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A q-ANALOGUE OF APPELL'S F_1 FUNCTION, ITS INTEGRAL REPRESENTATION AND TRANSFORMATIONS

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An extension of Askey and Wilson's q-beta integral is evaluated as a sum of two double series. The formula is then used to find a q-analogue of Appell's F_1 function via its integral representation as well as q-analogues of transformations of F_1 to another F_1 and F_3 functions.

1. Introduction. Appell's F_1 and F_3 functions are defined by the infinite series [**7**, **10**, **20**]

$$(1.1)\qquad F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[\alpha]_{m+n}[\beta]_m[\beta']_n}{m!n![\gamma]_{m+n}} x^m y^n$$

and

$$(1.2)\qquad F_3(\alpha,\ \alpha';\ \beta, \beta';\ \gamma;\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[\alpha, \beta]_m\, [\alpha', \beta']_n}{m!n![\gamma]_{m+n}} x^m y^n$$

subject to usual convergence restrictions, where the shifted factorials are defined by $[a]_0 = 1, [a]_m = a(a+1)\cdots(a+m-1), m = 1, 2, \ldots,$ and $[a, b]_m = [a]_m[b]_m$. The F_1 function is the only one of the four Appell functions that has an integral representation in terms of a single integral [**10**, 9.3(4)]

$$(1.3)\qquad F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}du,$$

where $0 <$ Re $\alpha <$ Re γ. Letting $u = 1 - v$ leaves the form of the integral in (1.3) unchanged and gives [**10**, 9.4(1)]

$$(1.4)\qquad F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) = (1-x)^{-\beta}(1-y)^{-\beta'}\ F_1\left(\gamma-\alpha;\ \beta,\ \beta';\ \gamma;\ \frac{x}{x-1}, \frac{y}{y-1}\right).$$

When $\beta' = 0$, (1.4) reduces to [**10**, 2.4(1)]

$$(1.5)\qquad {}_2F_1(\alpha, \beta; \gamma; x) = (1-x)^{-\beta}\ {}_2F_1\left(\gamma-\alpha, \beta;\ \gamma;\ \frac{x}{x-1}\right),$$

where the ${}_2F_1$ function is Gauss's function defined by

$$ {}_2F_1(a,b;\ c;\ x) = \sum_{m=0}^{\infty} \frac{[a,b]_m}{m![c]_m} x^m, \qquad |x| < 1. \tag{1.6}$$

From the definition of F_1 and with the use of (1.5) one can show [**10**, 9.5(4)]

$$\begin{aligned} F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) \\ = (1-y)^{-\beta'} F_3\left(\alpha, \gamma-\alpha;\ \beta, \beta'; \gamma; x, \frac{y}{y-1}\right), \end{aligned} \tag{1.7}$$

expressing F_1 in terms of an F_3.

The integral representation (1.3) can be utilized to give four more equations like (1.4), [**10**, 9.4], of which one is

$$\begin{aligned} F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) \\ = (1-y)^{-\alpha} F_1\left(\alpha; \beta, \gamma-\beta-\beta';\ \gamma;\ \frac{x-y}{1-y}, \frac{-y}{1-y}\right). \end{aligned} \tag{1.8}$$

When $\gamma = \beta + \beta'$, this becomes

$$\begin{aligned} F_1(\alpha;\ \beta, \beta'; \beta+\beta';\ x, y) \\ = (1-y)^{-\alpha}\ {}_2F_1\left(\alpha, \beta;\ \beta+\beta';\ \frac{x-y}{1-y}\right). \end{aligned} \tag{1.9}$$

Equation (1.9) shows that, in a special case, the F_1 function reduces to an ordinary hypergeometric function. This is not the case in general, otherwise the study of this function will not be of the same interest as such.

As has been the case with hypergeometric functions of one variable, Appell's functions have been extended to basic (or q-) analogues. The first to look into such analogues was F.H. Jackson [**12, 13**] who defined four q-functions corresponding to Appell's and gave, among other things, a q-analogue of (1.3) using q-integrals. Other studies on the subject include Agarwal's [**1, 2**], Jain's [**14**], Slater's [**20**, Ch. 9] and Andrews' [**4, 5**] who gave some summation and transformation formulas of Jackson's functions as well as showed that the first of these functions can be expressed as a multiple of a single series which is not the case with F_1 as mentioned above. Yet, it remains true that the study of q-Appell functions has not exactly paralleled that of the ordinary ones (compare the above references with [**10**, Ch.9 and **20**, Ch.8]). This may be because Jackson's functions are "direct" analogues rather than being "natural" ones.

In a recent study of basic hypergeometric polynomials **[9]**, Askey and Wilson gave a q-analogue of the beta integral (see also Rahman **[17]** for an elementary proof), namely

$$(1.10)\quad \int_{-1}^{1} dx\, w(x;a,b,c,d) = \frac{2\pi(abcd)_\infty}{(q,ab,ac,ad,bc,bd,cd)_\infty} \equiv \kappa(a,b,c,d),$$

$\max(|a|,|b|,|c|,|d|,|q|) < 1$, where

$$(1.11)\quad w(x;a,b,c,d) = (1-x^2)^{-1/2}\frac{h(x;1)h(x;-1)h(x;\sqrt{q})h(x;-\sqrt{q})}{h(x;a)h(x;b)h(x;c)h(x;d)}$$

with

$$(1.12)\quad h(x;a) = \prod_{n=0}^{\infty}(1-2axq^n + a^2q^{2n}) = |(ae^{i\theta})_\infty|^2, \qquad x=\cos\theta,$$

$$(1.13)\quad (a)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad \text{whenever the product converges,}$$

and $(a_1,a_2,\ldots,a_k)_\infty = (a_1)_\infty(a_2)_\infty\cdots(a_k)_\infty$.

Unlike the other q-analogues of the beta integral **[3, 6, 8]** which are q-integrals, (1.10) is a Riemann integral. This integral has been generalized by Nassrallah and Rahman **[15, 16]** and Rahman **[18]** who showed that if

$$(1.14)\quad S(a,b,c,d,f,g;\lambda,\mu) \equiv \int_{-1}^{1} w(x;a,b,c,d)\,\frac{h(x;\lambda)h(x;\mu)}{h(x;f)h(x;g)}\,dx,$$

and if $\max(|a|,|b|,|c|,|d|,|f|,|g|,|q|) < 1$, then **[18]**

$$(1.15)\quad S(a,b,c,d,f,g;\lambda,\mu) = \frac{2\pi(\lambda\mu/af,\lambda\mu/cf,\lambda\mu/df,\lambda\mu/fg,\lambda b,\lambda/b,\mu b,\mu/b)_\infty}{(q,ab,ac,ad,ag,bc,bd,bg,cd,cg,dg,bf,f/b,\lambda\mu b/f)_\infty} \cdot {}_{10}\phi_9\left[\begin{matrix}\lambda\mu b/fq, & q\surd, & -q\surd, & ab, & bc, & bd, & bg, & \lambda/f, & \mu/f, & \lambda\mu/q \\ & \surd, & -\surd, & \lambda\mu/af, & \lambda\mu/cf, & \lambda\mu/df, & \lambda\mu/gf, & \mu b, & \lambda b, & bq/f\end{matrix};q\right] + \text{idem}(b;f),$$

provided

$$\lambda\mu = abcdfg. \tag{1.16}$$

If condition (1.16) doesn't hold then **[16]**

$$\begin{aligned}(1.17)\quad & S(a,b,c,d,f,g;\lambda,\mu) \\ &= \kappa(a,b,c,d)\frac{(q/abcd, aq/b, aq/c, aq/d, \lambda a, \lambda/a, \mu a, \mu/a)_\infty}{(qa^2, q/bc, q/bd, q/cd, af, f/a, ag, g/a)_\infty} \\ &\cdot {}_{10}\phi_9\left[\begin{matrix} a^2, & qa, & -qa, & ab, & ac, & ad, & af, & ag, & aq/\lambda, & aq/\mu \\ & a, & -a, & aq/b, & aq/c, & aq/d, & aq/f, & aq/g, & \lambda a, & \mu a \end{matrix}; \lambda\mu q/abcdfg\right] \\ &+ \text{idem}(a; f, g),\end{aligned}$$

provided $\mid \lambda\mu q/abcdfg \mid < 1$, in the case the ${}_{10}\phi_9$ series doesn't terminate, for convergence purposes. The expression idem$(a; f, g)$ means idem$(a; f)$+ idem$(f; g)$ where idem$(a; f)$ means an expression similar to the previous one with a and f interchanged.

The ${}_{10}\phi_9$ series appearing on the r.h.s. of (1.15) and (1.17) are special cases of the basic hypergeometric series defined by

$$\begin{aligned}(1.18)\quad & {}_r\phi_s\left[\begin{matrix} a_1, & a_2, \dots, & a_r \\ b_1, & b_2, \dots, & b_s \end{matrix}; z\right] \\ &= \sum_{m=0}^{\infty} \frac{(a_1, a_2, \dots, a_r)_m}{(q, b_1, \dots, b_s)_m} (-1)^{(1+s-r)m} q^{(1+s-r)\binom{m}{2}} z^m,\end{aligned}$$

where

$$(1.19)\quad (a)_m = \begin{cases} (a)_\infty/(aq^m)_\infty, & \text{for any } m \text{ being a complex number,} \\ \prod_{n=0}^{m-1}(1-aq^n), & \text{for } m \text{ being a positive integer.} \end{cases}$$

The open square roots in the ${}_{10}\phi_9$ series in (1.15) are over the upper left-hand term, in this case $\lambda\mu b/fq$.

It is not too hard to show from (1.14) that as $q \to 1-$

$$(1.20)\quad S(q^{\alpha/2-1/4}, q^{\alpha/2+1/4}, -q^{\gamma/2-\alpha/2-1/4}, -q^{\gamma/2-\alpha/2+1/4}, \lambda q^{-\beta}, \mu q^{-\beta'}; \lambda, \mu)$$
$$\longrightarrow 2^{2\gamma-2}(1-\lambda)^{-2\beta}(1-\mu)^{-2\beta'} \cdot \int_0^1 z^{\alpha-1}(1-z)^{\gamma-\alpha-1}(1-uz)^{-\beta}(1-vz)^{-\beta'}dz$$
$$= 2^{2\gamma-2}(1-\lambda)^{-2\beta}(1-\mu)^{-2\beta'}\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} \cdot F_1(\alpha; \beta, \beta'; \gamma; u, v),$$

where $u = -4\lambda/(1-\lambda)^2$, $v = -4\mu/(1-\mu)^2$, $|u|, |v| < 1$. Yet neither one of the right-hand sides of (1.15) and (1.17) seems to go to the same limit as in (1.20) for the same values of the parameters. In fact the r.h.s. of (1.15) would go to a multiple of the r.h.s. of (1.9) and that is what should happen because it is the case $\gamma = \beta + \beta'$. Also, (1.17) seems to be equivalent to Andrews' [**4**, Th. 1], that is to say that the q-Appell function as far as Askey and Wilson's integral (1.10) is concerned can always be expressed as a sum of three single ${}_{10}\phi_9$ series which doesn't have an equivalence in the ordinary case. So it would be natural to ask the question: what does this q-Appell function look like?

In this note we shall prove our main result

$$(1.21)\quad S(a,b,c,d,f,g;\lambda,\mu) = \kappa(a,b,c,d)\frac{(a\mu, b\mu, c\mu, \lambda a, \lambda/a, abcg)_\infty}{(ag, bg, cg, fa, f/a, abc\mu)_\infty} \cdot \sum_{m=0}^{\infty}\frac{(\lambda/f, ab, ac, ad, ag, abc\mu)_m}{(q, aq/f, \lambda a, \mu a, abcd, abcg)_m}q^m \cdot {}_8\phi_7\left[\begin{matrix} abc\mu q^{m-1}, & q\surd, & -q\surd, & \mu/d, & \mu/g, & bc, & abq^m, & acq^m \\ & \surd, & -\surd, & abcdq^m, & abcgq^m, & \mu aq^m, & c\mu, & b\mu \end{matrix}; dg\right] + \text{idem}(a; f)$$

and show that for the same values of the parameters as in (1.20), the r.h.s. of (1.21) will go to the same limit as in (1.20) when $q \to 1-$.

In the next section we derive (1.21). In §3 we shall show that as $q \to 1-$, the r.h.s. of (1.21) indeed gives F_1. Finally in §§4 and 5 we look at the transformations and special cases of this new q-Appell function giving q-analogues of (1.4) and (1.7).

2. Derivation of (1.21). The starting point of our process is, as in **[16]**, Sears' formula for the sum of two balanced non-terminating ${}_3\phi_2$'s **[19**, (5.2)]

$$(1.21)\quad {}_3\phi_2\left[\begin{matrix}\alpha_i, & \beta_i, & \gamma_i \\ & \eta_i, & \delta_i\end{matrix};q\right] + \frac{(q/\eta_i, \alpha_i, \beta_i, \gamma_i, q\delta_i/\eta_i)_\infty}{(\eta_i/q, \delta_i, \alpha_i q/\eta_i, \beta_i q/\eta_i, \gamma_i q/\eta_i)_\infty} \cdot {}_3\phi_2\left[\begin{matrix}\alpha_i q/\eta_i, & \beta_i q/\eta_i, & \gamma_i q/\eta_i \\ & q^2/\eta_i, & \delta_i q/\eta_i\end{matrix};q\right] = \frac{(q/\eta_i, \delta_i/\alpha_i, \delta_i/\beta_i, \delta_i/\gamma_i)_\infty}{(\delta_i, \alpha_i q/\eta_i, \beta_i q/\eta_i, \gamma_i q/\eta_i)_\infty},$$

provided $\eta_i\delta_i = q\alpha_i\beta_i\gamma_i$ to guarantee balancedness. The process here differs completely from that in **[16]**. Consider a product of (2.1) with itself, that is if we view (2.1) as an equation $l_i = r_i$, say, then the product is $l_1 l_2 = r_1 r_2$. Doing so and letting $\beta_1 = ae^{i\theta}, \gamma_1 = ae^{-i\theta}, \beta_2 = be^{i\theta}, \gamma_2 = be^{-i\theta}$ then multiplying by $w(x; a, b, c, d)$ and integrating with respect to x from -1 to 1 yield, after simplification,

$$(2.2)\quad \frac{(q/\eta_1, \delta_1/\alpha_1, q/\eta_2, \delta_2/\alpha_2)_\infty}{(\delta_1, \alpha_1 q/\eta_1, \delta_2, \alpha_2 q/\eta_2)_\infty} \cdot \int_{-1}^{1} dx\, w(x; a, b, c, d) \left|\frac{(\delta_1 e^{i\theta}/a, \delta_2 e^{i\theta}/b)_\infty}{(aqe^{i\theta}/\eta_1, bqe^{i\theta}/\eta_2)_\infty}\right|^2$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n}{(q, \eta_1, \delta_1)_m(q, \eta_2, \delta_2)_n} q^{m+n} \cdot \int_{-1}^{1} dx\, w(x; aq^m, bq^n, c, d) + \frac{(q/\eta_2, \alpha_2, \delta_2 q/\eta_2)_\infty}{(\eta_2/q, \delta_2, \alpha_2 q/\eta_2)_\infty}$$
$$\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2 q/\eta_2)_n}{(q, \eta_1, \delta_1)_m(q, q^2/\eta_2, \delta_2 q/\eta_2)_n} q^{m+n} \cdot \int_{-1}^{1} dx\, w(x; aq^m, bq^{1+n}/\eta_2, c, d) + \frac{(q/\eta_1, \alpha_1, \delta_1 q/\eta_1)_\infty}{(\eta_1/q, \delta_1, \alpha_1 q/\eta_1)_\infty}$$
$$\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha_1 q/\eta_1)_m(\alpha_2)_n}{(q, q^2/\eta_1, \delta_1 q/\eta_1)_m(q, \eta_2, \delta_2)_m} q^{m+n} \cdot \int_{-1}^{1} dx\, w(x; aq^{1+m}/\eta_1, bq^n, c, d)$$

$$+\frac{(q/\eta_1,\alpha_1,\delta_1 q/\eta_1,q/\eta_2,\alpha_2,\delta_2 q/\eta_2)_\infty}{(\eta_1/q,\delta_1,\alpha_1 q/\eta_1,\eta_2/q,\delta_2,\alpha_2 q/\eta_2)_\infty}$$
$$\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_1 q/\eta_1)_m(\alpha_2 q/\eta_2)_n}{(q,q^2/\eta_1,\delta_1 q/\eta_1)_m(q,q^2/\eta_2,\delta_2 q/\eta_2)_n}q^{m+n}$$
$$\cdot\int_{-1}^{1}dx\, w(x;aq^{1+m}/\eta_1,bq^{1+n}/\eta_2,c,d).$$

We next apply (1.10) to the integrals on the r.h.s. of (2.2) and let $aq/\eta_1=f$, $bq/\eta_2=g$, $\delta_1/a=\lambda$, $\delta_2/b=\mu$, $\alpha_1=\lambda/f$ and $\alpha_2=\mu/g$. Doing so and simplifying lead to

$$(2.3)\quad S(a,b,c,d,f,g;\lambda,\mu)$$
$$=\kappa(a,b,c,d)\frac{(\lambda a,\lambda/a,\mu b,\mu/b)_\infty}{(af,f/a,bg,g/b)_\infty}\sum_{m=0}^{\infty}\frac{(\lambda/f,ab,ac,ad)_m}{(q,aq/f,\lambda a,abcd)_m}q^m$$
$$\cdot\,{}_4\phi_3\left[\begin{matrix}\mu/g, & bc, & bd, & abq^m\\ & bq/g, & \mu b, & abcdq^m\end{matrix};q\right]$$
$$+\kappa(a,g,c,d)\frac{(\lambda a,\lambda/a,\mu g,\mu/g)_\infty}{(af,f/a,bg,b/g)}\sum_{m=0}^{\infty}\frac{(\lambda/f,ag,ac,ad)_m}{(q,aq/f,\lambda a,acdg)_m}q^m$$
$$\cdot\,{}_4\phi_3\left[\begin{matrix}\mu/b, & gc, & gd, & agq^m\\ & gq/b, & \mu g, & acdgq^m\end{matrix};q\right]$$
$$+\kappa(f,b,c,d)\frac{(\lambda f,\lambda/f,\mu b,\mu/b)_\infty}{(af,a/f,bg,g/b)_\infty}\sum_{m=0}^{\infty}\frac{(\lambda/a,fb,fc,fd)_m}{(q,fq/a,\lambda f,bcdf)_m}q^m$$
$$\cdot\,{}_4\phi_3\left[\begin{matrix}\mu/g, & bc, & bd, & bfq^m\\ & bq/g, & \mu b, & bcdfq^m\end{matrix};q\right]$$
$$+\kappa(f,g,c,d)\frac{(\lambda f,\lambda/f,\mu g,\mu/g)_\infty}{(af,a/f,bg,b/g)_\infty}\sum_{m=0}^{\infty}\frac{(\lambda/a,fg,fc,fd)_m}{(q,fq/a,\lambda f,cdfg)_m}q^m$$
$$\cdot\,{}_4\phi_3\left[\begin{matrix}\mu/b, & gc, & gd, & gfq^m\\ & gq/b, & \mu g, & cdfgq^m\end{matrix};q\right].$$

From (2.3) we obtain (1.21) by applying Bailey's transformation formula between a very well-posed ${}_8\phi_7$ and two balanced ${}_4\phi_3$'s [**10**, 8.5 (3)].

3. q-Analogue of F_1. In this section we show that the r.h.s. of (1.21) gives a q-analogue of F_1. To see this, substitute the values for the parameters as in (1.20) into (1.21) to get

$$(3.1)\quad \begin{aligned} &S(q^{\alpha/2-1/4}, q^{\alpha/2+1/4}, -q^{\gamma/2-\alpha/2-1/4}, -q^{\gamma/2-\alpha/2+1/4}, \\ &\qquad\qquad \lambda q^{-\beta}, \mu q^{-\beta'}; \lambda, \mu) \\ &= \frac{2\pi\Gamma_q(\alpha)\Gamma_q(\gamma-\alpha)(-q^{1/2},-q)_{\gamma/2-1/2}(-q^{1/2},-q)_{\gamma/2-1}}{[\Gamma_q(1/2)]^2\Gamma_q(\gamma)(-\mu q^{\gamma/2+\alpha/2-1/4})_{-\beta'}} \\ &\cdot(-\mu q^{\gamma/2-\alpha/2-1/4}, \mu q^{\alpha/2-1/4}, \mu q^{\alpha/2+1/4})_{-\beta'} \\ &\cdot(\lambda q^{\alpha/2-1/4}, \lambda q^{1/4-\alpha/2})_{-\beta} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^\alpha)_{m+n}(q^\beta)_m(q^{\beta'})_n}{(q)_m(q)_n(q^\gamma)_{m+n}} \\ &\cdot\frac{(\mu q^{\alpha/2-\beta'-1/4}, -\mu q^{\gamma/2+\alpha/2-1/4}, -q^{\gamma/2})_m(-q^{\gamma/2-1/2})_{m+n}}{(\lambda^{-1}q^{\alpha/2+\beta+3/4}, \lambda q^{\alpha/2-1/4})_m(-\mu q^{\gamma/2+\alpha/2-\beta'-1/4}, \mu q^{\alpha/2-1/4})_{m+n}} \\ &\cdot\frac{(-\mu q^{\gamma/2+\alpha/2+m-5/4}, -\mu q^{\alpha/2-\gamma/2-1/4}, -q^{\gamma/2})_n}{(\mu q^{\alpha/2+1/4}, -\mu q^{\gamma/2-\alpha/2-1/4})_n} \\ &\cdot\frac{(1+\mu q^{\gamma/2+\alpha/2+m+2n-5/4})}{(1+\mu q^{\gamma/2+\alpha/2+m-5/4})}(-\mu q^{\gamma/2-\alpha/2-\beta'+1/4})^n q^m + L, \end{aligned}$$

where

$$(3.2)\quad \Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}, \quad 0<q<1, \quad \lim_{q\to 1-}\Gamma_q(x) = \Gamma(x),$$

see Askey **[8]**, and

$$(3.3)\quad \begin{aligned} L &= \frac{2\pi\Gamma_q(\gamma-\alpha)(1-q)^{1/2+\gamma-\alpha-\beta}(\lambda q^{1/2-\beta/2})_\infty}{\Gamma_q(1/2)\Gamma_q(\beta)(-\lambda^{-1}q^{\alpha/2+\beta-1/4})_\infty} \\ &\cdot\frac{(\lambda\mu q^{-\beta}, \mu q^{\alpha/2+1/4}, -\mu q^{\gamma/2-\alpha/2-1/4})_{-\beta'}}{(-\lambda\mu q^{\gamma/2-\beta})_{-\beta'}(\lambda q^{\alpha/2+1/4-\beta})_{\gamma-\alpha}} \\ &\cdot(-\lambda q^{-\beta/2}, -\lambda q^{1/2-\beta/2})_{\gamma/2-\alpha/2-\beta/2-1/4}(\lambda q^{-\beta/2})_{\gamma/2-\beta/2-1/4} \\ &\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda q^{1/4-\alpha/2}, -\lambda q^{\gamma/2-\alpha/2+1/4-\beta}, \lambda\mu q^{-\beta-\beta'})_m}{(q, \lambda q^{5/4-\alpha/2-\beta}, \lambda^2 q^{-\beta})_m} \\ &\cdot\frac{(-\lambda\mu q^{\gamma/2-\beta})_m(-\lambda q^{\gamma/2-\alpha/2-\beta-1/4}, \lambda q^{\alpha/2-\beta+1/4})_{m+n}}{(\lambda q^{\gamma-\alpha/2-\beta+1/4}, -\lambda\mu q^{\gamma/2-\beta-\beta'}, \lambda\mu q^{-\beta})_{m+n}} \\ &\cdot\frac{(-\lambda\mu q^{\gamma/2-\beta+m-1}, -\mu q^{\alpha/2-\gamma/2-1/4}, q^{\beta'}, -q^{\gamma/2})_n}{(q, \mu q^{\alpha/2+1/4}, -\mu q^{\gamma/2-\alpha/2-1/4})_n(1+\lambda\mu q^{\gamma/2-\beta-1+m})} \\ &\cdot(1+\lambda\mu q^{\gamma/2-\beta-1+m+2n})(-\mu q^{\gamma/2-\alpha/2+1/4-\beta'})^n q^m. \end{aligned}$$

Next we let $q \to 1-$ in (3.1) and we get

$$(3.4)\quad \begin{aligned} &2^{2\gamma-2}(1-\lambda)^{-2\beta}(1-\mu)^{-2\beta'} \\ &\cdot \int_0^1 z^{\alpha-1}(1-z)^{\gamma-\alpha-1}(1-uz)^{-\beta}(1-vz)^{-\beta'}dz \\ &= 2^{2\gamma-2}\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}(1-\lambda)^{-2\beta}(1-\mu)^{-2\beta'} \\ &\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[\alpha]_{m+n}[\beta]_m[\beta']_n}{m!n![\gamma]_{m+n}}u^m v^n + \lim_{q\to 1-} L, \end{aligned}$$

which upon simplifying gives

$$(3.5)\quad \begin{aligned} &\int_0^1 z^{\alpha-1}(1-z)^{\gamma-\alpha-1}(1-uz)^{-\beta}(1-vz)^{-\beta'}dz \\ &= \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F_1(\alpha;\beta,\beta';\gamma;u,v) \\ &\quad + 2^{2-2\gamma}(1-\lambda)^{2\beta}(1-\mu)^{2\beta'}\lim_{q\to 1-} L. \end{aligned}$$

Comparing (3.5) with (1.3) gives $\lim_{q\to 1-} L = 0$.

We have shown that $\lim_{q\to 1-} L = 0$ by the above reasoning, because it does not seem possible at this stage to take the limit using L'Hôpital's rule. It would be much nicer if we can prove it directly.

4. Transformations of (1.21). One of the advantages of the integral in (1.14) is the high degree of symmetry of the parameters. For example, if we interchange a and d, the l.h.s. of (1.21) does not change but the *r.h.s.* does. Doing so and using the same values for the parameters as in (1.20), then equating the result to the r.h.s. of (3.1) lead to

$$(4.1)\quad \begin{aligned} &\frac{2\pi\Gamma_q(\alpha)\Gamma_q(\gamma-\alpha)(-q^{1/2},-q)_{\gamma/2-1/2}}{[\Gamma_q(1/2)]^2\Gamma_q(\gamma)(-\mu q^{\gamma/2+\alpha/2-1/4})_{-\beta'}} \\ &\cdot(-q^{1/2},-q)_{\gamma/2-1}(\lambda q^{\alpha/2-1/4},\lambda q^{1/4-\alpha/2})_{-\beta} \\ &\cdot(-\mu q^{\gamma/2-\alpha/2-1/4},\mu q^{\alpha/2-1/4},\mu q^{\alpha/2+1/4})_{-\beta'} \\ &\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^\alpha)_{m+n}(q^\beta)_m(q^{\beta'})_n}{(q)_m(q)_n(q^\gamma)_{m+n}}q^m\cdot\frac{(-q^{\gamma/2-1/2})_{m+n}}{(-\mu q^{\gamma/2+\alpha/2-\beta'-1/4})_{m+n}} \end{aligned}$$

(continues)

(continued)

$$\cdot\frac{(\mu q^{\alpha/2-\beta'-1/4}, -\mu q^{\gamma/2+\alpha/2-1/4}, -q^{\gamma/2})_m}{(\mu q^{\alpha/2-1/4})_{m+n}(\lambda^{-1}q^{\alpha/2+\beta+3/4}, \lambda q^{\alpha/2-1/4})_m}$$
$$\cdot\frac{(-\mu q^{\gamma/2+\alpha/2+m-5/4}, -\mu q^{\alpha/2-\gamma/2-1/4}, -q^{\gamma/2})_n}{(\mu q^{\alpha/2+1/4}, -\mu q^{\gamma/2-\alpha/2-1/4})_n}$$
$$\cdot\frac{(1+\mu q^{\gamma/2+\alpha/2+m+2n-5/4})(-\mu q^{\gamma/2-\alpha/2-\beta'+1/4})^n}{(1+\mu q^{\gamma/2+\alpha/2+m-5/4})} + L$$
$$= \frac{2\pi\Gamma_q(\alpha)\Gamma_q(\gamma-\alpha)(-q^{1/2}, -q)_{\gamma/2-1/2}}{[\Gamma_q(1/2)]^2\Gamma_q(\gamma)(\mu q^{\gamma-\alpha/2+1/4})_{-\beta'}}$$
$$\cdot(-q^{1/2}, -q)_{\gamma/2-1}(-\lambda q^{\gamma/2-\alpha/2+1/4}, -\lambda q^{\alpha/2-\gamma/2-1/4})_{-\beta}$$
$$\cdot(-\mu q^{\gamma/2-\alpha/2+1/4}, -\mu q^{\gamma/2-\alpha/2-1/4}, \mu q^{\alpha/2+1/4})_{-\beta'}$$
$$\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^{\gamma-\alpha})_{m+n}(q^{\beta})_m(q^{\beta'})_n}{(q)_m(q)_n(q^{\gamma})_{m+n}}q^m\frac{(-q^{\gamma/2+1/2})_{m+n}}{(-\mu q^{\gamma/2-\alpha/2+1/4})_{m+n}}$$
$$\cdot\frac{(-q^{\gamma/2}, -\mu q^{\gamma/2-\alpha/2-\beta'+1/4}, \mu q^{\gamma-\alpha/2+1/4})_m}{(\mu q^{\gamma-\alpha/2-\beta'+1/4})_{m+n}(-\lambda^{-1}q^{\gamma/2-\alpha/2+\beta+5/4}, -\lambda q^{\gamma/2-\alpha/2+1/4})_m}$$
$$\cdot\frac{(\mu q^{\gamma-\alpha/2+m-3/4}, \mu q^{1/4-\alpha/2}, -q^{\gamma/2})_n}{(-\mu q^{\gamma/2-\alpha/2-1/4}, \mu q^{\alpha/2+1/4})_n}$$
$$\cdot\frac{(1-\mu q^{\gamma-\alpha/2+m+2n-3/4})(\mu q^{\alpha/2-\beta'-1/4})^n}{(1-\mu q^{\gamma-\alpha/2+m-3/4})} + G$$

where L is defined by (3.3) and G is an expression similar to L which goes to 0 as $q \to 1-$ in the same manner as L does. Taking the limit $q \to 1-$, (4.1) gives (1.4).

Next, if we look at L and G we see that each has the term $(\lambda/f)_\infty = (q^\beta)_\infty$. Hence if we let $\beta = 0$, (4.1) reduces to

$$(4.2)\quad {}_8\phi_7\left[\begin{matrix}-\mu q^{\gamma/2+\alpha/2-5/4}, & q\surd, -q\surd, & q^{\alpha}, & -q^{\gamma/2}, \\ & \surd, -\surd, & -\mu q^{\gamma/2-\alpha/2-1/4}, & \mu q^{\alpha/2-1/4},\end{matrix}\right.$$
$$\left.\begin{matrix}q^{\beta'}, & -q^{\gamma/2-1/2}, & -\mu q^{\alpha/2-\gamma/2-1/4} \\ -\mu q^{\gamma/2+\alpha/2-\beta'-1/4}, & \mu q^{\alpha/2+1/4}, & q^{\gamma}\end{matrix}; -\mu q^{\gamma-\alpha/2-\beta'+1/4}\right]$$

(continues)

(continued)

$$= \frac{(-\mu q^{\gamma/2-\alpha/2+1/4}, -\mu q^{\gamma-\alpha-1/4}, -\mu q^{\gamma/2+\alpha/2-1/4})_{-\beta'}}{(-\mu q^{\gamma/2-\alpha/2-1/4}, \mu q^{\alpha/2-1/4}, \mu q^{\gamma-\alpha/2+1/4})_{-\beta'}}$$

$$\cdot {}_8\phi_7 \left[\begin{matrix} \mu q^{\gamma-\alpha/2-3/4}, & q\surd, & -q\surd, & q^{\gamma-\alpha}, & q^{\beta'}, \\ & \surd, & -\surd, & \mu q^{\alpha/2+1/4}, & \mu q^{\gamma-\alpha/2-\beta'+1/4}, \end{matrix} \right.$$

$$\left. \begin{matrix} -q^{\gamma/2+1/2}, & \mu q^{1/4-\alpha/2}, & -q^{\gamma/2} \\ -\mu q^{\gamma/2-\alpha/2-1/4}, & q^{\gamma}, & -\mu q^{\gamma/2-\alpha/2+1/4} \end{matrix} ; \mu q^{\alpha/2-\beta'-1/4} \right],$$

which is actually a special case of Bailey's transformation

$$(4.3)\quad {}_8\phi_7 \left[\begin{matrix} a, & q\surd, & -q\surd, & b, & c, & d, \\ & \surd, & -\surd, & aq/b, & aq/c, & aq/d, \end{matrix} \right.$$

$$\left. \begin{matrix} e, & f \\ aq/e, & aq/f \end{matrix} ; a^2q^2/bcdef \right]$$

$$= \frac{(aq, aq/ef, a^2q^2/bcde, a^2q^2/bcdf)_\infty}{(aq/e, aq/f, a^2q^2/bcd, a^2q^2/bcdef)_\infty}$$

$$\cdot {}_8\phi_7 \left[\begin{matrix} a^2q/bcd, & q\surd, & -q\surd, & aq/bc, & aq/bd, & aq/cd, \\ & \surd, & -\surd, & aq/d, & aq/c, & aq/b, \end{matrix} \right.$$

$$\left. \begin{matrix} e, & f \\ a^2q^2/bcde, & a^2q^2/bcdf \end{matrix} ; aq/ef \right]$$

a limiting case of **[20**, (3.4.2.4)]. As $q \to 1-$, (4.2) goes to (1.5) upon some simplification and relabeling of the parameters.

Equation (4.2) is not the only q-analogue of (1.5), F.H. Jackson gave the following analogue **[11]**

$$(4.4)\qquad {}_2\phi_1 \left[\begin{matrix} \alpha, & \beta \\ & \gamma \end{matrix} ; x \right] = \frac{(\beta x)_\infty}{(x)_\infty} \, {}_2\phi_2 \left[\begin{matrix} \gamma/\alpha, & \beta \\ \gamma, & \beta x \end{matrix} ; \alpha x \right].$$

But in view of our result in **[15]**, it is not surprising that q-analogues of ${}_2F_1$'s in terms of ${}_8\phi_7$'s should exist.

Furthermore, if we apply (4.3) to the r.h.s. of (4.2), we get

$$(4.5)\quad {}_8\phi_7\left[\begin{matrix}-\mu q^{\gamma/2+\alpha/2-5/4}, & q\surd, & -q\surd, & q^{\alpha}, & -q^{\gamma/2-1/2},\\ & \surd, & -\surd, & -\mu q^{\gamma/2-\alpha/2-1/4}, & \mu q^{\alpha/2+1/4},\end{matrix}\right.$$
$$\left.\begin{matrix}q^{\beta'}, & -\mu q^{\alpha/2-\gamma/2-1/4}, & -q^{\gamma/2}\\ -\mu q^{\gamma/2+\alpha/2-\beta'-1/4}, & q^{\gamma}, & \mu q^{\alpha/2-1/4}\end{matrix}; -\mu q^{\gamma-\alpha/2-\beta'+1/4}\right]$$
$$= \frac{(-\mu q^{\gamma/2-\alpha/2+1/4}, -\mu q^{\gamma-\alpha-1/4}, -\mu q^{\gamma/2+\alpha/2-1/4})_{-\beta'}}{(-\mu q^{\gamma/2-\alpha/2-1/4}, \mu q^{\alpha/2-1/4}, \mu q^{\gamma-\alpha/2+1/4})_{-\beta'}}$$
$$\cdot \frac{(-\mu q^{\gamma/2+\alpha/2-\beta'-1/4})_{\gamma-\alpha}}{(\mu q^{\alpha/2-\beta'-1/4}, \mu q^{\gamma/2+1/4})_{\gamma-\alpha}(-\mu q^{\gamma/2-\alpha/2+1/4})_{\alpha-\gamma}}$$
$$\cdot\ {}_8\phi_7\left[\begin{matrix}-\mu q^{3\gamma/2-\alpha/2-\beta'-5/4}, & q\surd, & -q\surd, & q^{\gamma-\alpha},\\ & \surd, & -\surd, & -\mu q^{\gamma/2+\alpha/2-\beta'-1/4},\end{matrix}\right.$$
$$\begin{matrix}q^{\gamma-\beta'}, & -q^{\gamma/2-1/2},\\ -\mu q^{\gamma/2-\alpha/2-1/4}, & \mu q^{\gamma-\alpha/2-\beta'+1/4},\end{matrix}$$
$$\left.\begin{matrix}-\mu q^{\gamma/2-\alpha/2-\beta'-1/4}, & -q^{\gamma/2}\\ q^{\gamma}, & \mu q^{\gamma-\alpha/2-\beta'-1/4}\end{matrix}; -\mu q^{\alpha/2-\gamma/2+1/4}\right],$$

which is another q-analogue of Euler's relation **[10**, 1.2 (2)**]**

$$(4.6)\qquad {}_2F_1(\alpha, \beta'; \gamma; z) = (1-z)^{\gamma-\alpha-\beta'}\ {}_2F_1(\gamma-\alpha, \gamma-\beta'; \gamma; z),$$

the other one being Heine's

$$(4.7)\qquad {}_2\phi_1\left[\begin{matrix}\alpha, & \beta'\\ & \gamma\end{matrix}; z\right] = \frac{(\alpha\beta' z/\gamma)_\infty}{(z)_\infty}\ {}_2\phi_1\left[\begin{matrix}\gamma/\alpha, & \gamma/\beta'\\ & \gamma\end{matrix}; \alpha\beta' z/\gamma\right],$$

see **[10**, 8.4 (2)**]**.

5. q-Analogue of (1.7). To obtain a q-analogue of the transformation (1.7) from F_1 to F_3, we make use of (4.3) by applying it to the r.h.s. of (1.21) and thus get a relation between four double sums which as $q \to 1-$ will give (1.7). It turns out that we do not actually need the four-term relation and that if we consider only the first double sum on the r.h.s. of (1.21) with the values of the parameters as in (1.20) we get by (4.3)

$$(5.1)\quad \sum_{m=0}^{\infty} \frac{(q^\alpha, q^\beta, -q^{\gamma/2-1/2}, -q^{\gamma/2}, \mu q^{\alpha/2-\beta'-1/4})_m}{(q, q^\gamma, \lambda^{-1} q^{\alpha/2+\beta+3/4}, \lambda q^{\alpha/2-1/4}, \mu q^{\alpha/2-1/4})_m} \cdot \frac{(-\mu q^{\gamma/2+\alpha/2-1/4})_m}{(-\mu q^{\gamma/2+\alpha/2-\beta'-1/4})_m} q^m$$

$$\cdot\, {}_8\phi_7 \left[\begin{matrix} -\mu q^{\gamma/2+\alpha/2-5/4+m}, & q\surd, & -q\surd, & -\mu q^{\alpha/2-\gamma/2-1/4}, & q^{\beta'}, & q^{\alpha+m}, & -q^{\gamma/2}, & -q^{\gamma/2-1/2+m} \\ & \surd, & -\surd, & q^{\gamma+m}, & -\mu q^{\gamma/2+\alpha/2-\beta'-1/4+m}, & -\mu q^{\gamma/2-\alpha/2-1/4}, & \mu q^{\alpha/2-1/4+m}, & \mu q^{\alpha/2+1/4} \end{matrix} ; -\mu q^{\gamma/2-\alpha/2-\beta'+1/4} \right]$$

$$= \frac{(-\mu q^{\gamma/2+\alpha/2-1/4}, -\mu q^{\gamma/2-\alpha/2+1/4})_{-\beta'}}{(\mu q^{\alpha/2-1/4}, \mu q^{\gamma-\alpha/2+1/4})_{-\beta'}}$$

$$\cdot \sum_{m=0}^{\infty} \frac{(q^\alpha, q^\beta, -q^{\gamma/2-1/2}, -q^{\gamma/2})_m}{(q, q^\gamma, \lambda^{-1} q^{\alpha/2+\beta+3/4}, \lambda q^{\alpha/2-1/4})_m} q^m$$

$$\cdot\, {}_8\phi_7 \left[\begin{matrix} \mu q^{\gamma-\alpha/2-3/4}, & q\surd, & -q\surd, & q^{\beta'}, & q^{\gamma-\alpha}, & -q^{\gamma/2}, & -q^{\gamma/2+1/2}, & \mu q^{1/4-\alpha/2-m} \\ & \surd, & -\surd, & \mu q^{\gamma-\alpha/2-\beta'+1/4}, & \mu q^{\alpha/2+1/4}, & -\mu q^{\gamma/2-\alpha/2+1/4}, & -\mu q^{\gamma/2-\alpha/2-1/4}, & q^{\gamma+m} \end{matrix} ; \mu q^{\alpha/2-\beta'+m-1/4} \right]$$

which is a q-analogue of (1.7).

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