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# UNITARY COBORDISM OF CLASSIFYING SPACES OF QUATERNION GROUPS

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**The main purpose of this article is to prove that the complex cobordism ring of classifying spaces of quaternion groups $\Gamma_k(|\Gamma_k| = 2^k)$ is a quotient of the graded ring $U^*(pt)[[X, Y, Z]]$ ($\dim X = \dim Y = 2, \dim = Z = 4$) by a graded ideal generated by six homogeneous formal power series.**

**0. Introduction.** Let $\Gamma_k$ be the generalized quaternion group. $\Gamma_k$ is generated by $u, v$, subject to the relations $u^t = v^2$, $uvu = v$, $t = 2^{k-2}$. In order to calculate $U^*(B\Gamma_k)$ we first consider the case $k = 3$, i.e. $\Gamma_3 = \Gamma$. We recall that $\Gamma = \{\pm 1, \pm i, \pm j, \pm k\}$ with the relations $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$. We shall define $A \in \tilde{U}^2(B\Gamma)$, $B \in \tilde{U}^2(B\Gamma)$, $D \in \tilde{U}^4(B\Gamma)$ as Euler classes of complex vector bundles over $B\Gamma$ corresponding to unitary irreducible representations of $\Gamma$. Let $\Lambda_*$ be the graded $U^*(pt)$-algebra $U^*(pt)[[X, Y, Z]]$ with $\dim X = \dim Y = 2$, $\dim Z = 4$, $\Omega_* = U^*(pt)[[Z]] \subset \Lambda_*$ and $U^*(pt)[[D]] = \{P(D), P \in \Omega_*\}$. Then by using the Atiyah-Hirzebruch spectral sequence we obtain the following results where $T(Z) \in \Omega_4$, $J(Z) \in \Omega_0$ are well defined formal power series.

THEOREM 2.18. (a) *As graded $U^*(pt)$-algebras we have:*

$$U^*(pt)[[D]] \simeq \Omega_*/(T(Z)).$$

(b) *As graded $U^*(pt)[[D]]$-modules we have: $U^*(B\Gamma) \simeq U^*(pt)[[D]] \oplus U^*(pt)[[D]] \cdot A \oplus U^*(pt)[[D]] \cdot B$ and $A$, $B$ have the same annihilator $(2 + J(D)) \cdot U^*(pt)[[D]]$.*

THEOREM 2.17. *The graded $U^*(pt)$-algebra $U^*(B\Gamma)$ is isomorphic to $\Lambda_*/I_*$ where $I_*$ is a graded ideal generated by six homogeneous formal power series.*

The method used for $\Gamma$ is extended to $\Gamma_k$, $k \geq 4$. As before we shall define $B_k \in \tilde{U}^2(B\Gamma_K)$, $C_k \in \tilde{U}^2(B\Gamma_k)$, $D_k \in \tilde{U}^4(B\Gamma_k)$ as Euler classes of complex vector bundles over $B\Gamma_k$ corresponding to unitary irreducible representations of $\Gamma_k$ and elements $G'(Z) \in \Omega_2$,

$T_k(Z) \in \Omega_4$. If $B'_k = B_k + G'_k(D_k)$, $C'_k = C_k + G'_k(D_k)$ then we get:

THEOREM 3.14. (a) $U^*(pt)[[D_k]] \simeq \Omega_*/(T_k)$ *as graded* $U^*(pt)$*-algebras.*

(b) *As graded* $U^*(pt)[[D_k]]$*-modules we have*:

$$U^*(B\Gamma_k) \simeq U^*(pt)[[D_k]] \oplus U^*(pt)[[D_k]] \cdot B'_k \oplus U^*(pt)[[D_k]] \cdot C'_k$$

*and* $B'_k$, $C'_k$ *have the same annihilator* $(2 + J(D_k)) \cdot U^*(pt)[[D_k]]$.

THEOREM 3.12. *The graded* $U^*(pt)$*-algebra* $U^*(B\Gamma_k)$ *is isomorphic to* $\Lambda_*/\tilde{I}_*$ *where* $\tilde{I}_*$ *is a graded ideal of* $\Lambda_*$ *generated by six homogeneous formal power series.*

In the appendix, part A, we give a new method of calculating $U^*(B\mathbb{Z}_m)$. Let $\Lambda'_*$ be the graded algebra $U^*(pt)[[Z]]$, $\dim Z = 2$.

THEOREM A.1. $U^*(B\mathbb{Z}_m) \simeq \Lambda'_*/([m](Z))$ *as graded* $U^*(pt)$*-algebras.*

In part B we show that:

THEOREM B.2.

$$U^{2i+2}(BSU(n)) \simeq U^{2i+2}(BU(n))/e(\Lambda^n\gamma(n)) \cdot U^{2i}(BU(n))$$

*and* $U^{2i+1}(BSU(n)) = 0$, $i \in \mathbb{Z}$.

*In this theorem* $e(\Lambda^n\gamma(n))$ *is the Euler class of* $\Lambda^n\gamma(n)$ *where* $\gamma(n)$ *denotes the universal bundle over* $BU(n)$.

In part C we calculate $H^*(B\Gamma_k)$, $k \geq 4$.

THEOREM C. *If* $k \geq 4$ *then we have* $H^*(B\Gamma_k) = \mathbb{Z}[x_k, y_k, z_k]$ *with* $\dim x_k = \dim y_k = 2$, $\dim z_k = 4$, *subject to the relations*:

$$2x_k = 2y_k = x_k y_k = 2^k z_k = 0, \quad x_k^2 = y_k^2 = 2^{k-1} z_k.$$

Theorem C is certainly known to workers in the field.

The layout is as follows:

I Preliminaries and notations.
II Calculation of $U^*(B\Gamma)$.
III Calculation of $U^*(B\Gamma_k)$, $k \geq 4$.
IV Appendix.

In the course of the computations we have determined the leading coefficients of some formal power series with the purpose of using them in a subsequent paper where the bordism groups $\tilde{U}_*(B\Gamma_k)$ are calculated.

We shall use the same notation for unitary irreducible representations of $\Gamma_k$ and corresponding complex vector bundles over $B\Gamma_k$. The notation $\gamma(n)$ will be used for the universal complex vector bundle over $BU(n)$. The notation $\mathbb{Z}$ will be for the ring of integers and $\mathbb{C}$ for the complex number field.

The results of this paper have been obtained in 1983 under the supervision of Dr. L. Hodgkin, University of London. I thank him sincerely for having proposed the subject, for his advice and encouragement. I would like to express my deep thanks to the referee who made many useful suggestions; they helped to improve the exposition of this paper and the statement of some results, particularly Theorems 2.18 and 3.14.

**I. Preliminaries and notations.** 1. Let $X$ be a CW-complex; we define a filtration on $U^n(X)$ by the subgroups

$$J^{p,q} = \operatorname{Ker}(i^*\colon U^n(X) \to U^n(X_{p-1})),$$

$X_p$ being the $p$-skeleton of $X$, $i\colon X_{p-1} \subset X$, $p+q=n$; $U^n(X)$ is a topological group, the subgroups $J^{p,q}$ being a fundamental system of neighbourhoods of 0; we denote this topology by $T$. If the $U^*$-Atiyah-Hirzebruch spectral sequence (denoted by $U^*$-AHSS) for $X$ collapses then $T$ is complete and Hausdorff (see [3]). The edge homomorphism $\mu\colon U^n(X) - H^n(X)$ is defined by $\mu = 0$ if $n < 0$ and if $n \geq 0$ it is the projection $U^n(X) = J^{0,n} = J^{n,0} \to J^{n,0}/J^{n+1,-1} = E_\infty^{n,0} \subset E_2^{n,0} = H^n(X)$. By easy arguments involving spectral sequences we have the following basic result:

THEOREM 1.1. *Let $X$ be a $CW$-complex such that:*

(a) *The $U^*$-AHSS for $X$ collapses.*

(b) *For each $n \geq 0$ there are elements $a_{in}$ generating the $\mathbb{Z}$-module $H^n(X)$.*

*Then for each $n \geq 0$ there are elements $A_{in} \in U^n(X)$ such that:*

(a) $\mu(A_{in}) = a_{in}$.

(b) *If $E$ denotes the $U^*(pt)$-submodule of $U^*(X)$ generated by the system $(A_{in})$ and if $E_n$ is the $n$-component of $E$ then $\overline{E}_n = U^n(X)$, $\overline{E}_n$ being the closure of $E_n$ for $T$.*

*Moreover* (b) *is valid of we take any system* $(A'_{in})$, $A'_{in} \in U^n(X)$ *such that* $\mu(A'_{in}) = a_{in}$ *for each* $(i,n)$. □

(See Theorem 2.5 for a proof of this result in a special case.)

2. Let $X$ be a skeleton-finite CW-complex, which is the case we are interested in. There is a ring spectra map $f\colon MU \to H$ (see [1]); by naturality of AHSS the map $f^{\#}(X)\colon U^*(X) \to H^*(X)$ induced by $f$ is identical to the edge-homomorphism described above. Let $\xi$ be a complex vector bundle over $X$ of dimension $n$; the Conner-Floyd characteristic classes of $\xi$ will be denoted by $cf_i(\xi)$; the Euler class $e(\xi)$ of $\xi$ for $MU$ is $cf_n(\xi)$ and the Euler class $e_1(\xi)$ for $H$ is the Chern class $c_n(\xi)$. As $f^{\#}(X)$ maps Euler classes on Euler classes we have $\mu(e(\xi)) = e_1(\xi)$ (see [7]).

3. Consider the formal power series ring $E_* = U^*(pt)[[c_1, c_2, \dots, c_r]]$ graded by taking $\dim c_1 = n_1 > 0, \dots, \dim c_r = n_r > 0$. Given $P(c_1, \dots, c_r) \in E_n$ with $P \neq 0$,

$$P = \sum a_u \cdot c_1^{u_1} \cdots c_r^{u_r}, \quad u = (u_1, \dots, u_r),$$

we define $\nu(P) = \{\inf(n_1u_1 + \cdots + n_ru_r), a_u \neq 0\}$ and $\nu(0) = +\infty$. Let $J_p$ be $\{P \in E_n | \nu(P) \geq p\}$; we have $E_n = J_0 \supset J_1 \supset \cdots$, and since $\bigcap_{p\geq 0} J_p = 0$, $E_n = \varprojlim E_n/J_p$, it follows that $E_n$ is complete and Hausdorff for the topology defined by the filtration $(J_p)$.

Suppose that $B$ is a CW complex such that the associated $U^*$-AHSS collapses; if $A_i \in U^{n_i}(B)$, $i = 1, 2, \dots, r$, then there is a unique continuous homomorphism $\psi\colon E_* \to U^*(B)$ such that $\psi(c_i) = A_i$, $i = 1, 2, \dots, r$.

Now in a different situation consider the case where $B_1$ is a CW-complex such that $U^*(B_1) = E_*$. There are two topologies on $U^*(B_1)$ defined respectively by the filtration $(J_p)$ on $E_*$ and by the filtration $(J_1^{p,q})$ deduced from the $U^*$-AHSS for $B_1$. If $B$ is a CW-complex such that the $U^*$-AHSS for $B$ collapses, $(J^{p,q})$ the corresponding filtration on $U^*(B)$ (see §I) and $g$ a continuous map: $B \to B_1$ then from $J_p \subset J_1^{p,q}$, $g^*(J_1^{p,q}) \subset J^{p,q}$ it follows that $g^*\colon E_n \to U^n(B)$ is continuous for the topologies defined by $\nu$ on $E_n$ and $(J^{p,q})$ on $U^*(B)$. As a consequence if $(P_m)$ is a sequence of polynomials such that $(P_m) \to P$ in $E_n$ and if $g^*(c_i) = A_i$ then $P_m(A_1, \dots, A_r) \to g^*(P)$ in $U^*(B)$; so if $P = \sum a_u c_1^{u_1} \cdots c_r^{u_r} \in E_n$ we can write $g^*(P) = \sum a_u A_1^{u_1} \cdots A_r^{u_r}$.

In the sequel we shall also be concerned with $\Lambda_* = U^*(pt)[[X, Y, Z]]$, $\dim X = \dim Y = 2$, $\dim Z = 4$; $\Lambda_*$ has the topology defined by $\nu$.

The following assertions are clear:

(a) In $\Lambda_{2n}$: $(R_p) \to 0$ iff $\nu(R_p) \to \infty$.

(b) If $P(X, Y, Z) \in \Lambda_{2m+2n}$, $Q(X, Y, Z) \in \Lambda_{2n}$ and $(R_p)$ a sequence in $\Lambda_{2m}$ such that $R_p \to R$ and $\nu(P - R_p Q) \to \infty$ then $RQ = P$.

(c) If $\nu(R_p) \to \infty$ then the sequence $(M_p)$ defined by $M_p = R_0 + \cdots + R_p$ converges to a unique limit denoted by $\sum_{p\geq 0} R_p$.

In Sections II and III we shall define three elements $A_k \in \tilde{U}^2(B\Gamma_k)$, $B_k \in \tilde{U}^2(B\Gamma_k)$, $D_k \in \tilde{U}^4(B\Gamma_k)$; as the $U^*$-AHSS for $B\Gamma_k$ collapses there is a unique continuous homomorphism $\varphi$ of graded $U^*(pt)$-algebras: $\Lambda_* \to U^*(B\Gamma_k)$ such that $\varphi(X) = A_k$, $\varphi(Y) = B_k$, $\varphi(Z) = D_k$.

The next well known result will be useful:

PROPOSITION 1.2. *Suppose $X$ a $CW$-complex such that $H^*(X) = \mathbb{Z}[a]$. Then there is an element $A \in U^*(X)$ such that $\mu(A) = a$ and $U^*(X) = H^*(X)\hat{\otimes}U^*(pt) = U^*(pt)[[A]]$. Moreover for any $A' \in U^*(X)$ such that $\mu(A') = a$ we have $U^*(X) = U^*(pt)[[A']]$.* □

**II. Computation of $U^*(B\Gamma)$.** We recall that the quaternion group $\Gamma$ consists of $\{1, \pm i, \pm j, \pm k\}$ subject to the relations $ij = k$, $jk = i$, $ki = j$, $i^2 = k^2 = -1$. The irreducible unitary representations of $\Gamma$ are $1\colon i \to 1$, $j \to 1$, $\xi_i\colon i \to 1$, $j \to -1$, $\xi_j\colon i \to -1$, $j \to 1$, $\xi_k\colon i \to -1$, $j \to -1$, $\eta\colon i \to \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $j \to \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$; the character table of $\Gamma$ is:

(Conjugacy classes)

| | $1$ | $-1$ | $\pm i$ | $\pm j$ | $\pm k$ |
|---|---|---|---|---|---|
| $1$ | 1 | 1 | 1 | 1 | 1 |
| $\xi_i$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\xi_j$ | 1 | 1 | $-1$ | 1 | $-1$ |
| $\xi_k$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\eta$ | 2 | $-2$ | 0 | 0 | 0 |

We have the following relations in the representation ring $R(\Gamma)$:

$$\xi_i^2 = \xi_j^2 = \xi_k^2 = 1, \quad \xi_i \cdot \xi_j = \xi_k, \quad \xi_j \cdot \xi_k = \xi_i, \quad \xi_k \xi_i = \xi_j,$$
$$\eta \cdot \xi_i = \eta \cdot \xi_j = \eta, \quad \eta^2 = 1 + \xi_i + \xi_j + \xi_k \qquad \text{(see [6], [2])}.$$

We have $H^0(B\Gamma) = \mathbb{Z}$, $H^{4n}(B\Gamma) = \mathbb{Z}_8$, $n \geq 1$, $H^{4n+2}(B\Gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $n \geq 0$, $H^{2n+1}(B\Gamma) = 0$. Moreover if $d$ is a generator of $H^4(B\Gamma)$ and if $a$, $b$ are generators of $H^2(B\Gamma)$ then $d^n$ is a generator of $H^{4n}(B\Gamma)$, $n \geq 1$, and $ad^n$, $bd^n$ are generators of $H^{4n+2}(B\Gamma)$, $n \geq 0$ (see **[5]**).

Since $H^m(B\Gamma) = 0$, $m$ odd we have:

PROPOSITION 2.1. *The $U^*$-AHSS for $B\Gamma$ collapses.* □

There are four important complex vector-bundles $\xi_i, \xi_j, \xi_k \colon E\Gamma \times_\Gamma \mathbb{C} \to B\Gamma$ and $\eta \colon E\Gamma \times_\Gamma \mathbb{C}^2 \to B\Gamma$ where the actions of $\Gamma$ on $\mathbb{C}$ and $\mathbb{C}^2$ are induced by the representations $\xi_i$, $\xi_j$, $\xi_k$ and $\eta$. We have a canonical inclusion $q \colon \mathbb{Z}_2 \subset \Gamma$ obtained by identifying $\{1, i^2\}$ with $\mathbb{Z}_2$; let $\rho$ be the unitary representation of $\mathbb{Z}_2$ given by $\rho(1) = 1$, $\rho(i^2) = -1$; the restriction map: $R(\Gamma) \to R(\mathbb{Z}_2)$ sends $\xi_i$, $\xi_j$, $\xi_k$ to 1 and $\eta$ to $2\rho$; so:

PROPOSITION 2.2. *$(Bq)^*(\xi_h), h = i, j, k$, are trivial and $(Bq)^*(\eta) = 2\rho$.* □

1. *Chern Classes of $\xi_i$, $\xi_j$, $\eta$.* The canonical isomorphism

$$\mathrm{Hom}(\Gamma, U(1)) \to H^2(\Gamma)$$

is given by $\delta \to c_1(g(\delta))$ where $g$ denotes the canonical map: $R(\Gamma) \to K^0(B\Gamma)$ and $c_1$ the first Chern class (Sec. **[2]**). Since $\mathrm{Hom}(\Gamma, U(1)) = \{1, \xi_i, \xi_j, \xi_k\}$ and $H^2(B\Gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ we have:

PROPOSITION 2.3. *$H^2(B\Gamma)$ is generated by $\{c_1(\xi_i), c_1(\xi_j)\}$.* □

Now we consider the topological group $\mathrm{Sp}(1)$ of quaternions of absolute value 1; $\mathrm{Sp}(1)$ is homeomorphic to $S^3$ and $H^*(BS^3) = \mathbb{Z}[u]$, $\dim u = 4$, $u$ being the first symplectic Pontrjagin class of the universal $\mathrm{Sp}(1)$-vector bundle $\theta$. If we consider $\theta$ as a $U(2)$-vector bundle, then $u = c_2(\theta)$ (see **[12]**, page 179). Let $p \colon \Gamma \subset \mathrm{Sp}(1) = S^3$ be the natural inclusion; then it is easily seen that $(Bp)^*(\theta) = \eta$, $\theta$ being regarded as a $U(2)$-vector bundle.

PROPOSITION 2.4. *We have $c_1(\eta) = 0$ and $H^4(B\Gamma)$ is generated by $c_2(\eta)$.*

*Proof.* Since $\det \eta = 1$ we have $c_1(\eta) = 0$. From the transgression exact sequence of the fibration: $S^3/\Gamma \to B\Gamma \xrightarrow{Bp} BS^3$ we get the exact sequence: $H^4(BS^3) \overset{(Bp)^*}{\to} H^4(B\Gamma) \to H^4(S^3/\Gamma) = 0$ and the result follows (see **[11]**, page 519). □

From 2.3, 2.4 we may take the Euler classes $e_1(\eta) = d$ as a generator of $H^4(B\Gamma)$ and $\{a = e_1(\xi_i), b = e_1(\xi_j)\}$ as a system of generators of $H^2(B\Gamma)$. Moreover $e_1(n \cdot \eta) = e_1(\eta)^n = d^n$ and $\{e_1(\xi_i + n \cdot \eta) = ad^n$, $e_1(\xi_j + n\cdot\eta) = bd^n\}$ are generators of $H^{4n}(B\Gamma), n \geq 1$ and $H^{4n+2}(B\Gamma)$, $n \geq 0$, respectively.

2. *Computation of* $U^*(B\Gamma)$. Let $A$, $B$, $D$ be the Euler classes for $MU$ of $\xi_i$, $\xi_j$, $\eta$: $e(\xi_i) = A \in \tilde{U}^2(B\Gamma)$, $e(\xi_j) = B \in \tilde{U}^2(B\Gamma)$, $e(\eta) = D \in \tilde{U}^4(B\Gamma)$. We recall that $\Lambda_* = U^*(pt)[[X, Y, Z]]$ is graded by taking $\dim X = \dim Y = 2$, $\dim Z = 4$; there is a unique continuous homomorphism $\varphi\colon \Lambda_* \to U^*(B\Gamma)$ of graded $U^*(pt)$-algebras such that $\varphi(X) = A$, $\varphi(Y) = B$, $\varphi(Z) = D$. In particular if $P(Z) = \alpha_0 + \alpha_1 Z + \cdots + \alpha_i Z^i + \cdots \in \Lambda_{2n}$ then $\varphi(P) = P(D) = \mathrm{Lim}_{n\to\infty}(\alpha_0 + \cdots + \alpha_n \cdot D^n)$ in $U^{2n}(B\Gamma)$. If $U^*(pt)[[D]] = \{R(D), R(Z) \in \Omega_*\}$, then $U^*(pt)[[D]]$ is a sub-$U^*(pt)$-algebra of $U^*(B\Gamma)$.

THEOREM 2.5. *$U^*(B\Gamma)$ is concentrated in even dimensions and as a $U^*(pt)[[D]]$-module $U^*(B\Gamma)$ is generated by* 1, $A$, $B$.

*Proof*. We have $U^{2n+1}(B\Gamma) = 0$ because $J^{p,q} = J^{p+1,q-1}$ if $p + q = 2n + 1$ and then $U^{2n+1}(B\Gamma) = J^{0,2n+1} = \bigcap_{p+q=2n+1} J^{p,q} = 0$ (see Section I).

*Suppose* $2n = 4m + 2 > 0$. If $x \in U^{4m+2}(B\Gamma) = J^{0,4m+2} = J^{4m+2,0}$ then $\mu(x) = \alpha_m ad^m + \beta_m bd^m = \mu(\alpha_m AD^m + \beta_m BD^m)$, $\alpha_m \in U^0(pt) = \mathbb{Z}$, $\beta_m \in U^0(pt) = \mathbb{Z}$. It follows that $\mu(x - (\alpha_m AD^m + \beta_m BD^m)) = 0$ and $x_1 = x - (\alpha_m AD^m + \beta_m BD^m) \in J^{4m+3,-1} = J^{4m+4,-2}$. Let $s_1$ be the quotient map: $J^{4m+4,-2} \to J^{4m+4,-2}/J^{4m+5,-3} = H^{4m+4}(B\Gamma, U^{-2}(pt)) = U^{-2}(pt) \otimes H^{4m+4}(B\Gamma)$. Then $s_1(x_1) = \gamma_{m+1} \otimes d^{m+1}$, $\gamma_{m+1} \in U^{-2}(pt)$. From the following commutative diagram where $\chi$ is induced by the $U^*(pt)$-module-structure:

$$\begin{array}{ccc} U^{-2}(pt) \otimes U^{4m+4}(B\Gamma) = U^{-2}(pt) \otimes J^{4m+4,0} & \xrightarrow{\chi} & J^{4m+4,-2} \\ {\scriptstyle 1\otimes\mu} \downarrow & & \downarrow {\scriptstyle s_1} \\ U^{-2}(pt) \otimes H^{4m+4}(B\Gamma) & \xrightarrow{\sim} & H^{4m+4}(B\Gamma, U^{-2}(pt)) \end{array}$$

it follows that $s_1(x_1) = s_1(\gamma_{m+1} D^{m+1})$ and then $s_1(x_1 - \gamma_{m+1} D^{m+1}) = 0$; so $(x_1 - \gamma_{m+1})D^{m+1} \in J^{4m+5-3} = J^{4(m+1)+2,-4}$. We have $x_2 = x_1 - \gamma_{m+1} D^{m+1} = x - (A \cdot \alpha_m D^m + B \cdot \beta_m D^m + \gamma_{m+1} D^{m+1}) \in J^{4(m+1)+2,-4}$.

By using again the products $\chi$ we see that after a finite number of steps there are three polynomials in $Z$:

$$\begin{aligned} P_q(Z) &= \alpha_m Z^m + \alpha_{m+1} Z^{m+1} + \cdots + \alpha_{m+q-1} Z^{m+q-1}, \\ Q_q(Z) &= \beta_m Z^m + \beta_{m+1} Z^{m+1} + \cdots + \beta_{m+q-1} Z^{m+q-1}, \\ R_q(Z) &= \gamma_{m+1} Z^{m+1} + \cdots + \gamma_{m+q} Z^{m+q}, \quad \text{with} \\ &\deg P_q = m + (q-1), \quad \deg Q_q = m + (q-1), \\ &\deg R_q = m + q \quad \text{such that} \end{aligned}$$

$$(1)\ x - (A \cdot P_q(D) + B Q_q(D) + R_q(D)) \in J^{4(m+q)+2,-4q}.$$

Furthermore

$$\begin{aligned} P_{q+1}(Z) &= P_q(Z) + \alpha_{m+q} Z^{m+q}, \\ Q_{q+1}(Z) &= Q_q(Z) + \beta_{m+q} Z^{m+q}, \\ R_{q+1}(Z) &= R_q(Z) + \gamma_{m+q+1} Z^{m+q+1}. \end{aligned}$$

If

$$\begin{aligned} P(Z) &= \sum_{i=m}^{\infty} \alpha_i Z^i \in \Lambda_{4m} \\ Q(Z) &= \sum_{i=m}^{\infty} \beta_i Z^i \in \Lambda_{4m} \\ R(Z) &= \sum_{i=m+1}^{\infty} \gamma_i Z^i \in \Lambda_{4m+2} \end{aligned}$$

then by using (1) and Section I we have $x = AP(D) + BQ(D) + R(D)$.

The cases $2n = 4m + 2 < 0$ and $2n = 4m$ are similar. □

The next two propositions will be used later on.

PROPOSITION 2.6. *If*

$$H(Z) = \sum_{i=0} \alpha_i Z^i \in \Lambda_{2n}$$

*is such that* $H(D) = 0$, *then* $\alpha_0 = 0$ *and if* $\alpha_p$ *is the leading coefficient, we have* $\alpha_p \in 8 \cdot U^*(pt)$.

*Proof*. Since $D \in \tilde{U}^*(B\Gamma)$ we have

$$\sum_{i=1}^{\infty} \alpha_i D^i = D \left( \sum_{i=1}^{\infty} \alpha_i D^{i-1} \right) \in \widetilde{U^*}(B\Gamma);$$

then $\alpha_0 \cdot 1 \in \tilde{U}^*(B\Gamma) \cap U^*(pt) = \{0\}$ and $\alpha_0 \cdot 1 = 0$. If $i$ denotes the inclusion $\{*\} \subset B\Gamma$ we have $i^*(\alpha_0 \cdot 1) = \alpha_0 = 0$. Then $H(Z) =$

$\alpha_p Z^p + \cdots + \alpha_m Z^m + \cdots$, $\alpha_p \neq 0$, $p \geq 1$. From $\alpha_q D^q \in J^{4q,2n-4q} \subset J^{4p+4,2n-(4p+4)}$, $q \geq p+1$, it follows that $t_q = \alpha_{p+1} D^{p+2} + \cdots + \alpha_q D^q \in J^{4p+4,2n-(4p+4)}$, $q \geq p+1$. Since $J^{4p+4,2n-(4p+4)}$ is closed for the topology $T$ of $U^{2n}(B\Gamma)$ we have

$$\sum_{i=p+1}^{\infty} \alpha_i D^i \in J^{4p+4,2n-(4p+4)} \subset J^{4p+1,2n-(4p+1)}.$$

Let $s$ be the quotient map

$$\begin{aligned} J^{4p,2n-4p} &\to J^{4p,2n-4p}/J^{4p+1,2n-(4p+1)} \\ &= H^{4p}(B\Gamma, U^{2n-4p}(pt)) = H^{4p}(B\Gamma) \otimes U^{2n-4p}(pt) \\ &= \mathbb{Z}_8 \otimes U^{2n-4p}(pt) = U^{2n-4p}(pt)/8 \cdot U^{2n-4p}(pt). \end{aligned}$$

Then:

$$0 = s(H(D)) = s(\alpha_p D^p) + s\left(\sum_{i=p+1} \alpha_i D^i\right) = s(\alpha_p D^p) = \alpha_p \otimes d^p;$$

since $d^p$ is a generator of $H^{4p}(B\Gamma)$ we have $\alpha_p \in 8U^{2n-4}(pt)$. $\square$

Let $F$ be the formal group law and $[2](Y) = F(Y, Y)$; if $\rho$ is the nontrivial unitary irreducible representation for $\mathbb{Z}_2$ then we get (see **[9]**):

PROPOSITION 2.7. *$U^*(B\mathbb{Z}_2) = U^*(pt)[[Y]]/([2](Y))$ and the image of $Y$ by the quotient map: $U^*(pt)[[Y]] \to U^*(B\mathbb{Z}_2)$ is the Euler class $e(\rho)$.* $\square$

We have adopted the following graduation in 2.7: if

$$F(X, Y) = X + Y + a_{11}XY + \sum_{i\geq 1, j\geq 1} a_{ij} X^i Y^j,$$

then $|a_{ij}| = 2(1-i-j)$, $|X| = |Y| = 2$; so $F(X, Y) \in \Lambda_2$. We shall often make use of the coefficient $a_{11}$. We know that there is a unique formal power series $[-1](Y) \in U^*(pt)[[Y]](\subset \Lambda_2)$ such that: $F(Y, [-1](Y)) = 0$.

PROPOSITION 2.8. *There is $P_0(Z) \in \Omega_2$, $P_0(Z) = b_1 Z + \sum_{i\geq 1} b_i Z^i$ such that $cf_1(\eta) = P_0(D)$. The coefficients $b_i$, $i \geq 1$, are determined by the relation $\sum_{i\geq 1} b_i (Y \cdot [-1](Y))^i = Y + [-1]Y$; in particular $b_1 = -a_{11}$.*

*Proof*. We have seen that if $\theta$ is the universal $Sp(1)$-bundle over $Sp(1) = BS^3$ considered as a $U(2)$-vector bundle then $\eta = (Bp)^*(\theta)$,

$p\colon \Gamma \subset Sp(1)$. As $H^*(BS^3) = \mathbb{Z}[u]$, $u = c_2(\theta)$, we have $U^*(BS^3) = U^*(pt)[[V]]$, $V = e(\theta)$, the Euler class of $\theta$ for $MU$. Hence there is $P_0(Z) = \sum_{i\geq 1} b_i Z^i \in \Omega_2$ such that $P_0(V) = cf_1(\theta)$; it follows that

$$cf_1(\eta) = (Bp)^*(cf_1(\theta)) = (Bp)^* \left(\sum_{i\geq 1} b_i V^i\right) = \sum_{i\geq 1} b_i D^i = P_0(D).$$

The relation $\sum_{i\geq 1} b_i(Y \cdot [-1]Y)^i = Y + [-1](Y)$ is proved in the Appendix part B and gives $b_1 = -a_{11}$. $\square$

We recall that $A = cf_1(\xi_i) \in \tilde{U}^2(B\Gamma)$, $B = cf_1(\xi_j) \in \tilde{U}^2(B\Gamma)$, $D = cf_2(\eta) \in \tilde{U}^4(B\Gamma)$; let $C \in \tilde{U}^2(B\Gamma)$ be $cf_1(\xi_k)$.

PROPOSITION 2.9. (a) *There are $P(Z) \in \Omega_2$, $Q(Z) \in \Omega_4$, $P(Z) = -4a_{11}Z + \sum_{i\geq 2} \alpha_i Z^i$, $Q(Z) = 4Z + \sum_{i\geq 2} \beta_i Z^r$, $\beta_2 \notin 2U^*(pt)$, such that $cf_1(\eta^2) = P(D) = A + B + C$, $cf_2(\eta^2) = Q(D) = AB + BC + CA$.*

(b) $cf_3(\eta^2) = ABC = 0$,

(c) $A^3 = -AQ(D) + A^2P(D)$, $B^3 = -BQ(D) + B^2P(D)$.

*Proof*. (a) Let $g\colon B\Gamma \to BU(2)$ be a map classifying $\eta$; then $\eta^2$ is classified by the composite: $B\Gamma \xrightarrow{\Delta} B\Gamma \times B\Gamma \xrightarrow{g\times g} BU(2) \times BU(2) \xrightarrow{m} BU(4)$, where $m$ is a map classifying $\gamma(2) \otimes \gamma(2)$ and $\Delta$ the diagonal map. We have $U^*(BU(2) \times BU(2)) = U^*(pt)[[c_1, c_2, c_1', c_2']]$, $c_1$, $c_2$, $c_1'$, $c_2'$ being respectively the images of $cf_1(\gamma(2)) \otimes 1$, $cf_2(\gamma(2)) \otimes 1$, $1 \otimes cf_1(\gamma(2))$, $1 \otimes cf_2(\gamma(2))$ by the canonical map: $U^*(BU(2)) \otimes U^*(BU(2)) \xrightarrow{X} U^*(BU(2) \times BU(2))$. Since the following diagram commutes:

$$\begin{tikzcd} U^*(BU(4)) \arrow[r, "m^*"] & U^*(BU(2) \times BU(2)) \arrow[r, "(g\times g)^*"] & U^*(B\Gamma \times B\Gamma) \arrow[r, "\Delta^*"] & U^*(B\Gamma) \\ & U^*(BU(2)) \otimes B^*(BU(2)) \arrow[u, "X"] \arrow[r, "X"] & U^*(B\Gamma) \otimes U^*(B\Gamma) \arrow[u] \arrow[ur, "\cup"'] & \end{tikzcd}$$

we must substitute $cf_1(\eta)$ for $c_1$, $c_1'$, $cf_2(\eta)$ for $c_2$, $c_2'$ in $m^*(cf_1(\gamma(4)))$, $m^*(cf_2(\gamma(4)))$, $m^*(cf_3(\gamma(4)))$ in order to calculate $cf_1(\eta^2)$, $cf_2(\eta^2)$, $cf_3(\eta^2)$ (see Sec. I).

We have $m^*(cf_1\gamma(4)) = \sum a_{(u,v)} c_1^{u_1} c_2^{u_2} c_1'^{v_1} c_2'^{v_2}$, $u = (u_1, u_2)$, $v = (v_1, v_2)$, $u_1 \geq 0$, $u_2 \geq 0$, $v_1 \geq 0$, $v_2 \geq 0$. It is important to calculate $a_{(u,v)}$ when $u_1 = u_2 = 0$, or $v_1 = v_2 = 0$.

*Suppose* $u_1 = u_2 = 0$. We denote by 0 the pair $(0, 0)$. Then the coefficients $a_{(0,v)}$ are given by $i^* \circ m^*(cf_1(\gamma(4)))$, $i$ being the natural inclusion:

$$\{*\} \times BU(2) \xrightarrow{i} BU(2) \times BU(2).$$

Since $i^* \circ m^*(\gamma(4)) = \gamma(2) + \gamma(2)$ we have $i^* \circ m^*(cf_1(\gamma(4))) = 2c_1'$. Similarly $a_{(u,0)} = 2c_1$. Hence

$$m^*(cf_1(\gamma(4))) = 2(c_1 + c_1') + \sum_{\substack{\|u\|\geq 1\\ \|v\|\geq 1}} a_{(u,v)} c_1^{u_1} c_2^{u_2} {c_1'}^{v_1} {c_2'}^{v_2}$$

where $\|u\| = u_1 + u_2$, $\|v\| = v_1 + v_2$.

We recall that $cf_1(\eta) = P_0(D)$, $P_0(Z) \in \Omega_2$, $\nu'(P_0) = 1$, $\nu' = \frac{1}{4}\nu$ (see Sec. I). Consider

$$\begin{aligned} P(Z) &= 2(P_0(Z) + P_0(Z)) + \sum_{\substack{\|u\|\geq 1\\ \|v\|\geq 1}} a_{(u,v)} P_0^{u_1+v_1}(Z) Z^{u_2+v_2} \\ &= 4b_1 Z + \sum_{i\geq 2} \alpha_i Z^i, \end{aligned}$$

$b_1$ being the first coefficient $\neq 0$ of $P_0(Z)$ because $u_1 + v_1 + u_2 + v_2 \geq 2$ when $\|u\| \geq 1$, $\|v\| \geq 1$. Hence $cf_1(\eta^2) = P(D)$. We remark that $P(Z) \in \Omega_2$.

There are unique elements $b_{(u,v)} \in U^*(pt)$ such that $m^*(cf_2(\gamma(4))) = \sum b_{(u,v)} c_1^{u_1} c_2^{u_2} {c_1'}^{v_1} {c_2'}^{v_2}$. Then the coefficients $b_{(u,0)}$ and $b_{(0,v)}$ are given by $cf_2(\gamma(2) + \gamma(2)) = cf_1^2(\gamma(2)) + 2cf_2(\gamma(2))$. Hence

$$m^*(cf_2(\gamma(4)) = c_1^2 + c_1^2 + 2(c_2 + c_2^1) + \sum_{\|u\|\geq 1, \|v\|\geq 1} b_{u,v} c_1^{u_1} c_2^{u_2} {c_1'}^{v_1} {c_2'}^{v_2}.$$

Consider

$$\begin{aligned} Q(Z) &= 4Z + 2P_0^2(Z) + \sum_{\|u\|\geq 1, \|v\|\geq 1} b_{(u,v)} P_0^{u_1+v_1}(Z) Z^{u_2+v_2} \\ &= 4Z + \sum_{i\geq 2} \beta_i Z^i. \end{aligned}$$

Then $cf_2(\eta^2) = Q(D)$, $Q(Z) \in \Omega_4$.

Let $q$ be the inclusion $\mathbb{Z}_2 \subset \Gamma$; since $(Bq)^*(\xi_h)$, $h = i, j, k$, are trivial by 2.2 we have $(Bq)^*(A) = (Bq)^*(B) = (Bq)^*(C) = 0$ and since $Q(D) = cf_2(\eta^2) = AB + BC + CA$ we have $(Bq)^*(Q(D)) = 0$. It follows by 2.7 that $(Bq)^*(D) = d^2$, $d$ being the image of $Y$ by the quotient map:

$$U^*(pt)[[Y]] \to U^*(pt)[[Y]]/([2](Y)).$$

Thus:

$$\begin{aligned} 4Y^2 + \sum_{i\geq 2} \beta_i \cdot Y^{2i} &= [2](Y) \cdot G(Y) \\ &= (2Y + a_{11}Y^2 + a_3 Y^3 + \cdots)(\varepsilon_0 Y + \varepsilon_1 Y^2 + \varepsilon_2 Y^3 + \cdots) \quad \text{and} \end{aligned}$$

$$\varepsilon_0 = 2, \quad 0 = 2\varepsilon_1 + a_{11}\varepsilon_0 = 2(\varepsilon_1 + a_{11}); \quad \text{so}$$
$$\varepsilon_1 = -a_{11}, \quad \beta_2 = 2\varepsilon_2 - a_{11}^2 + 2a_3;$$

since $a_{11}^2 \notin 2U^*(pt)$ (because $U^*(pt) = [x_1, x_2, \dots]$, $a_{11} = -x_1$) it follows that $\beta_2 \notin 2U^*(pt)$. The relations $P(D) = A + B + C$, $Q(D) = AB + BC + CA$ are easy consequences of the relation $\eta^2 = 1 + \xi_i + \xi_j + \xi_k$.

(b) The above relation gives $cf_3(\eta^2) = ABC$; in order to show that $ABC = 0$ we consider the Boardman map $Bd\colon U^*(B\Gamma) \to K^*(B\Gamma)\hat{\otimes}\mathbb{Z}[a_1, a_2, \dots]$ (see [**8**], page 358). This map is a ring-homomorphism which is injective because $B\Gamma$ has a periodic cohomology; furthermore if $\tau$ is a line complex vector bundle over $B\Gamma$ we have:

$$Bd(e(\tau)) = (\tau - 1) + (\tau - 1)^2 \otimes a_1 + (\tau - 1)^3 \otimes a_2 + \cdots;$$

as $(\xi_i - 1)(\xi_j - 1)(\xi_k - 1) = 0$ we get $Bd(ABC) = 0$ and $ABC = 0$.

(c) We have $Q(D) = A(B+C)+BC = A(P(D)-A)+BC$; as $ABC = 0$ we obtain $A^3 = -AQ(D) + A^2P(D)$; similarly $B^3 = -AQ(D) + A^2P(D)$. □

Proposition 2.10. *There is* $S(Z) = -a_{11}Z + \sum_{i\geq 2} s_i \cdot Z^i \in \Omega_2$ *such that* $A^2 = AS(D)$, $B^2 = BS(D)$. *Moreover:*

$$AB = (A + B)(P(D) - S(D)) - Q(D),$$

$P(Z)$, $Q(Z)$ *being as in* 2.9.

*Proof*. Consider the relation $\eta\xi_i = \eta$. If the vector bundle $\gamma(2)\otimes\gamma(1)$ over $BU(2)\times BU(1)$ is classified by $m_1\colon BU(2)\times BU(1) \to BU(2)$ and if $g\colon B \to BU(2)$, $h\colon B \to BU(1)$ are classifying maps for $\eta$ and $\xi_i$, then $\eta\xi_i$ is classified by:

$$B\Gamma \xrightarrow{\Delta} B\Gamma \times B\Gamma \xrightarrow{g\times h} BU(2) \times BU(1) \xrightarrow{m_1} BU(2).$$

We have the following commutative diagram:

$$\begin{tikzcd}
U^*(BU(2)) \arrow[r, "m_1^*"] & U^*(BU(2)) \times BU(1) \arrow[r, "(g\times h)^*"] & U^*(B\Gamma \times B\Gamma) \arrow[r, "\Delta^*"] & U^*(B\Gamma) \\
& U^*(BU(2)) \otimes U(BU(1)) \arrow[u, "X"] \arrow[r, "g^*\times h^*"] & U^*(B\Gamma) \otimes U^*(B\Gamma). \arrow[u, "X"] \arrow[ur, "\text{cup-product}"'] &
\end{tikzcd}$$

Moreover $U^*(BU(2) \times BU(1)) = U^*(pt)[[c_1, c_2, c_1']]$ where $c_1$, $c_2$, $c_1'$ are the images respectively of $cf_1\gamma(2) \otimes 1$, $cf_2\gamma(2) \otimes 1$, $1 \otimes cf_1\gamma(1)$ by the canonical map: $U^*(BU(2)) \times U^*(BU(1)) \xrightarrow{X} U^*(BU(2) \times BU(1))$. Then

$$m_1^*(cf_2(\gamma(2))) = \sum e_{(u,v)} c_1^{u_1} c_2^{u_2} c_1'^{\,v}, \quad u = (u_1, u_2).$$

If $i$ and $j$ are the natural inclusions: $BU(2)\times\{*\}\to BU(2)\times BU(1)$ and $\{*\}\times BU(1)\to BU(2)\times BU(1)$, then the coefficients $e_{(u,0)}$ and $e_{(0,v)}$ are given respectively by $i^*\circ m^*(cf_2(\gamma(2)))=cf_2(\gamma(2))=c_2$ and $j^*\circ m^*(cf_2(\gamma(2)))=cf_2(\gamma(1)+\gamma(1))=c_1'^2$. Hence

$$\begin{aligned} m_1^*(cf_2(\gamma(2))) &= c_2+c_1'^2+\sum_{\substack{\|u\|\geq 1\\ v\geq 1}} e_{(u,v)}c_1^{u_1}c_2^{u_2}c_1'^{v}\\ &= c_2+c_1'^2+c_1'N_1(c_1,c_2)+c_1'^2N_2(c_1,c_2)\\ &\quad+\cdots+c_1'^{m}N_m(c_1,c_2)+\cdots. \end{aligned}$$

To calculate $cf_2(\eta\cdot\xi_i)$ we substitute $cf_1(\eta)$, $cf_2(\eta)$, $cf_1(\xi_i)$, respectively for $c_1$, $c_2$, $c_1'$. We recall that $cf_1(\eta)=P_0(D)$, $\nu'(P_0)=1$ ($\nu'=\frac{1}{4}\nu$; see Sec. I). We can substitute $P_0(Z)$ for $c_1$ and $Z$ for $c_2$ in $N_m(c_1,c_2)$ to obtain $M_m(Z)\in\Omega_*$, $\nu'(M_m)\geq 1$, $m\geq 1$. We need to calculate the leading coefficient of $M_1(Z)$. To this purpose consider $T=BU(1)\times BU(1)$ and $r\colon T\to BU(2)$ a map classifying $\pi_1^*(\gamma(1))+\pi_2^*(\gamma(1))$, $\pi_1$, $\pi_2$ being respectively the first and second projections $T\to BU(1)$; we have $U^*(T\times BU(1))=U^*(pt)[[e_1,f_1,e_1']]$ with $(r\times 1)^*(c_1)=e_1+f_1$, $(r\times 1)^*(c_2)=e_1f_1$, $(r\times 1)^*(c_1')=e_1'$; it is easily seen that $(r\times 1)^*(m_1^*cf_2(\gamma(2)))=F(e_1,e_1')F(f_1,e_1')$ where $F$ denotes the formal group law. It follows that $e_{((1,0),1)}=1$, $e_{((0,1),1)}=2a_{11}$ and $M_1(Z)=a_{11}Z+\sum_{i\geq 2}b_i'Z^i$, $\nu'(M_1)=1$.

Now from the relation $A^3=-AQ(D)+A^2P(D)$ we deduce that $A^n=AQ_n(D)+A^2P_n(D)$, $n\geq 3$, with $Q_n(Z)\in\Omega_{2n-2}$, $P_n(Z)\in\Omega_{2n-4}$, $Q_3(Z)=-Q(Z)$, $P_3(Z)=P(Z)$, $Q_{n+1}(Z)=-Q(Z)P_n(Z)$, $P_{n+1}(Z)=P(Z)P_n(Z)+Q_n(Z)$. Then $\nu'(P_{n+1})\geq\inf(\nu'(P_n),\nu'(P_{n-1}))$ and $\nu'(P_{n+1})\geq(n+1)/2$; so:

$$\operatorname*{Lim}_{n\to\infty}\nu'(P_n)=\operatorname*{Lim}_{n\to\infty}\nu'(Q_n)=+\infty.$$

Consider

$$\begin{aligned} M_n(X,Z) &= Z+X^2[1+M_2(Z)+P(Z)M_3(Z)+\cdots+P_n(Z)M_n(Z)]\\ &\quad+X[M_1(Z)+Q_3(Z)M_3(Z)+\cdots+Q_n(Z)M_n(Z)]\in\Lambda_4. \end{aligned}$$

As

$$\operatorname*{Lim}_{n\to\infty}\nu(P_nM_n)=\operatorname*{Lim}_{n\to\infty}\nu(Q_nM_n)=+\infty$$

it follows that $\operatorname{Lim}_{n\to\infty}M_n(X,Z)$ exists (see Sec. I) and may be written as: $Z+X^2[1+H(Z)]+XH_1(Z)$ with $H(Z)\in\Omega_0$, $\nu'(H)\geq 1$. We remark that the leading coefficient of $H_1(Z)$ is that of $M_1(Z)$; so: $H_1(Z)=a_{11}Z+\sum_{i\geq 2}d_iZ^i\in\Omega_2$. Thus: $cf_2(\eta\xi_i)=D+A^2[1+H(D)]+AH_1(D)=cf_2(\eta)=D$ and $A^2[1+H(D)]=$

$-AH_1(D)$. Let $E(Z) \in \Omega_0$ be such that $E(Z)(1+H(Z)) = 1$; hence $A^2 = AS(D)$ with $S(Z) = -H_1(Z)E(Z) = -a_{11}Z + \sum_{i\geq 2} s_i Z^i \in \Omega_2$. Similarly $B^2 = BS(D)$. Now

$$\begin{aligned} AB &= AB + BC + CA - C(A+B) \\ &= Q(D) - [P(D) - (A+B)] \cdot (A+B) \\ &= Q(D) - P(D) \cdot (A+B) + 2AB + (A+B)S(D) \\ &= 2AB + Q(D) + (A+B)(S(D) - P(D)). \end{aligned}$$

Then:

$$AB = (A+B)[P(D) - S(D)] - Q(D). \qquad \square$$

LEMMA 2.11. *There is* $T(Z) = 8Z + 2\lambda_2 Z^2 + \sum_{i\geq 3} \lambda_i Z^i \in \Omega_4$, $\lambda_2 \notin 2U^*(pt)$ *and* $T(D) = 0$.

*Proof*. From $\eta^2 = 1 + \xi_i + \xi_j + \xi_k$ we get $\eta^3 = 4\eta$. Let $g_1 \colon B\Gamma \to BU(4)$ and $g \colon B\Gamma \to BU(2)$ be classifying maps (respectively) for $\eta^2$ and $\eta$; then $\eta^3$ is classified by: $B\Gamma \xrightarrow{\Delta} B\Gamma \times B\Gamma \xrightarrow{g_1 \times g} BU(4) \times BU(2) \xrightarrow{m_2} BU(8)$, $m_2$ being a map classifying $\gamma(4) \otimes \gamma(2)$. Then we get $m_2^*(cf_2(\gamma(8))) = \sum f_{(u,v)} c_1^{u_1} c_2^{u_2} c_3^{u_3} c_4^{u_4} c_1'^{v_1} c_2'^{v_2}$, with $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2)$. The coefficients $f_{(u,0)}$ and $f_{(0,v)}$ are given respectively by $cf_2(\gamma(4) + \gamma(4)) = c_1^2 + 2c_2$ and $cf_2(4\gamma(2)) = 6c_1'^2 + 4c_2'$. Thus

$$\begin{aligned} m_2^*(cf_2(\gamma(8))) &= c_1^2 + 2c_2 + 6c_1'^2 + 4c_2' \\ &\quad + \sum_{\substack{\|u\|\geq 1 \\ \|v\|\geq 1}} f_{(u,v)} c_1^{u_1} c_2^{u_2} c_3^{u_3} c_4^{u_4} c_1'^{v_1} c_2'^{v_2}. \end{aligned}$$

In order to calculate $cf_2(\eta^3)$ we must substitute $cf_1(\eta^2) = P(D)$, $cf_2(\eta^2) = Q(D)$, $cf_3(\eta^2) = 0$, $cf_4(\eta^2) = 0$, $cf_1(\eta) = P_0(D)$, $cf_2(\eta) = D$ respectively for $c_1$, $c_2$, $c_3$, $c_4$, $c_1'$, $c_2'$. Consider

$$\begin{aligned} E(Z) &= P^2(Z) + 2Q(Z) + 6P_0^2(Z) + 4Z \\ &\quad + \sum_{\|u\|\geq 1, \|v\|\geq 1} f_{(u,v)} P^{u_1}(Z) Q^{u_2}(Z) P_0^{v_1}(Z) \cdot Z^{v_2}, \end{aligned}$$

$u = (u_1, u_2, 0, 0)$, $v = (v_1, v_2)$. Hence $E(D) = cf_2(\eta^3)$; but as the leading coefficients of $P(Z)$ and $Q(Z)$ belong to $4U^*(pt)$, $E(Z)$ has the form: $2Q(Z) + 6P_0^2(Z) + 4Z + 4\tau Z^2 + \sum_{i\geq 3} \tau_i Z^i$. So: $E(D) = 2Q(D) + 6P_0^2(D) + 4D + 4\tau D^2 + \sum_{i\geq 3} \tau_i D^i = cf_2(\eta^3) = cf_2(4\eta) = 6cf_1^2(\eta) + 4cf_2(\eta) = 6P_0^2(D) + 4D$. Hence if $T(Z) = 2Q(Z) + 4\tau Z^2 + \sum_{i\geq 3} \tau_i Z^i \in \Omega_4$, then $T(D) = 0$. As $Q(Z) = 4Z + \sum_{i\geq 2} \beta_i Z^i$, $\beta_2 \notin 2U^*(pt)$, we have: $T(D) = 8Z + 2\lambda_2 Z^2 + \sum_{i\geq 3} \lambda_i Z^i$, $\lambda_2 \notin 2U^*(pt)$. $\square$

THEOREM 2.12. *If $M(Z) \in \Omega_*$ is such that $M(D) = 0$, then $M(Z) \in \Omega_* T(Z)$.*

*Proof*. We may suppose $M(Z) \in \Omega_{2n}$, $n \in \mathbb{Z}$. If $M(Z) = \omega_0 + \sum_{i \geq 1} \omega_i Z^i$, then by 2.6 we have $\omega_0 = 0$ and the first coefficient $\omega_i \neq 0$, say $\omega_{P_0}$, is such that $P_0 \geq 1$, $\omega_{p_0} \in 8U^*(pt)$. Thus $M(Z) = 8\omega'_{p_0} Z^{p_0} + \sum_{i > P_0} \omega_i Z^i$. Consider $M_1(Z) = M(Z) - \omega'_{P_0} \cdot Z^{P_0 - 1} \cdot T(Z) \in \Omega_{2n}$. We have $\nu(M_1(Z)) > \nu(M(Z))$ and $M_1(D) = 0$. Then $M_1(Z) = 8\omega'_{P_1} Z^{P_1} + \sum_{i > P_1} \theta_i \cdot Z^i$, $P_1 > P_0$. We form

$$M_2(Z) = M_1(Z) - \omega'_{P_1} Z^{P_1 - 1} T(Z)$$

and then $\nu(M_2) > \nu(M_1)$, $M_2(D) = 0$. After a finite number of steps we get $M_{r+1}(Z) = M(Z) - (\omega'_{P_0} Z^{P_0 - 1} + \cdots + \omega'_{P_r} Z^{P_r - 1}) T(Z)$ such that $P_r > P_{r-1} > \cdots > P_1 > P_0$, $\nu(M_{r_1}) > \nu(M_r) > \cdots > \nu(M_1) > \nu(M)$ and $M_{r+1}(D) = 0$. Since $\mathrm{Lim}_{r \to \infty} \nu(M_r) = \infty$ it follows that $M(Z) = (\sum_{k \geq 0} \omega'_{P_k} \cdot Z^{p_k - 1}) \cdot T(Z)$ (see Sec. I). $\square$

LEMMA 2.13. *There is $J(Z) = \mu_1 Z + \sum_{i \geq 2} \mu_i Z^i \in \Omega_0$, $\mu_1 \notin 2U^*(pt)$, such that $A[2 + J(D)] = B[2 + J(D)] = 0$.*

*Proof*. We have $[2](Y) = 2Y + a_{11} Y^2 + \sum_{i \geq 3} a_i Y^i$. As $\xi_i^2$ is trivial we have $[2](A) = 0$ and from $A^2 = AS(D)$ $(S(Z) \in \Omega_2)$ we get $A^n = AS^{n-1}(D)$. Consider $H_n(X, Z) = X[2 + a_{11} S(Z) + \cdots + a_n S^{n-1}(Z)]$. Since $\mathrm{Lim}_{n \to \infty} \nu(S^n) = \infty$ it follows that $\mathrm{Lim}_{n \to \infty} H_n(X, Z)$ exists and has the form $X[2 + J(Z)]$, with

$$J(Z) = a_{11} S(Z) + \sum_{n \geq 3} a_n S^{n-1}(Z) = -a_{11}^2 Z + \sum_{i \geq 2} \mu_i Z^i.$$

If $\mu_1 = -a_{11}^2$ we see that $\mu_1 \notin 2U^*(pt)$. Thus $A(2 + J(D)) = [2](A) = 0$. Similarly $B(2 + J(D)) = 0$. $\square$

LEMMA 2.14. *Suppose $XM(Z) + YN(Z) + E(Z) \in \Omega_*$ is such that $AM(D) + BN(D) + E(D) = 0$. Then the first coefficient $\neq 0$ of $M(Z)$ and the first coefficient $\neq 0$ of $N(Z)$ belong to $2U^*(pt)$.*

*Proof*. We may suppose $XM(Z) \in \Omega_{2n}$, $YN(Z) \in \Omega_{2n}$, $E(Z) \in \Omega_{2n}$, $n \in \mathbb{Z}$. We shall give a proof in the case: $0 \neq M(Z) = a_p Z^p + a_{p+1} Z^{p+1} + \cdots$, $a_p \neq 0$, $0 \neq N(Z) = b_q Z^q + b_{q+1} Z^{q+1} + \cdots$, $b_q \neq 0$ and $p \leq q$. We observe that if $s \geq p$ then $A(a_p D^p + \cdots + a_{p+s} D^{p+s}) \in J^{4p+2, 2n-4p-2}$ and consequently $AM(D) \in J^{4p+2, 2n-4p-2}$ because the subgroups $J^{*,*}$ are closed in $U^*(B\Gamma)$. Similarly

$$A(a_{p+1} D^{p+1} + \cdots + a_r D^r + \cdots) \in J^{4p+6, 2n-4p-6}$$

and consequently

$$A(a_{p+1}D^{p+1} + \cdots + a_r D^r + \cdots) \in J^{4p+3,2n-4p-3}.$$

There are similar remarks concerning $BN(D)$. Since by hypothesis $p \leq q$ we have $4p+2 \leq 4q+2$ and $J^{4p+2,2n-4p-2} \supset J^{4q+2,2n-4q-2}$. We shall denote by $g$ the quotient map:

$$\begin{aligned} J^{4p+2,2n-4p-2} &\to J^{4p+2,2n-4p-2}/J^{4p+3,2n-4p-3} \\ &= [U^h(pt)/2U^h(pt)] \oplus [U^h(pt)/2U^h(pt)], \end{aligned}$$

with $h = 2n - 4p - 2$. Then $g(AM(D)) = \overline{a}_p$, $\overline{a}_p$ being the image of $a_p$ by the quotient map

$$U^h(pt) \to U^h(pt)/2U^h(pt),$$

$U^h(pt)/2U^h(pt)$ being the first summand.

(a) *Suppose* $E(D) = 0$.

(i) $p = q$. We have $g(AM(D)) = \overline{a}_p$ and $g(BM(D)) = \overline{b}_p$ respectively in the first and second summand of the sum $[U^h(pt)/2U^h(pt)] \oplus [U^h(pt)/2U^h(pt)]$. Since $AM(D)+BN(D) = 0$ we have $\overline{a}_p = 0$, $\overline{b}_p = 0$ and thus $a_p \in 2U^*(pt)$, $b_p \in 2U^*(pt)$.

(ii) $p < q$. From $J^{4p+2,2n-4p-2} \supset J^{4p+3,2n-4p-3} \supset J^{4q+2,2n-4q-2}$ it follows that $g(BN(D)) = 0$ and consequently $\overline{a}_p = 0$ which means that $a_p \in 2U^*(pt)$.

(b) *Suppose* $E(D) \neq 0$.

Take $E(Z) = d_0 + \sum_{i \geq 1} d_i Z^i$. As $E(D) = -(AM(D) + BN(D)) \in \tilde{U}^*(B\Gamma)$ we have $d_0 = 0$. Hence:

$$E(Z) = \sum_{i \geq r} d_i Z^i, \quad d_r \neq 0, \quad r \geq 1.$$

If $d_r = 8e_{r_1}$, we form

$$\begin{aligned} E_1(Z) &= E(Z) - e_{r_1} Z^{r-1} T(Z) \\ &= \sum_{i \geq r'} d'_i Z^i, \quad r' > r,\ d'_{r'} \neq 0 \text{ or } \nu(E_1) > \nu(E). \end{aligned}$$

If $d'_{r'} = 8e_{r_2}$ we form $E_2(Z) = E_1(Z) - e_{r_2} Z^{r'-1} T(Z)$ and so on. But after a finite number of steps we have $E_{p_0}(Z) = \sum_{i \geq h} t_i Z^i$ and $t_h \notin 8U^*(pt)$ because, if not, we would have $E(Z) \in \Omega_* T(Z)$ and thus $E(D) = 0$ which contradicts the hypothesis (b): $E(D) \neq 0$ (see the proof of 2.12). Hence there is a formal power series $F(Z) \in \Omega_{2n}$ such that $F(D) = E(D)$ and $F(Z) = \sum_{i \geq h \geq 1} t_i Z^i$, $t_h \notin 8U^*(pt)$. This means that $E(D) \in J^{4h,2n-4h}$ and $E(D) \notin J^{4h+1,2n-4h-1}$.

(i) $p = q$, $4h < 4p + 2 = 4q + 2$.

Then: $J^{4h,2n-4h} \supset J^{4h+1,2n-4h-1} \supset J^{4p+2,2n-4p-2}$. Since $E(D) = -(AM(D) + BN(D))$ we have $E(D) \in J^{4h+1,2n-4h-1}$ which is impossible.

(ii) $p = q$, $4p + 2 = 4q + 2 < 4h$.

Then $J^{4p+2,2n-4p-2} \supset J^{4p+3,2n-4p-3} \supset J^{4h,2n-4h}$ and $AM(D) + BN(D) = -E(D) \in J^{4p+3,2n-4p-3}$. Consequently $\overline{a}_p = 0$, $\overline{b}_p = 0$ and thus $a_p \in 2U^*(pt)$, $b_p \in 2U^*(pt)$.

(iii) $p < q$, $4h < 4p + 2 < 4q + 2$.

Then $J^{4h,2n-4h} \supset J^{4p+2,2n-4p-2} \supset J^{4q+2,2n-4q-2}$. From $E(D) = -(AM(D) + BN(D))$ it follows that

$$E(D) \in J^{4p+2,2n-4p-2} \subset J^{4h+1,2n-4h-1} \ (\subset J^{4h,2h-4h})$$

which is impossible.

(iv) $p < q$, $4p + 2 < 4h < 4q + 2$ *or* $4p + 2 < 4q + 2 < 4h$.

We have either

$$J^{4p+2,2n-4p-2} \supset J^{4p+3,2n-4p-3} \supset J^{4h,2n-4h} \supset J^{4q+2,2n-4q-2}$$

or

$$J^{4p+2,2n-4p-2} \supset J^{4p+3,2n-4p-3} \supset J^{4q+2,2n-4q-2} \supset J^{4h,2n-4h}.$$

It follows in both cases that $\overline{a}_p = 0$ and $a_p \in 2U^*(pt)$. *Hence we have proved that if $p \leq q$ we have $a_p \in 2U^*(pt)$ in both cases $E(D) = 0$, $E(D) \neq 0$.* So $M(Z) = a_pZ^p + a_{p+1}Z^{p+1} + \cdots$, $a_p = 2e_p \neq 0$. By 2.13 if $K(X,Z) = X(2 + J(Z))$ then $K(A,D) = 0$. We form $XM(Z) - e_pZ^pK(X,Z) = XM_1(Z)$, $M_1(Z) = e_{p_1}Z^{p_1} + \cdots$, $p_1 > p$, and we get: $AM_1(D) + BN(D) + E(D) = 0$. If $p_1 < q$ we carry on the same process and after a finite number of steps there is $M_r(Z) \in \Lambda_{2n-2}$ such that $AM_r(D) + BN(D) + E(D) = 0$ and $q \leq p_r$, $p_r$ being such that $M_r(Z) = \omega_{p_r}Z^{p_r} + \omega_{p_r+1}Z^{p_r+1} + \cdots$, $\omega_{p_r} \neq 0$. Thus the argument used is the case $p \leq q$ (above) shows that $b_q \in 2U^*(pt)$. □

Let $I'_*$ the graded ideal of $\Lambda_*$ generated by $K(X,Z) = X(2+J(Z)) \in \Lambda_2$, $K(Y,Z) = Y \cdot (2 + J(Z)) \in \Lambda_2$ and $T(Z) \in \Omega_4$ (see 2.13, 2.12).

Lemma 2.15. *Let $M(Z)$, $N(Z)$, $E(Z)$ be elements of $\Omega_*$ such that $AM(D) + BN(D) + E(D) = 0$. Then: $XM(Z) + YN(Z) + E(Z) \in K(X,Z)\Omega_* + K(Y,Z)\Omega_* + T(Z)\Omega_* \subset I'_*$ and $AM(D) = BN(D) = E(D) = 0$.*

*Proof*. Suppose $XM(Z) \in \Lambda_{2n}$, $YN(Z) \in \Lambda_{2n}$, $E(Z) \in \Lambda_{2n}$, $n \in \mathbb{Z}$. We shall give a proof in the case $M(Z) \neq 0$, $N(Z) \neq 0$, the other cases being simpler. Take $P(X,Y,Z) = XM(Z) + YN(Z) + E(Z)$, $M(Z) = a_{p_0}Z^{p_0} + a_{p_0+1}Z^{p_0+1} + \cdots$, $a_{p_0} \neq 0$, $N(Z) = b_{q_0}Z^{q_0} + b_{q_0}Z^{q_0+1} + \cdots$, $b_{q_0} \neq 0$. By 2.14 we have $a_{p_0} = 2a'_{p_0}$, $b_{q_0} = 2b'_{q_0}$ and then: $P(X,Y,Z) - (a'_{p_0}Z^{p_0}K(X,Z) + b'_{q_0}Z^{q_0}K(Y,Z)) = X[M(Z) - a'_{p_0}Z^{p_0}(2 + J(Z))] + Y[N(Z) - b'_{q_0}Z^{q_0}(2 + J(Z))] + E(Z) = XM_1(Z) + YN_1(Z) + E(Z)$ with $\nu(M) < \nu(M_1)$, $\nu(N) < \nu(N_1)$. Moreover we have $AM_1(D) + BN_1(D) + E(D) = P(A,B,D) = 0$. The same process can be carried out for $XM_1(Z) + YN_1(Z) + E(Z)$ and after a finite number of operations we get $M_1(Z), M_2(Z), \ldots, M_{r+1}(Z)$, $N_1(Z), N_2(Z), \ldots, N_{r+1}(Z)$,

$$P(X,Y,Z) - \left[\left(\sum_{i=0}^{r} a'_{p_i}Z^{p_i}\right)K(X,Z) + \left(\sum_{i=0}^{r} b'_{q_i}Z^{q_i}\right)K(Y,Z)\right]$$
$$= XM_{r+1}(Z) + YN_{r+1}(Z) + E(Z)$$

with $p_0 = \nu'(M) < p_1 = \nu'(M_1) < \cdots < p_{r+1} = \nu'(M_{r+1})$, $q_0 = \nu'(N) < q_1 = \nu'(N_1) < \cdots < q_{r+1} = \nu'(N_{r+1})$. Take

$$H_1(Z) = \sum_{i=0}^{\infty} a'_{p_i}Z^{p_i}, \qquad H_2(Z) = \sum_{i=0}^{\infty} b'_{q_i}Z^{q_i}.$$

Since $\mathrm{Lim}_{r\to\infty}\nu(M_r) = \mathrm{Lim}_{r\to\infty}\nu(N_r) = +\infty$ we have $\mathrm{Lim}_{r\to\infty}XM_r(Z) = \mathrm{Lim}_{r\to\infty}YN_r(Z) = 0$ and there are $H_1(Z) \in \Omega_*$, $H_2(Z) \in \Omega_*$ such that: $P(X,Y,Z) - [H_1(Z)K(X,Z) + H_2(Z)K(Y,Z)] = E(Z)$. Since $P(A,B,D) = K(A,D) = K(B,D) = 0$ we have: $E(D) = 0$ and then by 2.12 there is $H_3(Z) \in \Omega_*$ such that $E(Z) = H_3(Z) \cdot T(Z)$. Finally we have $P(X,Y,Z) = H_1(Z)K(X,Z) + H_2(Z)K(Y,Z) + H_3(Z)T(Z) \in K(X,Z)\Omega_* + K(Y,Z)\Omega_* + T(Z)\Omega_* \subset I'_*$ and $XM(Z) = H_1(Z)K(X,Z)$, $YN(Z) = H_2(Z) \cdot K(Y,Z)$, $E(Z) = H_3(Z) \cdot T(Z)$. Consequently: $AM(D) = BN(D) = E(D) = 0$. □

Consider $S(X,Z) = X^2 - XS(Z) \in \Lambda_4$, $S(Y,Z) = Y^2 - YS(Z) \in \Lambda_4$, $R(X,Y,Z) = XY - (X+Y)(P(Z) - S(Z)) + Q(Z) \in \Lambda_4$. By 2.10 we have: $S(A,D) = S(B,D) = R(A,B,D) = 0$. Let $I''_*$ be the grade ideal of $\Lambda_*$ generated by $S(X,Z)$, $S(Y,Z)$, $R(X,Y,Z)$.

LEMMA 2.16. *For any $P(X,Y,Z) \in \Lambda_*$ there are $M(Z)$, $N(Z)$, $E(Z)$, elements of $\Omega_*$ such that $P(X,Y,Z) - [XM(Z) + YN(Z) + E(Z)] \in I''_*$.*

*Proof*. From $X^2 - XS(Z) = S(X,Z)$ we see that there is $M_n(X,Z) \in \Lambda_*$ such that $X^n - XS^{n-1}(Z) = S(X,Z)M_n(X,Z)$, $n \geq 2$, with $M_2(X,Z) = 1$ and $M_{n+1}(X,Z) = S^{n-1}(Z) + XM_n(X,Z)$, $n \geq 2$. It is easily seen that $\mathrm{Lim}_{n\to\infty}\nu(S^n) = \mathrm{Lim}_{n\to\infty}\nu(M_n) = +\infty$. If $P(X,Y,Z) \in \Lambda_{2m}$ we can write $P(X, Y, Z) = \sum_{i=0}^{\infty} X^i P_i(Y, Z)$ with $\dim P_i = 2(m-i)$. We have $X^i P_i(Y, Z) = XS^{i-1}(Z)P_i(Y, Z) + S(X, Z)M_i(X, Z)P_i(Y, Z)$, $i \geq 2$. From Section I and the fact that the multiplication by an element of $\Lambda_*$ is continuous we see that there are $H(Y, Z)$, $H_1(X, Y, Z)$ such that: $P(X, Y, Z) = XH(Y, Z) + S(X, Z)H_1(X, Y, Z) + P_0(Y, Z)$. Similarly there are $F_0(Z)$, $F_1(Z)$, $F_2(Y, Z)$ such that $H(Y, Z) = YF_1(Z) + S(Y, Z)F_2(Y, Z) + F_0(Z)$ and $G_0(Z)$, $G_1(Z)$, $G_2(Y, Z)$ such that $P_0(Y, Z) = YG_1(Z) + S(Y, Z)G_2(Y, Z) + G_0(Z)$. Then a straightforward calculation shows that with $M(Z) = F_0(Z) + F_1(Z) \cdot (P(Z) - S(Z))$, $N(Z) = G_1(Z) + F_1(Z) \cdot (P(Z) - S(Z))$, $E(Z) = G_0(Z) - Q(Z) \cdot F_1(Z)$ we get $P(X, Y, Z) - [XM(Z) + YN(Z) + E(Z)] \in I''_*$. □

Let $I_*$ be $I'_* + I''_*$.

THEOREM 2.17. *The graded $U^*(pt)$-algebra $U^*(B\Gamma)$ is isomorphic to $\Lambda_*/I_*$ where $I_*$ is a graded ideal generated by six homogeneous formal power series.*

*Proof*. Consider the map $\varphi: \Lambda_* \to U^*(B\Gamma)$ of graded $U^*(pt)$-algebras such that $\varphi(X) = A$, $\varphi(Y) = B$, $\varphi(Z) = D$. By Theorem 2.5 $\varphi$ is surjective and by Lemmas 2.15, 2.16 $\varphi$ is injective. □

REMARKS. (1) Consider the involution $h: \Lambda_* \to \Lambda_*$ such that $h(Y) = X$, $h(X) = Y$, $H(Z) = Z$. We have $h(I_*) = I_*$ and thus there is an isomorphism $\overline{h}$ of graded $U^*(pt)$-algebras: $U^*(B\Gamma) \to U^*(B\Gamma)$ such $\overline{h}(A) = B$, $\overline{h}(B) = A$, $\overline{h}(D) = D$. Consequently $\overline{h}^2 = \mathrm{Id}$.

(2) If $q: \mathbb{Z}_2 \subset \Gamma$ denotes the canonical inclusion, then $(Bq)^*: U^*(B\Gamma) \to U^*(B\mathbb{Z}_2)$ is neither injective nor surjective.

An important and easy consequences of Theorem 2.12 and Lemma 2.15 is the following theorem which gives the structure of $U^*(pt)[[D]]$-module of $U^*(B\Gamma)$.

THEOREM 2.18. (a) *As graded $U^*(pt)$-algebras we have:*

$$U^*(pt)[[D]] \simeq \Omega_*/(T(Z)).$$

(b) *As graded $U^*(pt)[[D]]$-modules we have: $U^*(B\Gamma) \simeq U^*(pt)[[D]] \oplus U^*(pt)[[D]]A \oplus U^*(pt)[[D]]$. $B$ and: $A$ and $B$ have the same annihilator*

$$(2 + J(D))U^*(pt)[[D]].$$ □

**III. Computation of $U^*(B\Gamma_k)$, $k \geq 4$.** The group $\Gamma_k$, $k \geq 4$, is generated by $u$, $v$, subject to the following relations $u^t = v^2$, $uvu = v$,

$t = 2^{k-2}$; $|\Gamma_k| = 2^k$. We have $H^0(B\Gamma_k) = \mathbb{Z}$, $H^{4p}(B\Gamma_k) = \mathbb{Z}_{2^k}$, $p > 0$, $H^{4p+2} = \mathbb{Z}_2 \oplus \mathbb{Z}_2, p \geq 0$, $H^{2p+1}(B\Gamma_k) = 0$, $p \geq 0$. Furthermore if $d_1$, $\{a_1, b_1\}$ are generators of respectively $H^4(B\Gamma_k)$ and $H^2(B\Gamma_k)$, then $d_1^p$, $\{a_1 d_1^p, b_1 d_1^p\}$ are generators of respectively $H^{4p}(B\Gamma_k)$ and $H^{4p+2}(B\Gamma_k), p \geq 0$ (see [5]). The irreducible unitary representations of $\Gamma_k$ are $1\colon u \to 1$, $v \to 1$, $\xi_1\colon u \to 1$, $v \to -1$, $\xi_2\colon u \to -1$, $v \to 1$, $\xi_3\colon u \to -1$, $v \to -1$,

$$\eta_r\colon u \to \begin{pmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \end{pmatrix}, \quad v \to \begin{pmatrix} 0 & (-1) \\ 1 & 0 \end{pmatrix}, \qquad r = 1, 2, \ldots, 2^{k-2} - 1$$

and $\omega$ a primitive $2^{k-1}$th root of unity (see [6]).

The relations between the irreducible unitary representations of $\Gamma_k$ are as follows: $\xi_1^2 = \xi_2^2 = \xi_3^2 = 1$, $\xi_1 \cdot \xi_2 = \xi_3$, $\xi_2\xi_3 = \xi_1$, $\xi_3 \cdot \xi_1 = \xi_2$; if we introduce $\eta_0 = 1 + \xi_1$, $\eta_{2^{k-2}} = \xi_2 + \xi_3$, then we can define $\eta_s, s \in \mathbb{Z}$, by the relations $\eta_{2^{k-2}+r} = \eta_{2^{k-2}-r}, \eta_r = \eta_{-r}$ and we have: $\eta_r \cdot \eta_s = \eta_{r+s} + \eta_{r-s}$, $r \in \mathbb{Z}$, $s \in \mathbb{Z}$ (see [10]). As in Section II we shall be working with $A_k = cf_1(\xi_1) \in \tilde{U}^2(B\Gamma_k)$, $B_k = cf_1(\xi_2) \in \tilde{U}^2(B\Gamma_k)$, $C_k = cf_1(\xi_3) \in \tilde{U}^2(B\Gamma_k)$, $D_k = cf_2(\eta_1) \in \tilde{U}^4(B\Gamma_k)$. We have as in 2.5 with $U^*(pt)[[D_k]] = \{R(D_k), R \in \Omega_*\}$:

**Theorem 3.1.** *$U^*(B\Gamma_k)$ is concentrated in even dimensions and as a module over $U^*(pt)[[D_k]]$, $U^*(B\Gamma_k)$ is generated by* 1, *$B_k$, $C_k$.* □

The following proposition is proved in the same way as 2.8 and 2.6, $P_0(Z)$ being the formal power series of 2.8:

**Proposition 3.2.** (a) *We have $cf_1(\eta_1) = P_0(D_k)$.*

(b) *If $H(Z) = \sum_{i \geq 0} \alpha_i Z^i \in \Omega_{2n}$ is such that $H(D_k) = 0$, then $\alpha_0 = 0$ and the leading coefficient of $H(Z)$ belongs to $2^k U^*(pt)$.* □

**Lemma 3.3.** *For each $n \in \mathbb{Z}$ there is a polynomial $P_{2n+1}(X) \in \mathbb{Z}[X]$ such that $P_{2n+1}(0) = 0$, $P_{2n+1}(2) = 2$, $P_{2n+1}(\eta_1) = \eta_{2n+1}$.*

*Proof*. Since $\eta_{-r} = \eta_r$, we may suppose $n \geq 0$. Then the assertion is evidently true if $n = 0$ with $P_1(X) = X$. Suppose that there are polynomials $P_{2i+1}(X) \in \mathbb{Z}[X]$, $0 \leq i \leq n-1$, such that $P_{2i+1}(\eta_1) = \eta_{2i+1}$, $P_{2i+1}(0) = 0$ and $P_{2i+1}(2) = 2$. Then $\eta_1^2 P_{2n-1}(\eta_1) = \eta_1^2 \eta_{2n-1} = (\eta_2 + \eta_0)\eta_{2n-1} = \eta_{2n+1} + 2\eta_{2n-1} + \eta_{2n-3}$. Hence if $P_{2n+1}(X) = (X^2 - 2)P_{2n-1}(X) - P_{2n-3}(X)$ we have $P_{2n+1}(X) \in \mathbb{Z}[X]$, $P_{2n+1}(0) = 0$, $P_{2n+1}(2) = 2$ and $P_{2n+1}(\eta_1) = \eta_{2n+1}$. □

In the sequel we shall consider the sequence $P_{2n+1}$, $n \geq 0$, determined by $P_1(X) = X$, $P_3(X) = X^3 - 3X$ and the relation

$$(X^2 - 2)P_{2n-1}(X) - P_{2n-3}(X) = P_{2n+1}(X).$$

If $P(X) \in \mathbb{Z}[X]$ we shall denote by $P'$ the derivatives of $P$.

PROPOSITION 3.4. *If $\zeta$ is a complex vector bundle over $B\Gamma_k$ such that $\zeta = P(\eta_1)$ where $P \in \mathbb{Z}[X]$, $P(0) = 0$, then there is a formal power series $P'(2)Z + \sum_{i\geq 2} \delta_i Z^i \in \Omega_4$ such that $cf_2(\zeta) = P'(2)D_k + \sum_{i\geq 2} \delta_i D_k^i$.*

*Proof*. For each $q \geq 1$ the complex bundle $\eta_1^q$ is classified by the composite: $\Gamma_k \xrightarrow{\Delta} (B\Gamma_k)^q \xrightarrow{X^q g} (BU(2))^q \xrightarrow{m_q} BU(2^q)$ where $\Delta$ is the diagonal map, $g$ a map classifying $\eta_1$ and $m_q$ a map classifying $\bigotimes^q \gamma(2)$. We have $U^*(BU(2)^q) = U^*(pt)[[c_1^{(1)}, c_2^{(1)}, c_1^{(2)}, c_2^{(2)}, \ldots, c_1^{(q)}, c_2^{(q)}]]$ where $c_k^{(i)}$, $k = 1$ or $k = 2$, is the image of $a_1 \otimes a_2 \cdots \otimes a_q$, $a_1 = a_2 = \cdots = a_{i-1} = 1$, $a_i = cf_k(\gamma(2))$ ($k = 1$ or $k = 2$), $a_{i+1} = \cdots = a_q = 1$, by the canonical product $\bigotimes^q U^*(BU(2)) \to U^*(BU(2^q))$. Then $m_q^*(cf_2\gamma(2^q)) = \sum a_u (c_1^{(1)})^{u_1^{(1)}} \cdot (c_2^{(1)})^{u_2^{(1)}} \cdots (c_1^{(q)})^{u_1^{(q)}} \cdot (c_2^{(q)})^{u_2^{(q)}}$. If we substitute $Z$ for $c_2^{(i)}$ and $P_0(Z)$ for $c_1^{(i)}$, $i = 1, 2, \ldots, q$, we have a formal power series $R_q(Z) \in \Omega_4$ such that $R_q(D_k) = cf_2(\eta_1^q)$. If $\{p_j\}$ denotes the base point of $BU(2)$ and $k_i$ the inclusion:

$$\{p_1\} \times \{p_2\} \times \cdots \times \{p_{i-1}\} \times BU(2) \times \{p_{i+1}\} \times \cdots \times \{p_q\} \subset (BU(2))^q,$$

we see that $k_i^* \circ m_q^*(cf_2(\gamma(2^q))) = cf_2(2^{q-1}\gamma(2)) = 2^{q-1}cf_2(\gamma(2)) + 2^{q-2}(2^{q-1} - 1)cf_1^2(\gamma(2))$. Consequently $R_q(Z) = q2^{q-1}Z + \sum_{i\geq 2} \varepsilon_i Z^i$. Similarly there are formal powers series $H_1(Z) \in \Omega_2$, $H_s(Z) \in \Omega_{2s}$, $s \geq 3$, such that $H_1(D_k) = cf_1(\eta_1^q)$ and $H_s(D_k) = cf_s(\eta_1^q)$, $s \geq 3$; we have $\nu'(H_1) \geq 1$, $\nu'(H_s) \geq 2$, $s \geq 3$. (We recall that $\nu'(P(Z)) = \frac{1}{4}\nu P(Z)$.) It follows that if $\zeta = \sum_{i=1}^r m_i \eta_1^i$, $m_i \geq 0$, there is a formal power series $H(Z) \in \Omega_4$ such that $H(D_k) = cf_2(\zeta)$ and $H(Z) = (\sum_{i=1}^r im_i 2^{i-1})Z + \sum_{i\geq 2} \varepsilon_i' Z^i$. Now suppose that $\zeta$ is a complex vector bundle such that $\zeta = \sum_{i=1}^r m_i \eta_1^i - \sum_{i=1} n_i \eta_1^i$, $m_i \geq 0$, $n_i \geq 0$. The above remarks show that

$$\begin{aligned} cf(\zeta) &= 1 + cf_1(\zeta) + cf_2(\zeta) + \cdots \\ &= [1 + M_1(D_k) + cf_2(\zeta_1) + M_2(D_k)] \\ &\quad \times [1 + M_1'(D_k) + cf_2(\zeta_2) + M_2'(D_k)]^{-1} \end{aligned}$$

with $\zeta_1 = \sum_{i=1}^r m_i \eta_1^i$, $\zeta_2 = \sum_{i=1}^r n_i \eta_1^i$, $M_1$, $M_2$, $M_1'$, $M_2'$ being elements of $\Omega_*$ such that $\nu'(M_1) \geq 1$, $\nu'(M_1') \geq 1$, $\nu'(M_2) \geq 2$, $\nu'(M_2') \geq 2$. It follows that $cf_2(\zeta) = M(D_k)$, with $M(Z) \in \Omega_4$ and $M(Z) =$

$\sum_{i=1}^{r}(im_i 2^{i-1} - in_i 2^{i-1})Z + \sum_{i\geq 2}\delta_i Z^i$. Then if $P(X) = \sum_{i=1}^{r} m_i X^i - \sum_{i=1}^{r} n_i X^i \in \mathbb{Z}[X]$ we see that $M(Z) = P'(2)Z + \sum_{i\geq 2}\delta_i Z^i, P'(X)$ being the derivative of $P(X)$. □

LEMMA 3.5. *There is a formal power series*

$$Q_1(Z) = (1 + 2^2 n(n+1))Z + \sum_{i\geq 2}\beta_i' Z^i \in \Omega_4$$

*such that* $Q_1(D_k) = cf_2(\eta_{2n+1})$.

*Proof*. Since $\eta_{2n+1} = P_{2n+1}(\eta_1)$ with $P_{2n+1} \in \mathbb{Z}[X]$, $P_{2n+1}(0) = 0$, then by 3.4 it is enough to prove that $P_{2n+1}'(2) = 1 + 2^2 n(n+1)$. This assertion is true when $n = 0$ because $P_1(X) = X$. Suppose that $P_{2i+1}'(2) = 1+2^2 i(i+1), 0 \leq i \leq n-1$. We have $P_{2n+1} = (X^2-2)P_{2n-1} - P_{2n-3}$ and then $P_{2n+1}'(2) = 2^2 P_{2n-1}(2) + 2P_{2n-1}'(2) - P_{2n-3}'(2) = 2^3 + 2[1+2^2(n-1)n] - [1+2^2(n-2)(n-1)] = 1+2^2 n(n+1)$ ($P_{2n-1}(2) = 2$ by 3.3). Hence the lemma has been proved. □

In Lemma 3.5 the coefficients $\beta_i'$ depend on $n$; however we have chosen not to complicate the notation.

PROPOSITION 3.6. *There is a formal power series*

$$T_k(Z) = 2^k Z + 2^{k-2}\lambda_2' Z^2 + 2^{k-3}\lambda_3' Z^3 + \cdots + 2\lambda_{k-1}' Z^{k-1} + \sum_{i\geq k}\lambda_i' Z^i \in \Omega_4,$$

*with* $\lambda_2' \notin 2U^*(pt)$, *such that* $T_k(D_k) = 0$. *Moreover if* $R(Z) \in \Omega_*$ *and* $R(D_k) = 0$ *then* $R(Z) \in T_k(Z)\Omega_*$.

*Proof*. From 3.5 there is a formal power series

$$Q_1(Z) = [1 + 2^2(2^{k-3} - 2)(2^{k-3} - 1)]Z + \sum_{i\geq 2}\beta_i' Z^i \in \Omega_4$$

such that $Q_1(D_k) = cf_2(\eta_{2^{k-2}-3})$. We have $1+2^2(2^{k-3}-2)(2^{k-3}-1) = 9 + 2^{2k-4} - 3\cdot 2^{k-1}$. Now

$$\begin{aligned}\eta_1^2\eta_{2^{k-2}-1} &= (\eta_2 + \eta_0)\eta_{2^{k-2}-1} \\ &= \eta_{2^{k-2}+1} + \eta_{2^{k-2}-3} + 2\eta_{2^{k-2}-1} = 3\eta_{2^{k-2}-1} + \eta_{2^{k-2}-3}\end{aligned}$$

and consequently if $P(X) = (X^2-3)P_{2^{k-2}-1}$, we have $P \in \mathbb{Z}[X]$, $P(0) = 0$ and $P(\eta_1) = \eta_{2^{k-2}-3}$. Then from 3.4 there is a formal power series $Q_2(Z) = P'(2)+\sum_{i\geq 2}\beta_i'' Z^i \in \Omega_4$ such that $Q_2(D_k) = cf_2(\eta_{2^{k-2}-3})$. We have $P'(2) = 2^2 P_{2^{k-2}-1}(2) + P'_{2^{k-2}-1}(2) = 2^3 + 1 + 2^2(2^{k-3}-1)2^{k-3} = 9 + 2^{2k-4} - 2^{k-1}$. Hence

$$\begin{aligned} 0 &= Q_2(D_k) - Q_1(D_k) \\ &= [9 + 2^{2k-4} - 2^{k-1} - (9 + 2^{2k-4} - 3 \cdot 2^{k-1})]D_k \\ &\quad + \sum_{i \geq 2} (\beta_i'' - \beta_i')D_k^i \\ &= 2^k D_k + \sum_{i \geq 2} \mu_i' D_k^i, \qquad \mu_i' = \beta_i'' - \beta'. \end{aligned}$$

Then if $T_k(Z) = 2^k Z + \sum_{i \geq 2} \mu_i' Z^i \in \Omega_4$ then we have $0 = T_k(D_k)$. By 3.2 and a proof similar to that of 2.12, Section II, if $R(Z) \in \Omega_*$ is such that $R(D_k) = 0$ then $R(Z) \in T_k(Z)\Omega_*$. Now we want to show that $\mu_2' = 2^{k-2}\lambda_2'$, $\lambda_2' \notin 2U^*(pt)$, $\mu_3' = 2^{k-3}\lambda_3', \ldots, \mu_{k-1}' = 2\lambda_{k-1}'$. Instead of $T_3(Z)$ we take the formal power series $T(Z)$ defined in Section II (see 2.11). We recall that $T(Z) = 2^3 Z + 2\lambda_2 Z^2 + \sum_{i \geq 3} \lambda_i Z^i$, $\lambda_2 \notin 2U^*(pt)$. Hence if $k = 3$ the assertion concerning the coefficients of $T_k(Z)$ is true. Suppose that

$$\begin{aligned} T_k(Z) &= 2^k Z + 2^{k-2}\lambda_2' Z^2 + 2^{k-3}\lambda_3' Z^3 \\ &\quad + \cdots + 2\lambda_{k-1}' Z^{k-1} + \sum_{i \geq k} \lambda_i' Z^i, \quad \lambda_2' \notin 2U^*(pt). \end{aligned}$$

Consider the inclusion

$$\begin{aligned} i_{k+1} \colon \Gamma_k &= \{(u^2)^m v^n,\ n = 0, 1,\ 0 \leq m \leq 2^{k-1} - 1\} \subset \Gamma_{k+1} \\ &= \{u^m v^n,\ n = 0, 1,\ 0 \leq m \leq 2^k - 1\}. \end{aligned}$$

It is easily seen that $(Bi_{k+1})^*(D_{k+1}) = D_k$. We have: $T_{k+1}(Z) = 2^{k+1}Z + \sum_{i \geq 2} \mu_i'' Z^i$ and $T_{k+1}(D_{k+1}) = 0$. It follows that $T_{k+1}(D_k) = 0$ and consequently there is an element $\alpha_0' + \alpha_1' Z + \alpha_2' Z^2 + \cdots \in \Omega_0$ such that:

$$\begin{aligned} &2^{k+1}Z + \sum_{i \geq 2} \mu_i'' Z^i \\ &= \left(2^k Z + 2^{k-2}\lambda_2' Z^2 + \cdots + 2\lambda_{k-1}' Z^{k-1} + \sum_{i \geq k} \lambda_i' Z^i\right)\left(\sum_{i \geq 0} \alpha_i' Z^i\right). \end{aligned}$$

Then $\alpha_0' = 2$; $\mu_2'' = 2^k \alpha_1' + 2^{k-2}\lambda_2'\alpha_0' = 2^{k-1}[2\alpha_1' + \lambda_2'] = 2^{k-1}\lambda_2''$, $\lambda_2'' \notin 2U^*(pt)$; if $2 \leq i \leq k$ we have:

$$\mu_i'' = 2^k \alpha_{i-1}' + 2^{k-2}\lambda_2'\alpha_{i-2}' + 2^{k-3}\lambda_3'\alpha_{i-3}' + \cdots + 2^{k-i}\lambda_i'\alpha_0' = 2^{(k+1)-i}\lambda_i''.$$

Hence the proposition has been proved. □

Suppose $k \geq 4$; the inclusions $i_k\colon \Gamma_{k-1} \subset \Gamma_k$ and $j_k\colon \Gamma \subset \Gamma_k$ are given respectively by $\{(u^2)^m v^n,\ 0 \leq m \leq 2^{k-2}-1,\ n = 0,1\} \subset \{u^m v^n,\ 0 \leq m \leq 2^{k-1}-1,\ n = 0,1\}$ and $j_k = i_k \circ \cdots \circ i_4$; $\Gamma_k$ is normal in $\Gamma_{k+1}$ and $\Gamma_{k+1}/\Gamma_k = \{1, \overline{u}\} \simeq \mathbb{Z}_2$; if $q_k\colon \Gamma_k \to \Gamma_k$ is the conjugation by $u \in \Gamma_{k+1} - \Gamma_k$ then $q_k(u^2) = u^2$, $q_k(v) = v(u^2)^{-1}$. Let $f_k\colon B\Gamma_k \to B\Gamma_{k-1}$, $g_k\colon B\Gamma \to B\Gamma_k$, $h_k\colon B\Gamma_k \to B\Gamma_k$ be respectively $Bi_k$, $Bj_k$ and $Bq_k$.

LEMMA 3.7. *Suppose* $k \geq 4$.
(a) $f_k^*(A_k) = A_{k-1}$, $f_k^*(B_k) = 0$, $f_k^*(C_k) = A_{k-1}$, $f_k^*(D_k) = D_{k-1}$.
(b) $g_k^*(A_k) = A$, $g_k^*(B_k) = 0$, $g_k^*(C_k) = A$, $g_k^*(D_k) = D$.
(c) $h_k^*(A_k) = A_k$, $h_k^*(B_k) = C_k$, $h_k^*(C_k) = B_k$.

*Proof*. The proof is easy; for example $f_k^*(A_k) = A_{k-1}$ because $i_k^*\colon R(\Gamma_k) \to R(\Gamma_{k-1})$ maps $\xi_1$ to the similar representation: $u^2 \to 1$, $v \to -1$. ($R(\Gamma_k)$ and $R(\Gamma_{k-1})$ denote the representation rings). □

The role played by $A$, $B$, $C$ in Section II was symmetrical. Unfortunately this is not the case for $A_k$, $B_k$, $C_k$ ($k \geq 4$) as we shall see in the forthcoming propositions. We recall that there are formal power series $S(Z) \in \Omega_2$, $J(Z) \in \Omega_0$ such that $A^2 = AS(D)$, $B^2 = BS(D)$, $C^2 = CS(D)$, $A(2+J(D)) = B(2+J(D)) = C(2+J(D)) = 0$ (see 2.10, 2.13).

The formal power series $S(Z)$, $J(Z)$ will play an important role in the calculations ahead.

PROPOSITION 3.8. *Suppose* $k \geq 4$.
(a) $A_k B_k C_k = 0$.
(b) $A_k(2+J(D_k)) = 0$.
(c) *There are* $E_k \in \Omega_2$, $F_k \in \Omega_4$ *such that* $A_k = B_k + C_k - E_k(D_k)$, $B_k C_k = F_k(D_k)$.

*Proof*. (a) The relation $A_k B_k C_k = 0$ is proved in exactly the same way as in 2.9(b).

(b) By 3.1 there are $H_0(Z) \in \Omega_2$, $H_1(Z) \in \Omega_2$, $H_2(Z) \in \Omega_4$ such that: $B_{k+1}^2 = B_{k+1}H_0(D_{k+1}) + C_{k+1}(D_{k+1}) + H_2(D_{k+1})$. By 3.7(c) we get $C_{k+1}^2 = C_{k+1}H_0(D_{k+1}) + B_{k+1}H_1(D_{k+1}) + H_2(D_{k+1})$ and $C_{k+1}^2 - B_{k+1}^2 = (C_{k+1} - B_{k+1})H_3(D_{k+1})$ with $H_3 = H_0 - H_1 \in \Omega_2$. By using 3.7(a) we see that: $A_k^2 = A_k \cdot H_3(D_k)$; as in 2.13 the relation $cf_1(\xi_1^2) = 0$ shows that there is $J_1(Z) \in \Omega_0$ depending on $H_3(Z)$ such that $A_k(2+J_1(D_k)) = 0$ and by 3.7(b) we get $A(2+J_1(D)) = 0$; so there is

$H_4(Z) \in \Omega_0$, $\nu'(H_4) \geq 1$ such that $2 + J_1(Z) = (2 + J(Z))(1 + H_4(Z))$ (see 2.15) and consequently $2 + J(Z) = (2 + J_1(Z))H_5(Z)$, $H_5(Z) \in \Omega_0$ being such that: $(1 + H_4(Z))(1 + H_5(Z)) = 1$. Hence $A_k(2 + J(D_k)) = 0$.

(c) By using the relations $\eta_r \cdot \eta_s = \eta_{r+s} + \eta_{r-s}$, $r \in \mathbb{Z}$, $s \in \mathbb{Z}$, $\eta_0 = 1 + \xi_1$, $\eta_{2^{k-2}} = \xi_2 + \xi_3$, then a straightforward calculation shows that there is a polynomial $R_m[X] \in \mathbb{Z}[X]$ such that $R_m(0) = 0$ and $\eta_{2^m} = R_m(\eta_1) + \eta_0$, $2 \leq m \leq k - 2$; in fact $R_m(X)$ is determined by $R_2(X) = X^4 - 4X$, $R_m(X) = R_{m-1}^2(X) + 4R_{m-1}(X)$; so: $\xi_2 + \xi_3 = \eta_{2^{k-2}} = R_{k-2}(\eta_1) + \eta_0 = R_{k-2}(\eta_1) + 1 + \xi_1$. Then the proof of 3.4 shows that there are $E_k(Z) \in \Omega_2$, $F_k(Z) \in \Omega_4$ such that: $B_k + C_k = cf_1(R_{k-2}(\eta_1)) + A_k = E_k(D_k) + A_k$ and $B_k C_k = A_k E_k(D_k) + cf_2(R_{k-2}(\eta_1)) = A_k E_k(D_k) + F_k(D_k)$. As $0 = AE_k(D) + F_k(D)$ by 3.7(b) we see that $E_k(Z) \in (2 + J(Z))\Omega_*$ and consequently $B_k C_k = F_k(D)$ since $A_k(2 + J(D_k)) = 0$. Hence (c) is proved. □

PROPOSITION 3.9. *Suppose $k \geq 4$.*

(a) *There is $M(Z) \in \Omega_2$ such that: $B_k(2 + J(D_k)) + M(D_k) = C_k(2 + J(D_k)) + M(D_k) = 0$ and $M(D_k) \neq 0$.*

(b) *There is $N(Z) \in \Omega_4$, such that: $B_k^2 = B_k S(D_k) + N(D_k)$, $C_k^2 = C_k S(D_k) + N(D_k)$ and $N(D_k) \neq 0$.*

(c) *There are $G_k(Z) \in \Omega_2$, $L_k(Z) \in \Omega_4$ the coefficients of which can be computed from those of $J(Z)$, $S(Z)$, $E_k(Z)$, $F_k(Z)$ and satisfying $G_k(D_k) = M(D_k)$, $L_k(D_k) = N(D_k)$.*

*Proof*. (a) As in 3.1 there are $H_1(Z) \in \Omega_2$, $K_0(Z) \in \Omega_2$, $K_1(Z) \in \Omega_4$ such that: $B_k^2 = B_k H_1(D_k) + A_k K_0(D_k) + K_1(D_k)$; hence: $AK_0(D) = 0$ which imply by 2.15 that $K_0(Z) \in (2 + J(Z))\Omega_*$; so: $B_k^2 = B_k H_1(D_k) + K_1(D_k)$ because $A_k(2 + J(D_k)) = 0$ by 3.8(b). We have $B_k^{n+1} = B_k H_n(D_k) + K_n(D_k)$ with $H_n(Z) \in \Omega_{2n}$, $K_n(Z) \in \Omega_{2n+2}$ satisfying: $H_n(Z) = H_1(Z)H_{n-1}(Z) + K_{n-1}(Z), K_n(Z) = K_1(Z)H_{n-1}(Z)$, $n \geq 2$. It follows easily that $\text{Lim}_{n\to\infty}\nu(H_n) = \text{Lim}_{n\to\infty}\nu(K_n) = +\infty$; as $cf_1(\xi_2^2) = 0$ we have $2B_k + \sum_{n\geq 2} a_n B_k^n = 0$ with $a_n = \sum_{i+j=n} a_{ij}$, the $a_{ij}, i \geq 1, j \geq 1$, being the coefficients of the formal group law. A proof similar to that of 2.13 shows that there are $P_1(Z) \in \Omega_0$, $P_2(Z) \in \Omega_2$, $\nu'(P_1) \geq 1$, $\nu'(P_2) \geq 1$ such that $B_k(2 + P_1(D_k)) + P_2(D_k) = 0$; by 3.7(a) we have $C_k(2 + P_1(D_k)) + P_2(D_k) = 0$; hence $A(2 + P_1(D)) = 0$ and as a direct consequence of 2.15 there is $P_3(Z) \in \Omega_0$ such that $2 + J(Z) = (2 + P_1(Z))P_3(Z)$ and then: $B_k(2 + J(D_k)) + M(D_k) = C_k(2 + J(D_k)) + M(D_k) = 0$ with $M(Z) = P_2(Z)$. $P_3(Z) \in \Omega_2$. Suppose $M(D_k) = 0$; then $B_k(2 + J(D_k)) = C_k(2 + J(D_k)) = 0$; from

3.8(c) we have $A_k^2 = A_k(B_k + C_k) - A_k E_k(D_k)$ and consequently $AE_k(D) = 0$; so $E_k(Z) \in (2 + J(Z))\Omega_*$ and $A_k^2 = (B_k + C_k)^2$. Let $\theta: MU \to K$ being the canonical map between spectra; $\theta$ sends Euler classes to Euler classes; the relation $A_k^2 = (B_k + C_k)^2$ becomes by using $\theta$: $1 + \xi_1 - \xi_2 - \xi_3 = 0$ in $K^0(B\Gamma_k)$ which is impossible since $1+\xi_1-\xi_2-\xi_3 \neq 0$ in $R(\Gamma_k)$ (the canonical map from $R(\Gamma_k)$ to $K^0(B\Gamma_k)$ is injective). Hence $M(D_k) \neq 0$.

(b) We have seen in (a) that $B_k^2 = B_k H_1(D_k) + K_1(D_k)$; so: $C_k^2 = C_k H_1(D_k) + K_1(D_k)$ and: $A^2 = AH_1(D) + K_1(D) = AS(D)$; then $A[H_1(D) - S(D)] + K_1(D) = 0$ and there is $S_0(Z) \in \Omega_2$ such that $H_1(Z) = S(Z) + (2 + J(Z))S_0(Z)$; consequently: $B_k^2 = B_k S(D_k) - M(D_k)S_0(D_k) + K_1(D_k) = B_k S(D_k) + N(D_k)$ with: $N(Z) = K_1(Z) - M(Z)S_0(Z) \in \Omega_4$; by 3.7(c) $C_k^2 = C_k S(D_k) + N(D_k)$. If $N(D_k) = 0$ then as in 2.13 we would have $C_k(2+J(D_k)) = 0$ and then $M(D_k) = 0$ which is false by (a). Hence: $N(D_k) \neq 0$.

(c) We need to show first that $T_k(Z) \notin 2\Omega_*$ ($T_3(Z) = T(Z)$ and $T_k(Z)$ are defined respectively in 2.11 and 3.6). Suppose $k = 3$; from $AB + BC + CA = Q(D)$ and $A(2 + J(D)) = B(2 + J(D)) = 0$ (see 2.9 and 2.13) we get $(2 + J(D))\ Q(D) = 0$; so:

$$\begin{aligned}(2 + J(Z))Q(Z) &= (2 + \mu_1 Z + \mu_2 Z^2 + \cdots)(4Z + \beta_2 Z^2 + \beta_3 Z^3 + \cdots) \\ &= 8Z + (2\beta_2 + 4\mu)Z^2 + (2\beta_3 + \mu_1\beta_2 + 4\mu_2)Z^3 \\ &\quad + \cdots \in T(Z)\Omega_*;\end{aligned}$$

hence $T(Z) \notin 2\Omega_*$ since $\mu_1\beta_2 \notin 2U^*(pt)$ (see 2.9 and 2.13). Suppose that $T_i(Z) \notin 2\Omega_*$, $3 \leq i \leq k-1$, and $T_k(Z) \in 2\Omega_*$; as $A_k = B_k + C_k - E_k(D_k)$ (see 3.8(c)) we have $E_k(D_{k-1}) = 0$ and then $E_k(Z) \in T_{k-1}(Z)\Omega_*$; from $T_k(Z) \in T_{k-1}(Z)\Omega_*$, $T_k(Z) \in 2\Omega_*$ and $T_{k-1}(Z) \notin 2\Omega_*$ it follows easily that $2T_{k-1}(D_k) = 0$; consequently $2E_k(D_k) = 0$ and $2A_k = 2(B_k + C_k)$ which is impossible (it can be seen by using $\theta: MU \to K$ as in (a)). Hence $T_k(Z) \notin 2\Omega_*$, $k \geq 3$. Let $q: \Omega_* \to \Omega_*/2\Omega_* = (U^*(pt)/2U^*(pt))[[Z]]$ be the canonical projection and $\overline{R}(Z)$ the image of $R(Z)$ by $q$. Now it follows easily from 3.8(c) and (a) that: $2M(D_k) + E_k(D_k)(2 + J(D_k)) = 0$ and then $2M(Z) + E_k(Z)(2 + J(Z)) = T_k(Z) \cdot H(Z)$, $H(Z) \in \Omega_*$. Hence $\overline{E}_k(Z) \cdot \overline{J}(Z) = \overline{T}_k(Z) \cdot \overline{H}(Z)$; as $\overline{T}_k(Z) \neq 0$ the formal power series $\overline{H}(Z)$ is unique and its coefficients which belong to $U^*(pt)/2U^*(pt) = \mathbb{Z}_2[x_1, x_1, \ldots]$ ($|x_i| = -2i$) are computable from those of $\overline{E}_k$, $\overline{J}$ and $\overline{T}_k$; if $\overline{H}(Z) = \sum \overline{d}_i Z^i$, $\overline{d}_i \neq 0$, then there is a unique element $e_i \in \mathbb{Z}[x_1, \ldots, x_n, \ldots] = U^*(pt)$ whose coefficients as a polynomial in $x_1, \ldots, x_n, \ldots$, are odd and such that $\overline{e}_i = \overline{d}_i$; it follows that

$E_k(Z)(2+J(Z))-T_k(Z)\cdot(\sum e_i Z^i) = -2G_k(Z)$ and $G_k(D_k) = M(D_k)$. The same method can be used to determine $L_k(Z)$ by considering the relation $2N(D_k) = E_k^2(D_k) - E_k(D_k)S(D_k) - 2F(D_k)$ which is an easy consequence of (b) and 3.8(c). □

Let $\tilde{I}'_*$ be the graded ideal of $\Lambda_*$ generated by the homogeneous formal power series $G_k(X,Z) = X(2+J(Z))+G_k(Z) \in \Lambda_2, G_k(Y,Z) = Y(2+J(Z)) + G_k(Z) \in \Lambda_2$, $T_k(Z) \in \Lambda_4$ (see 3.6 and 3.9) and $\tilde{I}''_*$ the graded ideal of $\Lambda_*$ generated by the homogeneous formal power series $L_k(X,Z) = X^2 - XS(Z) - L_k(Z) \in \Lambda_4$, $L_k(Y,Z) = Y^2 - YS(Z) - L_k(Z) \in \Lambda_4$, $XY - F_k(Z) \in \Lambda_2$ (see 3.8(c) and 3.9). The proofs of the following lemmas are quite similar to those of 2.15, 2.16 and will be omitted.

LEMMA 3.10. *If $H_1(Z)$, $H_2(Z)$, $H_3(Z)$ are elements of $\Omega_*$ such that $B_k H_1(D_k)+C_k H_2(D_k)+H_3(D_k) = 0$ then $XH_1(Z)+YH_2(Z)+H_3(Z) \in G_k(X,Z)\Omega_* + G_k(Y,Z)\Omega_* + T_k(Z)\Omega_* \subset \tilde{I}'_*$.* □

LEMMA 3.11. *For any $P(X,Y,Z) \in \Lambda_*$ there are $H_1(Z)$, $H_2(Z)$, $H_3(Z)$ elements of $\Omega_*$ such that $P(X,Y,Z) - [XH_1(Z) + YH_2(Z) + H_3(Z)] \in \tilde{I}''_*$.* □

As a direct consequence of 3.10, 3.11 we get our main theorem where $\tilde{I}_* = \tilde{I}'_* + \tilde{I}''_*$ (see the proof of 2.17).

THEOREM 3.12. *The graded $U^*(pt)$-algebra $U^*(B\Gamma_k)$ is isomorphic to $\Lambda_*/\tilde{I}_*$ where $\tilde{I}_*$ is a graded ideal of $\Lambda_*$ generated by six homogeneous formal power series.* □

REMARK. The homomorphism $f_k^*$ induced by the inclusion $\Gamma_{k-1} \subset \Gamma_k$ (see 3.7) is such that $f_k^*(B_k) = 0$,

$$f_k^*(C_k) = B_{k-1} + C_{k-1} - E_{k-1}(D_{k-1})(E_{k-1}(D_{k-1}) \neq 0),$$

$f_k^*(D_k) = D_{k-1}$ if $k \geq 5$ (see 3.8). But $f_4^*(B_4) = 0$, $f_4^*(C_4) = P(D) - (B + C)$, $P(D) \neq 0$ (see 2.9, 2.6), $f_4^*(D_4) = D$.

Let $U^*(pt)[[D_k]]$ be $\{R(D_k), R(Z) \in \Omega_*\}$.

THEOREM 3.13. (a) *$U^*(pt)[[D_k]] \simeq \Omega_*/(T_k)$ as graded $U^*(pt)$-algebras.*

(b) *$U^*(B\Gamma_k)$ is generated by 1, $A_k$, $B_k$ as a $U^*(pt)[[D_k]]$-module. Moreover if $V_k = U^*(pt)[[D_k]]$ then:*

$$V_k \cap V_k B_k = V_k \cap V_k C_k = V_k B_k \cap V_k C_k = G_k(D_k) \cdot V_k.$$

*Proof*. The assertion (a) is a consequence of 3.6; the first part of (b) is proven in 3.1 and the second part is a consequence of 3.10. □

Now we are going to alter $B_k$, $C_k$ in order to improve 3.13(b). From $B_k(2+J(D_k))+G_k(D_k)=0$ it follows easily that $G_k(D)=0$; so $AG_k(D)=0$ and $G_k(Z)=(2+J(Z))G'_k(Z)$, $G'_k(Z)\in\Omega_2$; hence

$$(B_k+G'_k(D_k))(2+J(D_k))=(C_k+G'_k(D_k))(2+J(D_k))=0.$$

Furthermore if $\mu\colon U^*(B\Gamma_k)\to H^*(B\Gamma_k)$ is the edge homomorphism (in connection with the $U^*$-AHSS for $B\Gamma_k$) then $\mu(B_k+G'_k(D_k))=\mu(B_k)$, $\mu(C_k+G'_k(D_k))=\mu(C_k)$. This remark and Lemma 3.10 allow the following rearrangement of Theorem 3.13 with $B'_k=B_k+G'_k(D_k)$, $C'_k=C_k+G'_k(D_k)$.

THEOREM 3.14. (a) *$U^*(pt)[[D_k]]\simeq\Omega_*/(T_k)$ as graded $U^*(pt)$-algebras.*

(b) *As graded $U^*(pt)[[D_k]]$-modules we have:*

$$U^*(B\Gamma_k)\simeq U^*(pt)[[D_k]]\oplus U^*(pt)[[D_k]]\cdot B'_k\oplus U^*(pt)[[D_k]]\cdot C'_k$$

*and $B'_k$, $C'_k$ have the same annihilator $(2+J(D_k))\cdot U^*(pt)[[D_k]]$.* □

**Appendix.**

(A) *Calculation of $U^*(B\mathbb{Z}_m)$ by a new method.* The method used in the case $G=\Gamma_k$ applies more simply in the case $G=\mathbb{Z}_m$. Let $w$ be $\exp(2i/m)$ and $\rho$ the irreducible unitary representation of $\mathbb{Z}_m$ defined by $\rho(\overline{q})=w^q$, $\overline{q}\in\mathbb{Z}_m$. Let $\eta$ be the complex vector bundle over $B\mathbb{Z}_m$ corresponding to $\rho$ and $D_1=e(\eta)=cf_1(\eta)\in U^2(B\mathbb{Z}_m)$.

Let $\Lambda'_*$ be $U^*(pt)[[Z]]$, graded by taking $\dim Z=2$. There is a topology on $\Lambda'_{2n}$, $n\geq 0$, defined by the subgroups $J_r=\{P\in\Lambda'_{2n},\nu(P)\geq r\}$, with $\nu(P)=2s$ if $P(Z)=a_sZ^s+a_{s+1}Z^{s+1}+\cdots,a_s\neq 0$; $\Lambda'_{2n}$ is complete and Hausdorff. Furthermore, $U^{2n}(B\mathbb{Z}_m)$ is topologized by the subgroups $J^{r,2n-r}$ induced by the $U^*$-AHSS for $B\mathbb{Z}_m$, taken as a system of neighbourhoods of 0. The group $U^{2n}(B\mathbb{Z}_m)$ is complete and Hausdorff because the $U^*$-AHSS for $B\mathbb{Z}_m$ collapses. Moreover there is a unique continuous homomorphism of graded $U^*(pt)$-algebras $\varphi'\colon\Lambda'_*\to U^*(B\mathbb{Z}_m)$ such that $\varphi'(Z)=D_1$ and $\varphi'$ is surjective (see Sections I and II).

The complex vector bundle $\eta^m$ is trivial ($\dim\eta^m=1$) because $\rho^m=1$. Hence $cf_1(\eta^m)=0$. If $m_0$ denotes a map: $BU(1)^m\to BU(1)$ classifying $\bigotimes^m\gamma(1)$ ($\gamma(1)$ being a universal complex vector bundle over $BU(1)$) and if $c_1=cf_1(\gamma(1))$ then:

$$m_0^*(c_1)=\sum a_{(u)}e_1^{u_1}e_2^{u_2}\cdots e^{u_m},\quad u=(u_1,\ldots,u_m).$$

$u_1 \geq 0, \ldots, u_m \geq 0$, $e_i$ being the image of $a_1 \otimes a_2 \otimes \cdots \otimes a_m$ with $a_1 = a_2 = \cdots a_{i-1} = 1$, $a_i = c_1$, $a_{i+1} = \cdots = a_m = 1$, by the product: $\bigotimes^m U^*(BU(1)) \to UBU(1)^m)$. The vector bundle $\eta^m$ is classified by the composite:

$$\begin{tikzcd} B\mathbb{Z}_m \arrow[r, "d"] \arrow[rrr, bend left, "m"] & (B\mathbb{Z}_m) \arrow[r, "\overset{m}{\times} g"] & BU(1)^m \arrow[r, "m_0"] & BU(1), \end{tikzcd}$$

$d$ being the diagonal map and $g$ a map classifying $\eta$. It follows that if $T(Z) = \sum a_{(u)} Z^{u_1+u_2+\cdots u_m} \in \Lambda'_2$, we have $T(cf_1(\eta)) = T(e(\eta)) = T(D_1) = 0$. It is easily seen that $T(Z) = [m](Z)$.

THEOREM A.1. *$U^*(B\mathbb{Z}_m) \simeq \Lambda'_*/([m](Z))$ as graded $U^*(pt)$-algebras.*

*Proof*. Let $I_*$ be $([m](Z))$. The homomorphism $\varphi' \colon \Lambda'_* \to U^*(B\mathbb{Z}_m)$ of graded $U^*(pt)$-algebras, defined above, is surjective; moreover $\varphi'(I_*) = 0$. Hence $\varphi'$ gives rise to a homomorphism of graded $U^*(pt)$-algebras $\overline{\varphi}' \colon \Lambda'_*/I_* \to U^*(B\mathbb{Z}_m)$. Let $P(Z)$ be any element of $\Lambda'_{2n}$ $(n \geq 0)$ such that $P(D_1) = 0$; if $P(Z) = a_0 + a_1 Z + a_2 Z^2 + \cdots$, then $a_0 = 0$ because $a_0 = -(a_1 D_1 + a_2 D_1^2 + \cdots) \in \tilde{U}^*(B\mathbb{Z}_m) \cap U^*(pt) = 0$. It follows that $P(Z) = a_{p_0} Z^{p_0} + a_{p_0+1} Z^{p_0+1} + \cdots$, with $p_0 \geq 1$, $a_{p_0} \neq 0$. We have

$$a_{p_0+1} D_1^{p_0+1} + \cdots + a_{p_0+k} D_1^{p_0+k} \in J^{2(p_0+1),2(n-p_0-1)};$$

since this group is closed in $U^{2n}(B\mathbb{Z}_m)$, it follows that

$$\sum_{i=1}^{\infty} a_{p+i} D_1^{p_0+i} \in J^{2(p_0+1),2(n-p_0-1)} \subset J^{2p_0+1,2(n-p_0)-1}.$$

Let $s$ be the quotient map:

$$\begin{aligned} J^{2p_0,2(n-p_0)} &\to J^{2p_0,2(n-p_0)}/J^{2p_0+1,2(n-p_0)-1} \\ &= H^{2p_0}(B\mathbb{Z}_m) \otimes U^{2(n-p_0)}(pt) = \mathbb{Z}_m \otimes U^{2(n-p_0)}(pt) \\ &= U^{2(n-p_0)}(pt)/mU^{2(n-p_0)}(pt) \end{aligned}$$

$(H^{2p_0}(B\mathbb{Z}_m) = \mathbb{Z}_m$ because $p_0 \geq 1)$. It follows from $s(P(D_1)) = 0$ that $a_{p_0} = ma'_{p_0}$. We form $P_1(Z) = P(Z) - a'_{p_0} Z^{p_0-1} T(Z)$; then $P_1(D_1) = 0$ and $\nu(P_1) > \nu(P)$. We repeat the same process, and there is an element $P_{r+1}(Z) \in \Lambda'_{2n}$, $r \geq 1$, such that

$$P_{r+1}(Z) = P(Z) - (a'_{p_0} Z^{p_0-1} + a'_{p_1} Z^{p_1-1} + \cdots + a'_{p_r} Z^{p_r-1}) T(Z)$$

with the properties: $P_{r+1}(D_1) = 0$, $\nu(P_{r+1}) = p_{r+1} > p_r \cdots > p_1 > p_0$. Hence $\lim_{r\to\infty} \nu(P_{r+1}) = +\infty$ and by Sec. I we have $P(Z) =$

$(\sum_{i=0}^{\infty} a'_{p_i} Z^{p_i-1})T(Z) \in I_{2n}$. It follows that $\overline{\varphi}'$ is injective and the theorem has been proved. □

*Note.* P. S. Landweber has proved a similar result by using other methods (see **[13]**).

(B) *Calculation of* $U^*(BSU(n))$. *Particular case* $n = 2$: $SU(2) = Sp(1)$. Consider the $S^1$-bundle $U(n)/SU(n) = S^1 \to BSU(n) \xrightarrow{p} BU(n)$, $n \geq 2$, $p = Bi$ with $i\colon SU(n) \subset U(n)$; let $\xi$ be the complex vector bundle $E = BSU(n) \times_{S^1} \mathbb{C} \xrightarrow{\pi} BU(n)$, where $S^1$ acts on $\mathbb{C}$ by the multiplication in $\mathbb{C}$. If $E_0 = E - j(BU(n))$, $j$ being the zero-section of $\xi$, then we have the Gysin exact sequence (see **[4]**):

$$\cdots \to U^i(BU(n)) \xrightarrow{e(\xi)} U^{i+2}(BU(n)) \xrightarrow{\pi_0^*} U^{i+2}(E_0) \\ \to U^{i+1}(BU(n)) \to \cdots,$$

where $\pi_0$ denotes $\pi|E_0$. The map $g\colon BSU(n) \to E_0$ defined by $g(x) = [x, 1]$ is an embedding; take $E' = g(BSU(n))$, $j'$ the inclusion: $E' \subset E_0$ and $h\colon E_0 \to E'$ the map defined by $h[x, z] = [xz/|z|, 1]$; then by using $h$ and the homotopy $H\colon E_0 \times I \to E_0$ given by $H([x, z], t) = [x, tz + (1-t)z/|z|]$ we see that $E'$ is a strong deformation retract of $E_0$; it is easily seen that $\pi' \circ h = \pi_0$ and $\pi' \circ g = p$ with $\pi' = \pi|E'$, $g$ being considered as a homeomorphism: $BSU(n) \xrightarrow{\sim} g(BSU(n))$. So: $\pi_0^* = h^* \circ g^{*-1} \circ p^*$ and since $h^* \circ g^{*-1}$ is an isomorphism the above exact sequence gives the following one:

$$\cdots \to U^i(BU(n)) \xrightarrow{\cdot e(\xi)} U^{i+2}(BU(n)) \xrightarrow{p^*} U^{i+2}(BSU(n)) \\ \to U^{i+1}(BU(n)) \to \cdots.$$

Consider the canonical map of ring spectra $f\colon MU \to H$ (see **[1]**); $f^\#(-)$ maps Euler classes to Euler classes. Suppose $e(\xi) = 0$; then $f^\#(-)(e(\xi)) = 0$, which means that the Euler class of $\xi$ for $H$ is 0. From the Gysin exact sequence of $\xi$ for $H$ it follows easily that $H^2(BU(n)) \simeq H^2(BSU(n))$ which is impossible since $H^2(BU(n)) \neq 0$ and $H^2(BSU(n)) = 0$ (see **[12]**, page 237). Hence $e(\xi) \neq 0$ and the map $\cdot - e(\xi)$ is injective. Consequently the sequence:

$$0 \to U^{2i}(BU(n)) \xrightarrow{\cdot e(\xi)} U^{2i+2}(BU(n)) \xrightarrow{p^*} U^{2i+2}(BSU(n)) \to 0$$

is exact and $U^{2i+1}(BSU(n)) = 0$, $i \geq 0$. So we have:

**Theorem B.1.** *We have $U^{2i+1}(BSU(n)) = 0$, $i \geq 0$, and the map $p^*$ induces an isomorphism*:

$$U^{2i+2}(BU(n))/e(\xi)U^{2i}(BU(n)) \simeq U^{2i+2}(BSU(n)), \qquad i \in \mathbb{Z}.$$

Now let $(g_{ij})$ be a set of transition functions for a universal $U(n)$-bundle: $EU(n) \to BU(n)$. If $\overline{g}_{ij}$ denotes the image of $g_{ij}$ by the quotient map $q\colon U(n) \to U(n)/SU(n) = S^1$ then $(\overline{g}_{ij})$ is a set of transition functions for $\xi$; from $q(g_{ij}) = \det(g_{ij})$ and $\dim \xi = 1$, it follows that $\xi$ is isomorphic to the complex vector bundle $\Lambda^n \gamma(n), \gamma(n)$ being a universal vector bundle over $BU(n)$. Hence:

THEOREM B.2.

$$U^{2i+2}(BU(n))/e(\Lambda^n\gamma(n)) \cdot U^{2i}(BU(n)) \simeq U^{2i+2}(BSU(n)).$$

*and* $U^{2i+1}(BSU(n)) = 0$, $i \geq 0$. □

*Particular Case* $n = 2$; $Sp(1) = SU(2)$. By Section II we have $U^*(BSp(1)) = U^*(BSU(2)) = U^*(pt)[[V]]$, with $V = cf_2(\theta)$, $\theta$ being a universal $Sp(1)$-vector bundle over $BS(1)$, regarded as a $U(2)$-vector bundle. Then $cf_1(\theta) = P_0(V) = \sum_{i=1}^{\infty} b_i V^i \in U^2(BSU(2))$. If $p$ denotes the projection: $BSU(2) \to BU(2)$, we have seen that the following sequence is exact: $0 \to U^{2i}(BU(2)) \xrightarrow{\cdot e(\Lambda^2\gamma(2))} U^{2i+2}(BU(2)) \xrightarrow{p^*} U^{2i+2}(BSU(2)) \to 0$. We wish to calculate the coefficients $b_i$, $i \geq 1$. The $Sp(1)$-vector bundle $\theta$ considered as a $SU(2)$-vector-bundle is a universal $SU(2)$-vector-bundle over $BSU(2)$ isomorphic to $p^*(\gamma(2))$ as a complex vector bundle. We have $U^*(BU(2)) = U^*(pt)[[c_1, c_2]]$. $c_1 = cf_1(\gamma(2))$, $c_2 = cf_2(\gamma(2))$ and consequently

$$p^*(c_1) = \sum_{i \geq 1} b_i V^i = \sum_{i \geq 1} b_i (cf_2(\theta))^i = \sum_{i \geq 1} b_i p^*(c_2)^i$$
$$= p^* \left( \sum_{i \geq 1} b_i c_2^i \right).$$

It follows that: $c_1 - \sum_{i \geq 1} b_i c_2^i = e(\Lambda^2\gamma(2)) \cdot H(c_1, c_2)$ with $H(c_1 c_2) \in U^0(BU(2))$.

Let $k\colon BU(1) \times BU(1) \to BU(2)$ be a map classifying $\gamma(1) \times \gamma(1)$. Hence $k^*(\Lambda^2\gamma(2)) = \gamma(1) \otimes \gamma(1)$ and $k^*(e(\Lambda^2\gamma(2))) = F(X, Y)$, the formal group law. Then $k^*(c_1 - \sum_{i \geq 1} b_i c_2^i) = F(X, Y) k^*(H(c_1, c_2))$; as $k^*(c_1) = X + Y$ and $k^*(c_2) = XY$ we get:

$$X + Y - \sum_{i \geq 1} b_i (XY)^i = F(X, Y) G(X, Y) \in U^*(pt)[[X, Y]].$$

If $i(X) = [-1](X)$ then we have:

$$X + i(X) = \sum_{i \geq 1} b_i (X \cdot i(X))^i.$$

This relation determines completely the coefficients $b_i$, $i \geq 1$; for example $b_1 = -a_{11}$, $b_2 = a_{11}a_{11}a_{21} - a_{22} \cdots$ the $a_{ij}$ being the coefficients of the group law.

(C) *Ring Structure of* $H^*(B\Gamma_k), k \geq 3$. M. Atiyah has determined the ring-structure of $H^*(B\Gamma_3)$ by using $K$-theory (see [2]); namely $H^*(B\Gamma_3) = \mathbb{Z}[x,y,z]$ subject to the relations $xy = 4z$, $2x = 2y = x^2 = y^2 = 8z = 0$, $\dim x = 2$, $\dim y = 2$, $\dim z = 4$. We want to give another proof of this result using complex cobordism and determine the ring structure of $H^*(B\Gamma_k)$, $k \geq 4$.

We have $H^2(B\Gamma) = \mathbb{Z}x \oplus \mathbb{Z}y$, $H^4(B\Gamma) = \mathbb{Z} \cdot z$ with $x = c_1(\xi_j)$, $y = c_1(\xi_k)$, $z = c_2(\eta)$ (see Section II). Moreover: $2x = 2y = 8z = 0$. We have

$$B^2 = BS(D), \qquad C^2 = CS(D),$$
$$BC = (B+C)[P(D) - S(D)] - Q(D)$$

($A$, $B$, $C$ play a symmetrical role; see Section II). If $\mu$ is the edge homomorphism we have $x^2 = \mu(BS(D)) = 0$ ($\mu$: $J^{4,0} \to J^{4,0}/J^{5,-1} = H^4(B\Gamma_3)$; $BS(D) \in J^{6,-2} \subset J^{5,-1}$); similarly $y^2 = 0$; $xy = -\mu(Q(D)) = -4z_3 = -4z = 4z$ because $Q(D) = 4D + \sum_{i\geq 2} \beta_i Z^i$ (see 2.9).

*Suppose* $k \geq 4$. We have $H^2(B\Gamma_k) = \mathbb{Z}x_k \oplus \mathbb{Z}y_k$, $H^4(B\Gamma_k) = \mathbb{Z} \cdot z_k$ with $x_k = c_1(\xi_2)$, $y_k = c_1(\xi_3)$, $z_k = c_2(\eta_1)$ (see 2.3, 2.4). We have $2x_k = 2y_k = 2^k z_k = 0$. The proof of Proposition 3.8 shows that $x_k y_k = \mu(F_k(D_k)), \mu$ being the edge homomorphism, $F_k(D_k) = cf_2(R_{k-2}(\eta_1))$ with $R_{k-2}(X) \in \mathbb{Z}[X]$; $R_{k-2}(X)$ is determined inductively by $R_2(X) = X^4 - 4X^2$, $R_{m+1}(X) = R_m^2(X) + 4R_m(X)$, $m \geq 2$. By 3.4 we get $F_k(D_k) = R'_{k-2}(2) + \sum_{i\geq 2} \nu_i D_k^i$, $\nu_i \in U^*(pt)$, $R'_{k-2}(X)$ being the derivative of $R_{k-2}(X)$. An easy calculation shows that $R'_{k-2}(2) = 2^{2k-4}$. As $2k-4 \geq k$ we get $x_k y_k = 2^{2k-4} z_k = 0$. As a consequence of the relations in $R(\Gamma_k)$ stated in the beginning of Section III we get: $\xi_2 \eta_1 = \eta_{2^{k-2}-1}$. Hence $x_k^2 + c_2(\eta_1) = c_2(\eta_{2^{k-2}-1})$ because $c_1(\eta_1) = 0$. By 3.5 $cf_2(\eta_{2^{k-2}-1}) = [1 + 2^{k-1}(2^{k-3}-1)]D_k + \sum_{i\geq 2} \beta'_i D_k^i$ and consequently $c_2(\eta_{2^{k-2}-1}) = (1 - 2^{k-1})z_k$. Therefore: $x_k^2 = -2^{k-1}z_k = 2^{k-1}z_k$. Similarly: $y_k^2 = 2^{k-1}z_k$. Hence we have proved the following result:

THEOREM C. *If* $k \geq 4$ *we have* $H^*(B\Gamma_k) = \mathbb{Z}[x_k, y_k, z_k]$, $\dim x_k = \dim y_k = 2, \dim z_k = 4$ *subject to the relations*: $2x_k = 2y_k = x_k y_k = 2^k z_k = 0$, $x_k^2 = y_k^2 = 2^{k-1} z_k$. □

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