map under φ to a and its identity map.

(a) Each category $\mathcal{C}(a)$ is either empty or a groupoid.

(b) (*Source lifting property*, SLP) For each object $y \in \mathcal{C}$ and each morphism $\beta: a \to \varphi(y)$ in $h\mathcal{G}$, there is an object $x \in \mathcal{C}$ such that $\varphi(x) = a$ and a morphism $\gamma: x \to y$ in $\mathcal{C}$ such that $\varphi(\gamma) = \beta$.

(c) (*Divisibility*) For each commutative triangle in $h\mathcal{G}$ of the form

$$\begin{array}{ccc} & \varphi(y) & \\ {}^{\varphi(\gamma)}\nearrow & & \nwarrow^{\varphi(\gamma')} \\ \varphi(x) & \underset{\beta}{\longrightarrow} & \varphi(x'), \end{array}$$

there exists a morphism $\delta: x \to x'$ in $\mathcal{C}$ such that $\varphi(\delta) = \beta$ and $\gamma' \circ \delta = \gamma$.

We say that $\mathcal{C}$ is a *skeletal groupoid over* $h\mathcal{G}$ if each category $\mathcal{C}(a)$ is skeletal, and that $\mathcal{C}$ is a *faithful groupoid over* $h\mathcal{G}$ if the functor φ is faithful. A *map* $\eta: \mathcal{C} \to \mathcal{C}'$ of *groupoids over* $h\mathcal{G}$ is a functor η such that $\varphi' \circ \eta = \varphi$. We note that any groupoid $\mathcal{C}$ over $h\mathcal{G}$ retracts onto a skeletal subgroupoid of $\mathcal{C}$.

If X is a G-space, define the *fundamental groupoid* $\pi(X;G)$ as follows. The objects of $\pi(X;G)$ are the G-maps $x: G/H \to X$, where H ranges over the (closed) subgroups of G; equivalently, x is a point in X^H. A morphism $x \to y$, $y: G/K \to X$, is the equivalence class of a pair (σ, ω), where $\sigma: G/H \to G/K$ is a G-map, and where $\omega: G/H \times I \to X$ is a G-homotopy from x to $y \circ \sigma$. Two such maps are equivalent if there are G-homotopies $j\colon \sigma' \simeq \sigma'$ and $k\colon \omega \simeq \omega'$ such that

$$k(\alpha, 0, t) = x(\alpha) \quad \text{and} \quad k(\alpha, 1, t) = y \circ j(\alpha, t)$$

for $\alpha \in G/H$ and $t \in I$. If G is discrete, then $\sigma = \sigma'$ and j is constant. Composition is evident. The projection functor $\varphi\colon \pi(X; G) \to h\mathscr{G}$ is given by sending $x\colon G/H \to X$ to the underlying orbit space G/H, and the morphism (σ, ω) to σ.

Fix a small category $G\mathscr{B}$ of orthogonal G-vector bundles over objects in $h\mathscr{G}$ so that $G\mathscr{B}$ contains an isomorphic copy of each G-vector bundle over an orbit, and so that $G\mathscr{B}$ is closed under external Whitney sums and pullbacks over maps in $\mathscr{G}$. A morphism in $G\mathscr{B}$ is a map of G-vector bundles; let $hG\mathscr{B}$ be its homotopy category. The base-space map $\varphi\colon hG\mathscr{B} \to h\mathscr{G}$ gives $G\mathscr{B}$ the structure of a groupoid over $h\mathscr{G}$.

Similarly, let $G\mathscr{V}\mathscr{B}$ be the category of virtual orthogonal G-bundles, that is pairs (E, F) of objects in $G\mathscr{B}$, thought of as formal differences of G-bundles. In order to describe the morphisms in $G\mathscr{V}\mathscr{B}$, let B and B' denote the base-space G-orbits of the pairs (E, F) and (E', F') respectively. A morphism $(E, F) \to (E', F')$ in $G\mathscr{V}\mathscr{B}$ over a G-map $\sigma\colon B \to B'$ is then an equivalence class of pairs of morphisms $(f\colon E \oplus L \to E' \oplus L', g\colon F \oplus L \to F' \oplus L')$ in $G\mathscr{B}$ over σ, with $q\colon L \to B$ and $q'\colon L' \to B'$. The equivalence relation is given by two forms of elementary equivalence:

1. $(f, g) \approx (f \oplus h, g \oplus h)$ for arbitrary h in $G\mathscr{B}$ covering σ;
2. $(f\colon E \oplus L \to E' \oplus L', g\colon F \oplus L \to F' \oplus L') \approx (h\colon E \oplus K \to E' \oplus K', k\colon F \oplus K \to F' \oplus K')$ if there are isomorphisms $L \cong K$ and $L' \cong K'$ such that the diagrams

$$\begin{array}{ccc} E \oplus L & \longrightarrow & E' \oplus L' \\ \downarrow & & \downarrow \\ E \oplus K & \longrightarrow & E' \oplus K' \end{array} \quad \text{and} \quad \begin{array}{ccc} F \oplus L & \longrightarrow & F' \oplus L' \\ \downarrow & & \downarrow \\ F \oplus K & \longrightarrow & F' \oplus K' \end{array}$$

commute.

We then take the smallest equivalence relation, containing these relations, making composition well-defined. (Details appear in **[CW]**.) When the base spaces B and B' coincide, morphisms $(E, F) \to (E', F')$ are in one-to-one correspondence with stable G-bundle maps $E \oplus F' \to E' \oplus F$, where we allow stabilization using arbitrary representations of G. Let $hG\mathscr{V}\mathscr{B}$ be the homotopy category of $G\mathscr{V}\mathscr{B}$; $\varphi\colon hG\mathscr{V}\mathscr{B} \to h\mathscr{G}$ makes this a groupoid over $h\mathscr{G}$.

DEFINITION 2.2. An *(orthogonal) representation of the groupoid* $\mathscr{C}$, *or* $\mathscr{C}$*-representation*, is a map of groupoids $\rho: \mathscr{C} \to hG\mathscr{B}$; thus ρ assigns a G-vector bundle over G/H to each object x of $\mathscr{C}$ such that $\varphi(x) = G/H$, functorially up to homotopy. Similarly, a *virtual representation of* $\mathscr{C}$ is a map $\rho: \mathscr{C} \to hG\mathscr{V}\mathscr{B}$.

Strictly speaking, a representation is a natural isomorphism class of such maps, just as a group representation is a conjugacy class of homomorphisms, but we will usually think in terms of a particular representative map.

EXAMPLES 2.3. (a) A real G-module V determines a representation V of any groupoid $\mathscr{C}$. On objects x, $\mathsf{V}(x) = G/H \times V$ if $\varphi(x) = G/H$; on morphisms $\alpha: x \to y$, one takes $\mathsf{V}(\alpha) = \sigma \times 1$ if $\varphi(\alpha) = \sigma$. We usually take the groupoid underlying V to be $h\mathscr{G}$ itself. Similarly, any virtual representation $[V - W]$ determines an associated virtual groupoid representation $[\mathsf{V} - \mathsf{W}]$.

(b) If $p: E \to B$ is an orthogonal G-vector bundle, then p determines a $\pi(B; G)$-representation $\mathbb{P}$: pulling p back over G-maps $x: G/H \to B$, we obtain a system of G-vector bundles $\mathbb{P}(x) \to G/H$, and the G-covering homotopy property gives the action of $\mathbb{P}$ on morphisms in $\pi(B; G)$. A similar construction can be made with virtual G-vector bundles.

(c) If ρ and μ are groupoid representations, then the virtual groupoid representation $\rho - \mu$ is the object-by-object difference.

Groupoid representations are objects that combine "local" representation data (the objects $\rho(x)$), with orientation data (the morphisms $\rho(\alpha)$). In order to handle unoriented G-vector bundles, we weaken the notion of a $\mathscr{C}$-representation as follows. Let $wG\mathscr{B}$ be the category of G-vector bundles and homotopy classes of maps on base spaces induced by maps of G-vector bundles; thus we remember the existence but not the particular choices of maps of G-vector bundles. The base space functor $\beta: hG\mathscr{B} \to h\mathscr{G}$ factors through $wG\mathscr{B}$. We now define *weak representations of* $\mathscr{C}$ exactly as we defined $\mathscr{C}$-representations, but with $hG\mathscr{B}$ replaced by $wG\mathscr{B}$. The theory of weak representations closely parallels that of "strong" ones; in what follows we will concentrate more on the strong representations.

DEFINITION 2.4. A *map* (ζ, η) from a $\mathscr{C}$-representation ρ to a $\mathscr{C}'$-representation ρ' is a map $\zeta: \mathscr{C} \to \mathscr{C}'$ of groupoids over $h\mathscr{G}$ together with a natural isomorphism $\eta: \rho \to \rho' \circ \zeta$ of functors $\mathscr{C} \to hG\mathscr{B}$.

Maps between virtual representations are defined in the same way, replacing $hG\mathscr{B}$ by $hG\mathscr{V}\mathscr{B}$.

Maps between groupoid representations arise naturally from maps of G-vector bundles; a map $(\tilde{f}, f): p \to p'$ of G-vector bundles, where $\tilde{f}$ is the map of total spaces, and f is the base-space map, determines a natural isomorphism of functors $\tilde{f}_*: \mathbb{P} \to \mathbb{P}' \circ f_*$, and $(f_*, \tilde{f}_*)$ is a map from the $\pi(B; G)$-representation $\mathbb{P}$ to the $\pi(B'; G)$-representation $\mathbb{P}'$.

For the remainder of this section, unless otherwise stated, representations may be virtual or actual.

By the product $\mathscr{C} \times \mathscr{C}'$ of two groupoids over $h\mathscr{G}$, we understand the pullback of their forgetful functors to $h\mathscr{G}$. The direct sum $\rho \oplus \rho'$ and tensor product $\rho \otimes \rho'$ of a $\mathscr{C}$-representation ρ and a $\mathscr{C}'$-representation ρ' are then defined in the evident way and are $\mathscr{C} \times \mathscr{C}'$-representations. When $\mathscr{C} = \mathscr{C}'$, we internalize these operations by restricting along the diagonal functor $\mathscr{C} \to \mathscr{C} \times \mathscr{C}$.

By using these constructions, one may now define a ring structure on the set of isomorphism classes of virtual groupoid representations of a fixed groupoid $\mathscr{C}$ over $h\mathscr{G}$, and we denote this ring by $\mathrm{RO}(\mathscr{C}; G)$.

While all G-vector bundles determine associated groupoid representations, the groupoid representation associated with an *orientable* G-vector bundle should satisfy the following minimal requirement.

DEFINITION 2.6. A $\mathscr{C}$-representation ρ is *orientable* if it is constant on morphisms in the sense that, for any pair of objects x and y of $\mathscr{C}$, and any pair of morphisms μ and ν from x to y with $\varphi(\mu) = \varphi(\nu)$, one has $\rho(\mu) = \rho(\nu)$. The (virtual) G-vector bundle $p: E \to B$ is *orientable* if the $\pi(B; G)$-representation $\mathbb{P}$ is orientable. For example, if the components of each fixed set of B are simply connected, then any G-vector bundle over B is orientable. Further, if $\mathscr{C}$ is a faithful groupoid over $h\mathscr{G}$, then any representation of $\mathscr{C}$ is orientable.

Fix an orientable G-vector bundle $p: E \to B$ and an orientable groupoid representation ρ.

DEFINITION 2.7. An *orientation of p in dimension ρ* is a map $(\xi, \eta): (\pi(B; G), \mathbb{P}) \to (\mathscr{C}, \rho)$. A map $(\tilde{f}, f): p \to p'$ between oriented G-vector bundles is said to be *orientation preserving* if $(\xi', \eta') \circ (f_*, \tilde{f}_*) = (\xi, \eta)$. If p has a given orientation in dimension ρ, we shall refer to p as an *oriented ρ-dimensional G-vector bundle.*

If we ignore orientability, we can use weak groupoid representations to capture the local representation structure of G-bundles as follows.

If ρ is a weak groupoid representation, then a *ρ-dimensional structure* on the G-vector bundle $p\colon E \to B$ is a map $\mathbb{P}_W \to \rho$ of weak groupoid representations, where $\mathbb{P}_W$ is defined in the same way as $\mathbb{P}$, using $wG\mathscr{B}$ in place of $hG\mathscr{B}$.

It is desirable to have a notion of orientation that is free of reference to a specific groupoid representation. With this in mind, the following is proved in [CMW].

THEOREM 2.8. *For each n there exists a "universal" virtual groupoid representation $\mathbb{SO}_n$ such that each orientable virtual groupoid representation of dimension n maps into $\mathbb{SO}_n$. Further, $\mathbb{SO}_n$ is unique up to isomorphism, and $\mathbb{SO}_n \oplus \mathbb{V} \cong \mathbb{SO}_{n+\dim V}$ for any G-module V.* □

Similarly, there are universal weak virtual representations which we denote by $\mathbb{O}_n$, with $\mathbb{O}_n \oplus \mathbb{V} \cong \mathbb{O}_{n+\dim V}$. Moreover, $\mathbb{O}_n$ is a final object in the category of weak virtual groupoid representations of virtual dimension n. There is a similar result for actual (as opposed to virtual) groupoid representations. However, one no longer has $\mathbb{SO}_n \oplus \mathbb{V} \cong \mathbb{SO}_{n+\dim V}$ in general. Examples of universal virtual groupoid representations appear in [CMW].

In view of the existence of universal groupoid representations, one can now define an *oriented n-dimensional* virtual G-vector bundle as simply an oriented $\mathbb{SO}_n$-dimensional virtual G-bundle.

3. Oriented G-manifolds. We now describe the theory of oriented G-manifolds modelled on virtual groupoid representations, and consider the implications for the fixed-set data.

DEFINITIONS 3.1. Let $\mathscr{C}$ be a faithful groupoid, and let $\gamma \in RO(\mathscr{C}; G)$. An *orientation of a smooth G-manifold M in dimension γ* is an orientation of the stable tangent bundle τ_M of M in dimension γ. If M has a given orientation in dimension γ then we say that M is a *γ-dimensional oriented manifold.* Replacing orientable representations with weak representations, we have the notion of a *γ-dimensional (unoriented) manifold.*

When M is a smooth compact G-manifold, it can be embedded in some representation V of G, and we denote the normal bundle of this embedding by $\nu_M(V)$. We define the *virtual normal bundle ν_M* of M to be the virtual bundle $\nu_M(V) - V$, so that an orientation of M in dimension γ is equivalent to an orientation of ν_M in dimension $-\gamma$.

In order to describe the fixed set data associated with a given γ-dimensional structure, we require additional information on passage to fixed sets in the category of groupoid representations. The following is a summary of constructions given in more detail in **[CMW]**.

Suppose then that γ is a representation of the groupoid $\mathscr{C}$ over $h\mathscr{G}$. Let $H \subset G$ and let N be the normalizer of H in G. We can first form the representation $\gamma|N$ by pulling $\mathscr{C}$ back along the map $e: h\mathscr{N} \to h\mathscr{G}$ defined by sending N/K to $G \times_N (N/K) = G/K$, and pulling γ back along the map $hN\mathscr{V}\mathscr{B} \to hG\mathscr{V}\mathscr{B}$ defined similarly. Having done this, we can pull back again along the map $i: h\mathscr{N}/\mathscr{H} \to h\mathscr{N}$ given by considering an N/H-space to be an N-space with trivial action by H. The resulting functor $i^*e^*\mathscr{C} \to hN\mathscr{V}\mathscr{B}$ specifies N-bundles over H-trivial base spaces. Each such bundle is the direct sum of its fiberwise H-fixed points, and the fiberwise complements to the fixed points. The H-fixed point bundles can be taken as defining the N/H-groupoid representation γ^H.

For the complementary representation, we consider another functor $e: h\mathscr{N} \to h\mathscr{N}/\mathscr{H}$ given again by sending N/K to $(N/H) \times_N (N/K) = N/HK$. The composite $i \circ e$, is not the identity, but there is a natural transformation $p: 1 \to i \circ e$, given by projecting $N/K \to N/HK$. If we take the complementary bundles to γ^H and pull back along e, what we get is not a map into $hN\mathscr{V}\mathscr{B}$ over $h\mathscr{N}$, but we can fix this by pulling each bundle back along p. This gives the N-groupoid representation that we call γ_H. We should interpret this as a functor into the category of virtual bundles without H-trivial summands, where stabilization only involves representations without trivial summands. We can also pull γ^H back in this way in order to consider it as an N-representation.

If $\mathscr{C}$ is a skeletal groupoid, then so is $\mathscr{C}' = e^*\mathscr{C}$, and this allows us to define a map $e^*i^*\mathscr{C}' \to \mathscr{C}'$ over $h\mathscr{N}$ given by pulling back the map induced by $i \circ e$ along p, using the source lifting property. This is covered by a map $\gamma^H \oplus \gamma_H \to \gamma$, which tells us how a γ-orientation of a G-bundle ξ over an H-fixed space may be rebuilt from a γ^H-orientation of ξ^H and a γ_H-orientation of the complementary bundle ξ_H.

If M is oriented in dimension γ, then, using inward normals to the boundary, ∂M inherits an orientation in dimension $\gamma - 1$. Further, each M^H inherits the structure of an oriented γ^H-dimensional $N(H)/H$-manifold, while the normal bundle $\nu(M^H, M)$ of M^H in M has the structure of a γ_H-dimensional $N(H)$-bundle.

EXAMPLE 3.2. Let $G = \mathbb{Z}/2$ and let L denote the one-dimensional nontrivial real representation of G. If m and n are greater than one, let G act on the complex projective space $M = \mathbb{C}P(L^{2m} \oplus \mathbb{R}^{2n})$ by translation of planes. The fixed set of M is $M^G = \mathbb{C}P^{m-1} \amalg \mathbb{C}P^{n-1}$ with normal representations L^{2n} and L^{2m} on the respective components. The fundamental groupoid $\pi(M; G)$ has a skeleton consisting of a single object over the free orbit and two objects over the trivial orbit. An orientation of M in $\mathbb{SO}_{2(m+n-1)}$ consists of the following:

(a) A nonequivariant orientation of M.

(b) An equivariant identification of the tangent plane at a chosen point in each component of M^G with $L^{2n} \oplus \mathbb{R}^{2(m-1)}$ or $L^{2m} \oplus \mathbb{R}^{2(n-1)}$ respectively.

(c) Compatibility of the orientations in (a) and (b) with given nonequivariant identifications of $L^{2n} \oplus \mathbb{R}^{2(m-1)}$ and $L^{2m} \oplus \mathbb{R}^{2(n-1)}$ with $\mathbb{R}^{2(m+n-1)}$, these being part of the data in $\mathbb{SO}_{2(m+n-1)}$.

Given (a), there are now four choices for (b) satisfying (c). These choices restrict to four distinct orientations of the fixed-set, and force unique corresponding orientations of the normal bundle to the fixed set. Notice that a nonequivariant orientation alone does not determine an orientation of the fixed-set, while a collection of orientations for all the given data (fixed-sets and normal bundles) need not specify an equivariant orientation, as they may not satisfy (c).

Given two copies of M and a fixed point in each, lying in corresponding components, one can now take an equivariant connected sum using any two orientations of M. If we change one of the orientations at one of the fixed points without altering the nonequivariant orientation, we can form a second connected sum which is nonequivariantly homeomorphic to the first, but possesses a different involution. This illustrates the inadequacy of nonequivariant orientations alone, and is what led Rothenberg and Sondow, and others, to consider local versions of the theory we have described here.

4. Bordism of oriented manifolds. Continuing the discussion in §3, we describe the associated oriented and unoriented G-bordism groups. When the groupoid representations considered have the form $\mathbb{V} - \mathbb{W}$ for fixed G-modules V and W, these groups coincide with those in **[P2, K1, W1, C1]**.

If N is an oriented γ-dimensional manifold, define $-N$ to be the oriented γ-dimensional manifold obtained by reversing the orientation on the trivial summand of $\nu_N(V) \oplus \mathbb{R} = \nu_N(V \oplus \mathbb{R})$. Equivalently, we reverse the orientation on $\tau_N \oplus \mathbb{R} - \mathbb{R}$ by reversing the orientation of one of the trivial summands.

DEFINITION 4.1. Say that two closed oriented γ-dimensional manifolds, M and N, are *cobordant* if there is an oriented $\gamma+1$-dimensional manifold W such that ∂W is isomorphic, as an oriented γ-dimensional manifold, to $M \amalg -N$.

As usual, we can define cobordism of oriented γ-dimensional manifolds over a given G-space X. The resulting group of bordism classes is the γth *oriented bordism group* $\Omega_\gamma^G(X)$. Substituting unoriented manifolds for oriented ones gives the γth *unoriented bordism group* $\mathfrak{N}_\gamma^G(X)$. (Here γ is a weak virtual representation.) These extend naturally to theories defined on pairs of G-spaces, and we denote the corresponding reduced theories by $\tilde{\Omega}_\gamma^G(X)$ and $\tilde{\mathfrak{N}}_\gamma^G(X)$ respectively.

If M is an oriented γ-dimensional manifold and N is an oriented δ-manifold, then $M \times N$ is an oriented $\gamma \times \delta$-manifold. This induces a natural exterior product

$$\Omega_\gamma^G(X) \otimes \Omega_\delta^G(Y) \to \Omega_{\gamma\times\delta}^G(X \times Y),$$

and similarly for the unoriented case.

As an example, there is a distinguished element $D(V)/S(V) \to S^V$ in $\tilde{\Omega}_V^G(S^V)$. Multiplication by this element induces a map

$$\tilde{\Omega}_\gamma^G(X) \to \tilde{\Omega}_{\gamma+V}^G(\Sigma^V X).$$

Following tom Dieck **[D1]** and others **[BH, W1, C1]** we take the colimit, and define the γth *stable oriented G-bordism group* to be

$$\underline{\tilde{\Omega}}_\gamma^G(X) = \operatorname*{colim}_V \tilde{\Omega}_{\gamma+V}^G(\Sigma^V X),$$

taken over all finite G-subspaces of a universe $\mathscr{U}$. $\underline{\tilde{\mathfrak{N}}}_\gamma^G(X)$ is defined in an analogous way.

The usual geometric arguments show that $\tilde{\Omega}_{\gamma+*}^G(-)$ and $\tilde{\mathfrak{N}}_{\gamma+*}^G(-)$ are integer graded G-homology theories for any γ. The stable versions form $RO(G)$-graded theories in the sense of **[W5]** (with all suspension isomorphisms) for each γ.

We interpret the stable bordism groups as bordism groups of *stable manifolds*, where a stable manifold is the equivalence class of a map $f: (M, \partial M) \to (D(V), S(V))$. Here M is a G-manifold and V is a representation of G. The equivalence relation is generated by considering f to be equivalent to $f \times 1: M \times D(W) \to D(V \oplus W)$. This interpretation can be useful in transferring geometric arguments about geometric bordism to arguments about stable bordism. Note that, in the nonequivariant case, transversality says that any stable manifold is the class of an actual manifold, and this manifold is unique up to cobordism. In the equivariant case we do not have such a general transversality result, but the results we do have **[W3]** show at least that for G finite we can assume V above contains no trivial summands.

5. Representing oriented bordism. For the remainder of the paper we restrict G to be finite. In this section we show how the homology theories $\tilde{\Omega}^G_{\gamma+*}(-)$ and $\underline{\tilde{\Omega}}^G_{\gamma+*}(-)$ may be represented by G-spectra.

The following classification result is shown in **[CMW]**.

THEOREM 5.1. *Given any groupoid $\mathscr{C}$, there is a G-space $B\mathscr{C}$ such that $[X, B\mathscr{C}]_G \cong [\pi X, \mathscr{C}]_{h\mathscr{G}}$, where the latter indicates natural isomorphism classes of maps over $h\mathscr{G}$. For example, $BhG\mathscr{V}\mathscr{B}$ classifies virtual orthogonal representations of πX. Further, if $\rho\colon \mathscr{C} \to hG\mathscr{V}\mathscr{B}$ is a representation, then there is a fibration $B\rho\colon B\mathscr{C} \to BhG\mathscr{V}\mathscr{B}$ that classifies ρ-orientations, in that the set of homotopy classes of lifts of a given map $X \to BhG\mathscr{V}\mathscr{B}$ is isomorphic to the set of ρ-orientations of the corresponding representation of πX.* □

We use this to construct Grassmannian models of various ρ-dimensional bundles. Let $\mathscr{U}$ be a G-universe, and let U be a G-inner product space. If $n \geq 0$, define the Grassmannian

$$\mathrm{Gr}_n(U) = \{(Y, u) \mid u \in \mathscr{U} \text{ and } Y \subset U \text{ is a } G_u\text{-invariant } n\text{-plane}\},$$

where G acts in the evident way. If $n < 0$, let $\mathrm{Gr}_n(U) = \varnothing$. The extra coordinate u in $\mathrm{Gr}_n(U)$ is present to control isotropy; see **[W1]** or **[C1]** for an explanation of this point.

Let ρ be an orientable virtual representation of a groupoid $\mathscr{C}$. Then we have the fibration $B\rho\colon B\mathscr{C} \to BhG\mathscr{V}\mathscr{B}$ mentioned in Theorem 5.1. Let $\eta\colon \mathrm{Gr}_{|\rho|}(U) \to BhG\mathscr{V}\mathscr{B}$ classify the groupoid representation corresponding to the canonical G-bundle over the Grassmannian, and let $\mathscr{O}_\rho(U)$ be the pullback in the following diagram:

$$\begin{array}{ccc} \mathscr{O}_\rho(U) & \longrightarrow & B\mathscr{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle B\rho} \\ \mathrm{Gr}_{|\rho|}(U) & \underset{\eta}{\longrightarrow} & BhG\mathscr{V}B. \end{array}$$

Similarly, if ρ is a weak representation of $\mathscr{C}$, let $B\rho\colon B\mathscr{C} \to BwG\mathscr{V}\mathscr{B}$ be the associated fibration, and define $\mathscr{G}_\rho(U)$ to be the pullback in the diagram

$$\begin{array}{ccc} \mathscr{G}_\rho(U) & \longrightarrow & B\mathscr{C} \\ \downarrow & & \downarrow{\scriptstyle B\rho} \\ \mathrm{Gr}_{|\rho|}(U) & \underset{\eta}{\longrightarrow} & BwG\mathscr{V}B. \end{array}$$

We can think of $\mathscr{O}_\rho(U)$ as the space of oriented ρ-dimensional planes in U; $\mathscr{G}_\rho(U)$ is the space of ρ-dimensional planes in U. By pulling back the canonical bundle over $\mathrm{Gr}_{|\rho|}(U)$, the spaces $\mathscr{O}_\rho(U)$ and $\mathscr{G}_\rho(U)$ have bundles over them, the first having a ρ-orientation, and the last being ρ-dimensional. The following is clear from the universal property of $\mathrm{Gr}_n(\mathscr{U})$ (as given in **[W1]**), and the classification property of $B\rho$:

Proposition 5.2. *$\mathscr{O}_\rho(\mathscr{U})$ classifies oriented ρ-dimensional (actual) bundles, and $\mathscr{G}_\rho(\mathscr{U})$ classifies ρ-dimensional bundles.* □

Write

$$BO_G(\rho, V) = \mathscr{G}_\rho(\mathscr{U} \oplus V),$$
$$bO_G(\rho, V) = \mathscr{G}_\rho(\mathscr{U}^G \oplus V),$$

and define $BSO_G(\rho, V)$ and $bSO_G(\rho, V)$ similarly, using $\mathscr{O}_\rho$. We consider $bO_G(\rho, V) \subset BO_G(\rho, V)$ and $bSO_G(\rho, V) \subset BSO_G(\rho, V)$. When ρ is orientable, there is a natural map $BSO_G(\rho, V) \to BO_G(\rho, V)$, and the restriction to $bSO_G(\rho, V)$ gives a map $bSO_G(\rho, V) \to bO_G(\rho, V)$.

These spaces have ρ-dimensional G-bundles over them, which we call EO_G (over BO_G) and ESO_G (over BSO_G). Write TO_G for the Thom space of EO_G, and tO_G for the Thom space of the restriction of this bundle to bO_G. Define TSO_G and tSO_G similarly.The above maps induce G-maps $TSO_G \to TO_G$ and $tSO_G \to tO_G$.

If $V \subset W$, let $W - V$ denote the orthogonal complement of V in W. Then classification of the bundle $EO_G(\rho, V) \oplus (W - V)$ gives a map of Thom spaces

$$\sigma_{W-V}\colon \Sigma^{W-V} TO_G(\rho, V) \to TO_G(\rho + (W - V), W),$$

and this restricts to a map

$$\sigma_{W-V}\colon \Sigma^{W-V} tO_G(\rho, V) \to tO_G(\rho + (W - V), W).$$

Similarly, there are maps involving the oriented spaces.

Suppose now that γ is an orientable virtual groupoid representation. Define the G-prespectra **[LMS]** ${}_\gamma TO_G, {}_\gamma tO_G, {}_\gamma TSO_G$, and ${}_\gamma tSO_G$, by letting

$${}_\gamma TO_G(V) = TO_G(V - \gamma, V),$$

with the maps σ above giving the structural maps; the other three prespectra are defined similarly. The corresponding G-spectra will be called ${}_\gamma MO_G, {}_\gamma mO_G, {}_\gamma MSO_G$, and ${}_\gamma mSO_G$, respectively. The corresponding G-homology theories will be denoted ${}_\gamma MO_*^G(-)$, and so on.

THEOREM 5.3. *If X is any based $G-CW$ complex, there are natural isomorphisms*

$$\underline{\mathfrak{N}}_\gamma^G(X) \cong {}_\gamma MO_0^G(X),$$

$$\underline{\Omega}_\gamma^G(X) \cong {}_\gamma MSO_0^G(X),$$

$$\mathfrak{N}_\gamma^G(X) \cong {}_\gamma MO_0^G(X),$$

and

$$\Omega_\gamma^G(X) \cong {}_\gamma mSO_0^G(X).$$

Moreover, the diagram

$$\begin{array}{ccc} \Omega_\gamma^G(X) & \longrightarrow & \underline{\Omega}_\gamma^G(X) \\ \downarrow & & \downarrow \\ \mathfrak{N}_\gamma^G(X) & \longrightarrow & \underline{\mathfrak{N}}_\gamma^G(X) \end{array}$$

is represented by the diagram of spectra

$$\begin{array}{ccc} {}_\gamma mSO_G & \longrightarrow & {}_\gamma MSO_G \\ \downarrow & & \downarrow \\ {}_\gamma mO_G & \longrightarrow & {}_\gamma MO_G. \end{array}$$

Proof. For the stable results, see **[BH]**. For the geometric results, see **[W2]**, **[W1]**, or **[C1]**. The last two references also give the relation between the geometric and stable theories. The Grassmannian models for the classifying spaces are needed to allow the definition of the smaller spaces bO and bSO; these spaces are needed in order to use equivariant transversality results from **[W3]**. □

The definitions of the spectra imply that we have equivalences

$${}_\gamma MO_G \cong \Sigma^W({}_{\gamma+W}MO_G)$$

and

$${}_\gamma MSO_G \cong \Sigma^W({}_{\gamma+W}MSO_G).$$

Thus, each γ determines an RO(G)-graded theory, the theory of stable manifolds of dimension $\gamma + \alpha$, where $\alpha \in$ RO(G). The geometric theories are not quite as nice: we only have equivalences

$${}_\gamma mO_G \cong \Sigma^k({}_{\gamma+k}mO_G)$$

and

$$_{\gamma}mSO_G \cong \Sigma^k({}_{\gamma+k}mSO_G),$$

where $k \in \mathbb{Z}$. There are maps

$$_{\gamma}mO_G \to \Sigma^W({}_{\gamma+W}mO_G)$$

and

$$_{\gamma}mSO_G \to \Sigma^W({}_{\gamma+W}mSO_G);$$

these fail to be equivalences in general, but represent stabilization. As in **[C1]**, we can show

$$_{\gamma}MO_G \cong \mathrm{colim}_W \Sigma^W({}_{\gamma+W}mO_G)$$

and

$$_{\gamma}MSO_G \cong \mathrm{colim}_W \Sigma^W({}_{\gamma+W}mSO_G),$$

corresponding to the definitions of the stable theories.

6. A spectral sequence. Recall that a *family* of subgroups of G is a collection of subgroups that is closed under subconjugation. Two families $\mathscr{F}' \subset \mathscr{F}$ are *adjacent* if $\mathscr{F} - \mathscr{F}'$ consists of a single conjugacy class of subgroups. Corresponding to any groupoid $\mathscr{C}$ over $h\mathscr{G}$ is the family of subgroups $\varphi(\mathscr{C}) = \{H \mid \varphi^{-1}(G/H) \neq \varnothing\}$.

DEFINITION 6.1. Two groupoids $\mathscr{C}' \subset \mathscr{C}$ are *adjacent* if $(\varphi(\mathscr{C}), \mathscr{F}')$ is a pair of adjacent families, and $\mathscr{C}'$ is the part of $\mathscr{C}$ lying over $\mathscr{F}'$. Two groupoid representations ρ' and ρ are adjacent if the underlying groupoids $\mathscr{C}'$ and $\mathscr{C}$ are adjacent, and if ρ' is the restriction of ρ to $\mathscr{C}'$.

DEFINITION 6.2. If $\gamma' \subset \gamma$ are two virtual representations, then a *(γ, γ')-manifold* is an oriented γ-dimensional manifold M such that ∂M is $(\gamma' - 1)$-dimensional, in such a way that the structure on ∂M is the restriction of the $(\gamma - 1)$-structure inherited from M.

This gives rise to bordism of (γ, γ')-manifolds in the usual way, leading to relative G-bordism groups $\Omega^G_{(\gamma,\gamma')}(-)$. The usual geometric argument shows that we have a long exact sequence

$$\cdots \to \Omega^G_{\gamma'}(-) \to \Omega^G_{\gamma}(-) \to \Omega^G_{(\gamma,\gamma')}(-) \to \Omega^G_{\gamma'-1}(-) \to \cdots.$$

If γ and γ' are adjacent, notice that γ'-dimensional bordism is really γ-dimensional bordism with a restriction on the isotropy (as in **[CF]**). If $\mathscr{F}'$ is the family associated with γ', and $E\mathscr{F}'$ is the universal $\mathscr{F}'$-space **[P1]**, then $\Omega^G_{\gamma'}(X) \cong \Omega^G_{\gamma}(E\mathscr{F}' \times X)$, and the exact sequence is actually the exact sequence corresponding to the cofibration $(E\mathscr{F}')^+ \to S^0 \to \tilde{E}\mathscr{F}'$.

Since the exact sequence above is compatible with stabilization, we can do exactly the same thing for stable bordism, getting relative groups $\underline{\Omega}^G_{(\gamma,\gamma')}(-)$ and a long exact sequence. Again, this is the exact sequence corresponding to the cofibration above.

If γ is a virtual representation, and $\mathscr{F}$ is the corresponding family, then if $\emptyset = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n = \mathscr{F}$ is a sequence of adjacent families, there is a corresponding sequence $\gamma_0, \dots, \gamma_n = \gamma$ of adjacent representations. The resulting long exact sequences can be interpreted as an exact couple, giving a spectral sequence converging to $(\gamma + *)$-bordism, where $*$ is an integer. This is just the spectral sequence corresponding to the filtration of the family $\mathscr{F}$. The E^2-term of this spectral sequence is given by the relative bordism groups, which we proceed to compute.

Let (γ, γ') be an adjacent pair of representations with corresponding families $\mathscr{F}$ and $\mathscr{F}'$ and with $\mathscr{F} - \mathscr{F}' = (H)$. Let $\mathscr{C}$ be the groupoid underlying γ, which we may assume is skeletal and faithful. If x is an object in $\mathscr{C}(G/H)$, let $\varphi_x \colon \mathscr{C}_x \to h\mathscr{G}$ be the groupoid of $\mathscr{C}$ generated by x, and $\gamma_x = \gamma|\mathscr{C}_x$. Let us assume that $\gamma(x)$ is the class of an actual bundle, say $G \times_H V$. The normalizer $N(H)$ acts on $\mathscr{C}_x(G/H)$ because $\mathscr{C}_x$ is skeletal and faithful. Let $K \subset N(H)$ be the isotropy subgroup of x under this action. The representation γ_x then specifies a homomorphism $K/H \to hG\mathscr{V}\mathscr{B}(G \times_H V, G \times_H V)$ such that the composite with the map $hG\mathscr{V}\mathscr{B}(G \times_H V, G \times_H V) \to N(H)/H$ is the inclusion $K/H \to N(H)/H$. The restriction of these bundle maps to $G \times_H V_H$ gives a map into the set of virtual self-maps of this bundle, where now we can restrict the stabilization to allow only addition of representations with no trivial summands. We can also think of this as a map $K/H \to hK\mathscr{V}\mathscr{B}(K \times_H V_H, K \times_H V_H)$ (using the restricted notion of stabilization here). Now notice (since G is finite) that there are short exact sequences of groups

$$\begin{array}{ccccccccc} 1 \to & SO_H(V_H) & \to & K\mathscr{B}(K \times_H V_H, K \times_H V_H) & \to & hK\mathscr{B}(K \times_H V_H, K \times_H V_H) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & SO'_H & \to & K\mathscr{V}\mathscr{B}(K \times_H V_H, K \times_H V_H) & \to & hK\mathscr{V}\mathscr{B}(K \times_H V_H, K \times_H V_H) & \to 1 \end{array}$$

where $SO_H(V_H)$ is the identity component of the group $O_H(V_H)$ of H-isometries of V_H, and SO'_H is the colimit of the groups $SO_H(W)$ as W runs through all representations of H without trivial summands (O'_H is defined similarly). This gives us an action of the group $hK\mathscr{B}(G \times_H V_H, G \times_H V_H)$ on the classifying space $BSO_H(V_H)$. From this we get an action of K on the space $hK\mathscr{V}\mathscr{B} \times_{hKB} BSO_H(V_H)$, where we write hKB for $hK\mathscr{B}(G \times_H V_H, G \times_H V_H)$, and so on. Further, we have short exact sequences

$$\begin{array}{ccccccccc} 1 \to & \pi_0 O_H(V_H) & \to & hK\mathscr{B}(K \times_H V_H, K \times_H V_H) & \to & K/H & \to 1 \\ & \downarrow & & \downarrow & & \| & \\ 1 \to & \pi_0 O'_H & \to & hK\mathscr{V}\mathscr{B}(K \times_H V_H, K \times_H V_H) & \to & K/H & \to 1 \end{array}$$

From these, since $\pi_0 O_H(V_H) \to \pi_0 O'_H$ is a split inclusion (the quotient being those components corresponding to irreducible representations of H not appearing in V_H), we see that the inclusion $hK\mathscr{B} \to hK\mathscr{V}\mathscr{B}$ is also split, with the same quotient. Therefore, if we use the homomorphism $K \to hK\mathscr{V}\mathscr{B} \to hK\mathscr{B}$ to get an action of K on $BSO_H(V_H)$, the K-space $hK\mathscr{V}\mathscr{B} \times_{hKB} BSO_H(V_H)$ above is really the product

$$\pi_0 O'_H / \pi_0 O_H(V_H) \times BSO_H(V_H) = \pi_0(O'_H / O_H(V_H)) \times BSO_H(V_H).$$

This is the action used in the statement of the following proposition.

PROPOSITION 6.3. *There is a natural isomorphism*

$$\Omega^G_{(\gamma,\gamma')}(X) \cong \sum_{[x]\in\mathscr{C}(G/H)/N(H)} \Omega^{K/H}_{\gamma^H_x|K/H}(X^H \times \pi_0(O'_H/O_H(V_H)) \times BSO_H(V_H)),$$

where the action on the argument is the one specified above. The sum extends over those $[x]$ *for which* $\gamma_H(x)$ *is the class of an actual bundle of the form* $G \times_H V_H$. *Notice that these are really free bordism groups because of the structure of* $\gamma^H_x|K/H$.

Proof. By classical arguments (see for example [CF]), any (γ, γ')-manifold M is G-cobordant to a normal tube around the submanifold $M^{(H)}$ consisting of points with isotropy conjugate to H. This in turn is determined by the $N(H)$-equivariant normal tube about M^H. This tube determines an actual $N(H)$-bundle over the H-fixed space M^H which has a virtual γ_H-orientation. There is an $N(H)/H$-space B that classifies such bundles (over general H-fixed $N(H)$-spaces), and we claim that

$$B \simeq \coprod_{[x]\in\mathscr{C}(G/H)/N(H)} \pi_0(O'_H/O_H(V_H)) \times BSO_H(V_H)$$

where x runs only through the objects such that $\gamma_H(x)$ is the class of an actual bundle.

First we say that B is the disjoint union of $N(H)/H$-spaces B_x, one for each class $[x] \in \mathscr{C}(G/H)/N(H)$ such that $\gamma_H(x)$ is the class of an actual bundle. Indeed, if X is an $N(H)/H$-space such that $X/N(H)$ is connected, and if X has over it a bundle of the sort that we are talking about, then the γ_H-orientation of the bundle will induce, among other things, a groupoid map $\pi(X) \to \mathscr{C}$. By the connectivity of X, the objects over G/H that are in the image of this functor must all be isomorphic, in fact they must be a class $[x] \in \mathscr{C}(G/H)/N(H)$. Further, since the bundle is assumed to be an actual bundle, it must be that $\gamma_H(x)$ is the class of an actual bundle. Therefore B must decompose as stated, where each B_x classifies actual $N(H)$-bundles over H-fixed spaces, equipped with $(\gamma_x)_H$-orientations.

Next we say that

$$B_x \simeq N(H) \times_K B'_x$$

as $N(H)/H$-spaces, where B'_x classifies actual K-bundles over H-fixed spaces, with $(\gamma_x)_H|K$-orientations. Again, consider an $N(H)/H$-space X with a bundle of the sort classified by B_x, and consider the map $\pi(X) \to \mathscr{C}_x$ induced by the $(\gamma_x)_H$-orientation of the bundle. From this map, we see that X must break up into a disjoint union of components corresponding to the cosets of $N(H)/K$, and so we can write $X = N(H) \times_K X_0$, where X_0 is the K-subset of points mapping into the object x in $\mathscr{C}_x$. Clearly then, having a $(\gamma_x)_H$-orientation of a bundle over X is equivalent to having the $(\gamma_x)_H|K$-orientation of the bundle restricted to X_0, and so we get the equivalence that we claim.

Finally, we say that

$$B'_x \simeq \pi_0(O'_H/O_H(V_H)) \times BSO_H(V_H)$$

as K/H-spaces, where the action is that described before the proposition. In fact, B'_x must be the pullback in the diagram

$$\begin{array}{ccc} B'_x & \longrightarrow & BSO'_H \\ \downarrow & & \downarrow \\ BO_H(V_H) & \longrightarrow & BO'_H. \end{array}$$

This is a diagram of K-spaces, where K acts on the bottom row via its maps into

$$hK\mathscr{V}\mathscr{B}(K \times_H V_H, K \times_H V_H) \quad \text{and} \quad hK\mathscr{B}(K \times_H V_H, K \times_H V_H)$$

(see the discussion before the statement of the proposition). To see this, consider any bundle ξ over a K-space X on which H acts trivially, such that ξ has a $(\gamma_x)_H|K$-orientation. Each fiber of ξ is then equipped with a homotopy class of virtual maps into V_H, which is the orientation of the fiber. Form three principal bundles associated with ξ: P_a will be the principal $O_H(V_H)$-bundle formed by taking the space of all H-isometries from V_H into the fibers of ξ. P_V will be the principal SO'_H-bundle formed by taking the space of all virtual H-isometries of V_H into the fibers of ξ preserving the virtual orientation. Finally, let P be the principal O'_H-bundle formed by taking the space of all virtual H-isometries of V_H into the fibers of ξ. These spaces are acted on by various groups: P_a is acted on by the group $K\mathscr{B}(K \times_H V_H, K \times_H V_H)$ (thinking of an H-map from V_H as the same as a K-map from $K \times_H V_H$), P has a similar action by $K\mathscr{V}\mathscr{B}(K \times_H V_H, K \times_H V_H)$, and P_V has an action by the group of orientation-preserving maps in $K\mathscr{V}\mathscr{B}(K \times_H V_H, K \times_H V_H)$. All of these groups have K/H as a quotient, and the projections to X are compatible with these actions. Further, there are maps $P_a \to P$ and $P_V \to P$ compatible with the actions in the obvious way. Now, there are universal principal bundles of the same sort over the spaces $BO_H(V_H), BSO'_H$, and BO'_H. Universality yields maps of principal bundles, covering compatible K-maps of X into the three spaces $BO_H(V_H), BSO'_H$, and BO'_H. Thus, from the bundle ξ we get a K-map from X into the pullback of these three spaces. Conversely, if X maps into the pullback, then we obtain, from the map into $BO_H(V_H)$, an actual bundle over X, which is equipped with a virtual $(\gamma_x)_H|K$-orientation, via the map into BSO'_H. Thus the pullback is the space B'_x needed to classify such bundles. Finally, it is easy to identify this pullback with the K-space $\pi_0(O'_H/O_H(V_H)) \times BSO_H(V_H)$ described before this proposition. □

We summarize what we have done in the following theorem:

THEOREM 6.4. *If G is a finite group, γ is an orientable G-groupoid representation, and $\varnothing = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$ is a sequence of adjacent families of subgroups of G with $\mathscr{F}_p - \mathscr{F}_{p-1} = (H_p)$ and with $\mathscr{F}_n$ the family associated with γ, then there is a spectral sequence converging to $\Omega^G_{\gamma+*}(X)$, with the E^2 term being*

$$E^2_{p,q} = \sum_{[x]\in\mathscr{C}(G/H)/N(H)} \Omega^{K/H}_{\gamma^H_x|K/H+q}(X^H \times \pi_0(O'_H/O_H(V_H)) \times BSO_H(V_H)),$$

where H denotes H_p, and as usual γ_x denotes the restriction of γ to the groupoid generated by x (the other notations are as in Proposition 6.3).

REMARK 6.5. The E^2 term is expressible as suitably structured nonequivariant bordism of the spaces

$$E(K/H) \times_{K/H} (X^H \times \pi_0(O'_H/O_H(V_H)) \times BSO_H(V_H))$$

(which we abbreviate henceforth as $E(K/H) \times_{K/H} (X^H \times B)$). The suitable structure is described as follows. In the summand corresponding to x, the K/H-groupoid representation $\gamma_x^H|K/H$ is determined by the homomorphism $K/H \to hK\mathscr{V}\mathscr{B}(K \times_H V^H, K \times_H V^H)$ splitting the projection onto K/H (where stabilization is now restricted to addition of trivial summands because we are taking fixed-points). However, $hK\mathscr{V}\mathscr{B}(K \times_H V^H, K \times_H V^H) \cong \pi_0 O \times K/H \cong \mathbb{Z}/2 \times K/H$, and so we are really talking about a map $K/H \to \mathbb{Z}/2$. When this homomorphism is trivial, the K/H-action on a $\gamma_x^H|K/H$-oriented manifold preserves nonequivariant orientation. By passage to orbit spaces, the corresponding summand becomes the nonequivariant oriented bordism group $\Omega_n(E(K/H) \times_{K/H} (X^H \times B))$, where $n = \dim \gamma_x^H$.

In the case of a nontrivial homomorphism, the corresponding orbit manifold need not be oriented, but nevertheless inherits a "pseudo-oriented" structure as follows. Let M be a K/H-free $\gamma_x^H|K/H$-oriented manifold. Then the classification of the action of K/H, together with the map $K/H \to \mathbb{Z}/2$, determines a map $\theta\colon M/K \to B\mathbb{Z}/2$. The orientation on M determines a map $M/K \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} BSO(n)$ over $B\mathbb{Z}/2$, where again $n = \dim \gamma_x^H$ ($\mathbb{Z}/2$ acts on $BSO(n)$ as the group of covering transformations of the double cover $BSO(n) \to BO(n)$). The space $E(K/H) \times_{K/H} (X^H \times B)$ also maps into $B\mathbb{Z}/2$ via the natural projection onto $B(K/H)$. Conversely, all of this data allows us to reconstruct the $\gamma_x^H|K/H$-oriented free manifold M over the space $X^H \times B$. Thus, we can interpret the relative term as oriented bordism in the category of (nonequivariant) spaces over $B\mathbb{Z}/2$, where an orientation of a manifold $N \to B\mathbb{Z}/2$ is a lifting of the map $N \to BO(n)$ classifying the tangent bundle, to a map $N \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} BSO(n)$ over $B\mathbb{Z}/2$.

Taking the colimit in Proposition 6.4 as V_H gets large gives us the corresponding result for stable bordism. Recall that SO'_H denotes the colimit of the groups $SO_H(W)$ as W runs through the representations of H having no trivial summands. As in the remarks before Proposition 6.3, a homomorphism $K/H \to hK\mathscr{V}\mathscr{B}(K \times_H V_H, K \times_H V_H)$ gives an action of K/H on the space BSO'_H.

THEOREM 6.6. *If G is a finite group, γ is an orientable G-groupoid representation, and $\emptyset = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$ is a sequence of adjacent* *families of subgroups of G with $\mathscr{F}_n$ the family associated with γ, then there is a spectral sequence converging to $\underline{\Omega}^G_{\gamma+*}(X)$, with the E^2 term being*

$$E^2_{p,q} = \sum_{[x]\in\mathscr{C}(G/H)/N(H)} \Omega^{K/H}_{\gamma^H_x|K/H+q}(X^H \times BS\mathcal{O}'_H),$$

where H denotes H_p, and as usual γ_x denotes the restriction of γ to the groupoid generated by x. □

7. $\mathbb{Z}$-modelled oriented G-bordism. Recall that we have universal virtual oriented groupoid representations $\mathbb{SO}_n$ and universal virtual weak groupoid representations $\mathbb{O}_n$, one for each integer n. Recall also that $\mathbb{O}_n \oplus \mathbb{R} \cong \mathbb{O}_{n+1}$ and $\mathbb{SO}_n \oplus \mathbb{R} \cong \mathbb{SO}_{n+1}$.

DEFINITION 7.1. Define $\Omega^G_n = \Omega^G_{\mathbb{SO}_n}$, and similarly for the stable and unoriented theories.

From the comments before the definition, these define integer-graded G-bordism theories. It follows from the fact that $\mathbb{O}_n$ is final in the category of weak virtual n-dimensional groupoid representations, that the unoriented theory we have just defined is the usual unoriented bordism theory. Moreover, from §6 we have:

COROLLARY 7.2. *If $\varnothing = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n = \mathscr{A}$ is a sequence of adjacent families of subgroups of G, with $\mathscr{A}$ denoting the family of all subgroups, and $\mathscr{F}_{p+1} - \mathscr{F}_p = (H_p)$, then there is a spectral sequence converging to $\Omega^G_*(X)$, with the E^2 term being*

$$E^2_{p,q} = \sum_{[x]\in\mathscr{C}(G/H)/N(H)} \Omega^{K/H}_{\gamma^H_x|K/H}(X^H \times \pi_0(\mathcal{O}'_H/O_H(V_H)) \times BSO_H(V_H)),$$

where H denotes $H_p, \gamma = \mathbb{SO}_q$, and the rest of the notation is as in Proposition 6.4. There are similar spectral sequences for the other theories. □

The spectral sequence that we get here in the unoriented case is similar to the one that Kosniowski obtains in **[K2]** by considering slice types.

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