TIME-HARMONIC SOLUTIONS OF SOME DISSIPATIVE PROBLEMS FOR MAXWELL'S EQUATIONS IN A THREE-DIMENSIONAL HALF SPACE

John R. Schulenberger

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In constructing solutions of steady-state wave propagation problems, a common procedure is to assume that the frequency has a small imaginary part and, with an eventual solution in hand, to let this imaginary part go to zero — the principle of limiting absorption. There are three basic problems involved here. The first is to establish the principle of limiting absorption itself, i.e., to show in a rigorous manner that a steady-state solution can actually be obtained in this fashion. The second problem is to find a class of functions in which the solution so constructed is unique (a "radiation condition"). While in problems in exterior domains or with bounded perturbations of the coefficients uniqueness classes are essentially dictated by the asymptotic behavior at large spatial distances of the free-space Green functions, in the problems with infinite boundary considered below it is not immediately clear how to specify uniqueness classes. For example, must the asymptotic behavior at large distances of eventual surface-wave components of the solution be included in the conditions designating such classes? Finally, since, strictly speaking, steady-state solutions are physically meaningless (they fail to have finite energy), a third problem is to determine in what sense they are approximations for large times to the original time-dependent solutions—the principle of limiting amplitude. In this paper we study these questions for the steady-state versions of the dissipative problems for Maxwell's equations considered previously. While these problems are particular examples, the results obtained do provide a guide and generalize to other problems. For example, although the equations of elasticity are much more difficult to deal with, the steady-state Rayleigh surface wave of elasticity theory has basically the same properties as the surface wave in the selfadjoint Leontovich case considered below. As far as we know, questions of uniqueness for steady-state wave propagation problems in domains with infinite boundary have not been considered previously, although Eidus has recently established the principle of limiting absorption for some such domains. Unfortunately, the abstract approach used there apparently does not provide enough information to make it possible to define uniqueness classes for the solutions so constructed.

0. Introduction. To formulate the problem, let $E = \mathrm{diag}(\varepsilon I_3, \mu I_3)$ be the diagonal 6×6 matrix with diagonal elements ε, μ which are the electromagnetic constants of a homogeneous, isotropic, loss-free

medium filling the halfspace $R_+^3 = \{x \in R^3 : x_3 > 0\}$. Let $A(D)$, $D_j = -i\partial_j$, $j = 1, 2, 3$, be the 6×6 matrix operator

$$A(D) = \sum_{j=1}^3 A_j D_j = \begin{bmatrix} 0 & i \text{ rot} \\ -i \text{ rot} & 0 \end{bmatrix}, \qquad \text{rot} = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}.$$

With $\Lambda(D) = E^{-1}A(D)$ we now formulate the problem with the Leontovich boundary condition of finding time-harmonic solutions of

(0.2) $i\partial_t u(x,t) - \Lambda(D)u(x,t) = -f(x)\exp(-i\nu t), \qquad x \in R_+^3,$

$u_1(x',0,t) + \alpha u_5(x',0,t) = u_2(x',0,t) - \alpha u_4(x',0,t) = 0,$

$x' = (x_1, x_2) \in R^2, \quad t > 0, \quad \alpha \in C, \quad \operatorname{Re}\alpha \geq 0, \quad \nu \neq 0,$

$$f \in C_0^\infty(R_+^3, C^6),$$

(0.3) $i\partial_t u' + \Lambda(D)u'(x,t) = f(x)\exp(i\nu t), \qquad x \in R_+^3,$

$u_1'(x',0,t) - \overline{\alpha}u_5'(x',0,t) = u_2'(x',0,\overline{t}) + \overline{\alpha}u_4'(x',0,t) = 0.$

If $v(x)$, $v'(x)$ are solutions of

$$[\Lambda(D) - \nu I]v(x) = f(x), \qquad [\Lambda(D) - \nu I]v'(x) = f(x)$$

satisfying the respective boundary conditions indicated in (0.2), (0.3), then the functions

$$u(x,t) = \exp(-i\nu t)v(x), \qquad u'(x,t) = \exp(i\nu t)v'(x)$$

will be the desired time-harmonic solutions.

After recalling the required information from [1] in §1, we justify the principle of limiting absorption and construct such solutions in §2; we then determine their asymptotic behavior for $|x| \to \infty$ in R_+^3. In §3 we prove uniqueness of these solutions in appropriate "radiation" classes. It turns out that in the nonselfadjoint case ($\alpha_1 > 0$) we are able to prove uniqueness only for solutions $v(x)$, $v'(x)$ for which $\exp(-i\nu t)v(x)$, $\exp(i\nu t)v'(x)$ behave for large $|x|$ like outgoing hemispherical waves in R_+^3. We show via the limiting-amplitude principle that these are precisely the physically interesting solutions. Further, it is found that the asymptotic behavior of the surface-wave component of the solution must be included in the designation of a uniqueness class only in the selfadjoint Leontovich case $\alpha = i\alpha_2$, $\alpha_2 \neq 0$, i.e., when the surface modes from which the solution is formed have real

frequencies. Designation of uniqueness classes in this case is more difficult (and more interesting), since asymptotically spatial waves are hemispherical waves decaying like $|x|^{-1}$ while surface waves are cylindrical waves decaying only like $|x|^{-1/2}$ along the boundary $\{x_3 = 0\}$.

As mentioned above, versions of the limiting-absorption principle have been widely used. We note that for hyperbolic systems with elliptic spatial part the principle in a half space was established in a rigorous manner for selfadjoint problems by Matsumura [4]. For the same such systems the principle of limiting amplitude was established by Wakabayashi [10].

1. Background. In this section we recall the information from [2] needed below and record the principle of stationary phase in the form required in the next section. Below $^t M$ denotes the transpose of a matrix M, $^{\bar{t}} M$ denotes the conjugate transpose, and the adjoint of a matrix or operator is denoted by M^*.

With respect to the E inner product (see (0.2)) in C^6, $(\alpha, \beta) = {}^{\bar{t}}\alpha E \beta$, the symbol $\Lambda(\eta)$, $\eta \in R^3 \setminus \{0\}$ of the operator $\Lambda(D)$ of (0.2),

$$(1.1) \quad \Lambda(\eta) = \begin{bmatrix} 0 & -\varepsilon^{-1}\eta\wedge \\ \mu^{-1}\eta\wedge & 0 \end{bmatrix}, \qquad \eta\wedge = \begin{bmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{bmatrix}$$

is selfadjoint with distinct eigenvalues $\lambda_j(\eta) = jc|\eta|$, $j = 0, \pm 1$, $c^2 = (\varepsilon\mu)^{-1}$, each of multiplicity two. The resolution of the identity for $\Lambda(\eta)$ is

$$(1.2) \qquad I = P_0(\omega) + P_1(\omega) + P_{-1}(\omega)$$
$$\Lambda(\eta) = \lambda_1(\eta)P_1(\omega) + \lambda_{-1}(\eta)P_{-1}(\omega), \qquad \omega = \eta/|\eta|,$$

where the $P_k(\omega)$ are mutually orthogonal orthoprojectors with respect to the E inner product:

$$(1.3) \qquad {}^{\bar{t}}(EP_k) = EP_k, \qquad \delta_{jk}P_k = P_j P_k, \qquad k, j = 0, \pm 1,$$
$$P_0(\omega) = \begin{bmatrix} {}^t\omega\omega & 0 \\ 0 & {}^t\omega\omega \end{bmatrix},$$
$$P_j(\omega) = 2^{-1}\begin{bmatrix} -\omega\wedge\omega\wedge & -j\mu c\omega\wedge \\ j\varepsilon c\omega\wedge & -\omega\wedge\omega\wedge \end{bmatrix}$$
$$\omega\wedge\omega\wedge = (\omega\wedge)^2, \qquad \omega = \eta/|\eta| \in S^2, \qquad j = \pm 1.$$

In the space $\mathscr{H}$ of functions f, $g \in L_2(R_+^3, C^6)$ with linear product

$$(f, g) = \int_{R_+^3} {}^{\bar{t}} f E G$$

the operators Λ, Λ^* defined as the closure of $\Lambda(D)$ on $\{f \in C_0^\infty(\overline{R}_+^3, C^6):$ $Bf(x', 0) = 0$, $x' = (x_1, x_2)\}$, $\{f \in C_0^\infty(\overline{R}_+^3, C^6): B'f(x', 0) = 0\}$, where

(1.4) $Bf(x', 0) = {}^t(f_1 + \alpha f_5, f_2 - \alpha f_4)(x', 0^+),$

(1.4') $B'f(x', 0) = {}^t(f_1 - \overline{\alpha} f_5, f_2 + \overline{\alpha} f_4(x', 0^+),$

and $\alpha_1 + i\alpha_2$, $\alpha_1 > 0$, $\alpha_2 \neq 0$. The first case is that of the classical boundary condition [8], while the second is the case considered in [3]. In both these cases Λ is selfadjoint, and the spectrum of Λ, $\sigma(\Lambda)$ consists of the entire real line R. In the third case studied in [2] (which includes the second case) the spectrum of Λ consists of three parts:

(1.5) $\overline{\sigma}(\Lambda^*) = \sigma(\Lambda) = R \cup \{m\} \cup \{e\},$

$\{m\} = \{m(\lambda) = -p\lambda/\varepsilon, \ \lambda \in (0, \infty)\},$

$\{e\} = \{e(\lambda) = cq\lambda, \ \lambda \in (0, \infty)\},$

$p = p_1 + ip_2, \quad p_2 \geq 0, \quad \alpha_2 p_1 > 0, \quad p^2 = 1/(\mu\varepsilon^{-1} - \alpha^2),$

$q = q_1 + iq_2, \quad q_2 \leq 0, \quad \alpha_2 q_1 > 0, \quad q^2 = -\alpha^2 p^2.$

In the general case, the spectrum thus consists of R plus two lines in the lower half plane issuing from the origin to the left and right of the negative imaginary axis.

To points of R there correspond generalized eigenfunctions of $\Lambda(D)$, $\Lambda^*(D)$ satisfying $\Lambda(D)\Psi_j(x, \eta)_\pm = \lambda_j(\eta)\Psi_j(x, \eta)_\pm$, $\Lambda^*(D)\Psi_j'(x, \eta)_\pm = \lambda_j(\eta)\Psi_j'(x, \eta)_\pm$, $j = \pm 1$, and the boundary conditions (1.4), (1.4') given by

(1.6) $\Psi_j(x, \eta)_\pm$

$= (2\pi)^{-3/2} \chi_{\mp j}(\rho) e^{ix'\xi} [e^{i\rho x_3} P_j(\omega) - e^{-i\rho x_3} C_j(\omega) P_j(\omega)] E^{-1},$

$\Psi_j'(x, \eta)_\pm$

$= (2\pi)^{-3/2} \chi_{\mp j}(\rho) e^{ix'\xi} [e^{i\rho x_3} P_j(\omega) - e^{-i\rho x_3} C_j'(\omega) P_j(\omega)] E^{-1}$

where we have written $\eta = (\xi, \rho) \in R^3$, $\chi_{\mp j}$ are the characteristic functions of the half spaces $R_{\mp j}^3$, $j = \pm 1$ (e.g., $R_{\mp j}^3 = R_\pm$ for $j = -1$),

and

$$(1.7) \quad C(\omega) = \Delta(\omega)^{-1}[\varepsilon c\omega_3(\alpha^2 - \mu\varepsilon^{-1})Q + \alpha c(\omega')],$$
$$C'(\omega) = \Delta'(\omega)^{-1}[\varepsilon c\omega_3(\mu\varepsilon^{-1} - \overline{\alpha}^2)Q + \overline{\alpha}c(\omega')],$$
$$\Delta(\omega) = (\mu c - \alpha\omega_3)(\alpha\varepsilon c - \omega_3),$$
$$\Delta'(\omega) = (\mu c + \overline{\alpha}\omega_3)(\overline{\alpha}\varepsilon c + \omega_3),$$
$$Q = \mathrm{diag}[1, 1, -1, -1, -1, 1],$$
$$c(\omega') = \begin{bmatrix} 0 & \tilde{c}(\omega') \\ \tilde{c}(\omega') & 0 \end{bmatrix},$$
$$\tilde{c}(\omega') = \begin{bmatrix} \omega_2^2 - \omega_1^2 & -2\omega_1\omega_2 & 0 \\ -2\omega_1\omega_2 & \omega_2^2 - \omega_1^2 & 0 \\ 0 & 0 & |\omega'|^2 \end{bmatrix},$$
$$C_j(\omega) = C(\omega', j\omega_3), \qquad C'_j(\omega) = C'(\omega', j\omega_3).$$

We observe that

$$(1.8) \quad {}^tC_j(\omega) = C'_j(\tilde{\omega}), \qquad C_j(\omega)C_j(\tilde{\omega}) = C_j(\tilde{\omega})C_j(\omega) = I,$$
$$\tilde{\omega} = (\omega', -\omega_3),$$
$$C_j(\omega)P_j(\omega) = P_j(\tilde{\omega})C_j(\omega),$$
$$C'_j(\omega)P_j(\omega) = P_j(\tilde{\omega})C'_j(\omega), \qquad j = \pm 1.$$

To points of $\{e\}$ and $\{m\}$ there correspond generalized eigen-functions $\Sigma_S(x,\xi)$, $\Sigma'_S(x,\xi)$ satisfying $\Lambda(D)\Sigma_S(x,\xi) = s(\xi)\Sigma_S(x,\xi)$, $\Lambda^*(D)\Sigma'_S(x,\xi) = \overline{s}(\xi)\Sigma'_S(x,\xi)$, $S = E, M$, $s = e, m$, $e(\xi) = \varepsilon q|\xi|$, $m = -p|\xi|/\varepsilon$, and the boundary conditions (1.4), (1.4') given by

$$(1.9)$$
$$\Sigma_E(x,\xi) = (2\pi)^{-1}\mu\beta_E|\xi|^{-7/2}\exp[ix'\xi + i\tau_e(\xi)x_3]\mathscr{E}\,{}^t\mathscr{E}',$$
$$\Sigma_M(x,\xi) = (2\pi)^{-1}\varepsilon\beta_M|\xi|^{-7/2}\exp[ix'\xi + i\tau_m(\xi)x_3]\mathscr{M}\,{}^t\mathscr{M}',$$
$$\Sigma'_E(x,\xi) = (2\pi)^{-1}\mu\overline{\beta}_E|\xi|^{-7/2}\exp[ix'\xi - i\overline{\tau}_e(\xi)x_3]\overline{\mathscr{E}}\,{}^t\overline{\mathscr{E}},$$
$$\Sigma'_M(x,\xi) = (2\pi)^{-1}\mu\overline{\beta}_M|\xi|^{-7/2}\exp[ix'\xi - i\overline{\tau}_m(\xi)x_3]\overline{\mathscr{M}}\,{}^t\overline{\mathscr{M}},$$
$$\mathscr{E}(\xi) = {}^t(\xi_2 e(\xi), -\xi_1 e(\xi), 0, -\alpha^{-1}\xi_1 e(\xi), -\alpha^{-1}\xi_2 e(\xi), -\mu^{-1}|\xi|^2),$$
$$\mathscr{M}(\xi) = {}^t(\alpha\xi_1 m(\xi), \alpha\xi_2 m(\xi), \varepsilon^{-1}|\xi|^2, \xi_2 m(\xi), -\xi_1 m(\xi), 0),$$
$$\mathscr{E}'(\xi) = {}^t(\xi_2 e(\xi), -\xi_1 e(\xi), 0, \alpha^{-1}\xi_1 e(\xi), \alpha^{-1}\xi_2 e(\xi), -\mu^{-1}|\xi|^2),$$
$$\mathscr{M}'(\xi) = {}^t(\alpha\xi_1 m(\xi), \alpha\xi_2 m(\xi), -\varepsilon^{-1}|\xi|^2, -\varepsilon_2 m(\xi), \xi_1 m(\xi), 0),$$
$$\beta_E = (iq/2\alpha c\varepsilon)^{1/2}, \quad \beta_M = (-ip\alpha/2)^{1/2}, \quad \mathrm{Im}(\cdot)^{1/2} \geq 0,$$
$$\tau_m(\xi) = \alpha p|\xi|, \quad \tau_e(\xi) = -\alpha^{-1}\mu cq|\xi|, \quad \mathrm{Im}\,\tau_{m,e}(\xi) \geq 0.$$

Denoting by $\mathscr{K}_n$, $n = 2, 3$, the space of functions f, $g \in L_2(R^n, C^6)$ with the inner product $(f, g)_{\mathscr{K}_n} = \langle f, Eg \rangle$, the functions Ψ_j, Ψ'_j, Σ_S, Σ'_S, $S = E$, M, $j = \pm 1$, define bounded operators from $\mathscr{X}$ to $\mathscr{K}_n$ obtained by affixing the appropriate subscript and eventual prime to

$$(1.10) \qquad \Psi f(\eta) = \int_{R_+^3} {}^{\bar{\imath}}\Psi(x, \eta) E f(x)\, dx,$$

$$\Sigma f(\xi) = \int_{R_+^3} {}^{\bar{\imath}}\Sigma(x, \xi) E f(x)\, dx.$$

Their adjoints are

$$(1.11) \qquad \Psi^* g(x) = \int_{R^3} \Psi(x, \eta) E g(\eta)\, d\eta,$$

$$\Sigma^* h(x) = \int_{R^2} \Sigma(x, \xi) E h(\xi)\, d\xi.$$

The operators $\Pi_j = \Psi_j^* \Psi'_j$, $\Pi_S = \Sigma_S^* \Sigma'_S$, $j = \pm 1$, $S = E, M$, and their adjoints are bounded projections in $\mathscr{X}$ which are "orthogonal" in the sense that $0 = \Pi_j \Pi_{j'} = \Pi_j \Pi_S = \Pi_S \Pi_j = \Pi_E \Pi_M = \Pi_M \Pi_S$, $j \neq j' = 0, \pm 1$, $S = E, M$. Here Π_0 is the orthogonal projection onto the null space of Λ, $\mathscr{N}(\Lambda) = \mathscr{N}(\Lambda^*)$, defined by (see formula (A.9), p. 174 of [2])

(1.12) $\Phi_2 \Pi_0 f(\xi, x_3)$

$$= (2\pi)^{-1/2} \int_R \exp(i\rho x_3) P_0(\xi, \rho) \Phi_3 \chi_+ f(\xi, \rho)\, d\rho$$

$$- 2^{-1}(2\pi)^{1/2} \exp(-|\xi||x_3|) \widetilde{P}_0^r(\xi, -i|\xi|) \Phi_3 \chi_+ f(\xi, -i|\xi|)$$

where Φ_n, $n = 2, 3$, is the Fourier transform

$$\Phi_n f(\eta) = (2\pi)^{-n/2} \int_{R_n} \exp(-ix\eta) f(x)\, dx$$

with inverse $\Phi_n^* f(x) = \Phi_n f(-x)$, χ_+ is the characteristic function of the half space R_+^3, $P_0(\xi, \rho)$ is given in (1.3), and

$$(1.13) \qquad P_o^r(\xi, -i|\xi|) = \begin{bmatrix} {}^t d(\xi) \overline{d}(\xi) & 0 \\ 0 & {}^t d(\xi) \overline{d}(\xi) \end{bmatrix},$$

$$d(\xi) = (\xi_1, \xi_2, i|\xi|).$$

The semigroups $S(t) = \exp(-i\Lambda t)$, $S^*(t) = \exp(i\Lambda^* t)$ generated by Λ, Λ^* have the representations

$$(1.14) \quad S(t)f = \Pi_0 f + \Psi_1^* \exp(-ic|\cdot|t)\Psi_1' f + \Psi_{-1}^* \exp(ic|\cdot|t)\Psi_{-1}' f$$
$$+ \Sigma_E^* \exp(-ie(\cdot)t)\Sigma_E' f + \Sigma_M^* \exp(-im(\cdot)t)\Sigma_M' f,$$
$$S^*(t)f = \Pi_0 f + (\Psi_1')^* \exp(ic|\cdot|t)\Psi_1 f + (\Psi_{-1}')^* \exp(-ic|\cdot|t)\Psi_{-1} f$$
$$+ (\Sigma_E')^* \exp(i\overline{e}(\cdot)t)\Sigma_E f + (\Sigma_M')^* \exp(i\overline{m}(\cdot)t)\Sigma_M f$$

which for $t = 0$ are the Parseval identities for Λ, Λ^*. The last two terms on the right in (1.14) do not occur in the selfadjoint case $\alpha = 0$.

Suppose now that $R(s)$ is a smooth, matrix-valued function on the unit sphere $S^2 \ni s$, ω, $x = |x|\omega \in R_+^3$, $y \in R_+^3$, $r > 0$, and $\psi(r, y, s)$ is a smooth function. The principle of stationary phase in the form we require in the next section (see, e.g., [4], [9]) states that as $|x| \to \infty$, uniformly with respect to r in bounded intervals of R_+ and y in compact sets

$$(1.15) \quad \int_{S^2} \exp(irxs)\psi(r, y, s)R(s)\, dS$$
$$= (2\pi) \sum_{j=\pm 1} \{\exp(ijr|x| - ij\pi/2)\psi(r, y, j\omega)$$
$$R(j\omega)|rx|^{-1}\} + q(rx),$$
$$D^\alpha q(x) = O(|x|^{-2}) \text{ and smooth.}$$

For $R^2 \ni x' = |x'|\gamma$, $\gamma = (\cos\phi_x, \sin\phi_x)$, and $s = (\cos\phi, \sin\phi) \in S^1$ the corresponding statement is

$$(1.16) \quad \int_{S^1} \exp(irx's)\psi(r, y, s)R(s)\, d\phi$$
$$= (2\pi)^{1/2}|rx'|^{-1/2}$$
$$\times \sum_{j=\pm 1} \{\exp(ijr|x'| - ij\pi/4)\psi(r, y, j\gamma)R(j\gamma)\} + q(rx')$$
$$D^\alpha q(x') = O(|x'|^{-3/2}) \text{ and smooth.}$$

2. The limiting-absorption principle and asymptotics. It follows directly from (1.10)–(1.12), (1.14) that for $\zeta \notin \sigma(\Lambda)$, $f \in C_0^\infty(R_+^3, C^6)$

$$(2.1) \quad v(x, \zeta) = -\zeta^{-1}\Pi_0 f(x) + \sum_{j=\pm 1} \Psi_j^*[\lambda_j(\cdot) - \zeta]^{-1}\Psi_j' f(x)$$
$$+ \Sigma_E^*[e(\cdot) - \zeta]^{-1}\Sigma_E' f(x) + \Sigma_M^*[m(\cdot) - \zeta]^{-1}\Sigma_M' f(x)$$
$$\equiv v_0(x; \zeta) + v_{-1}(x; \zeta)$$
$$+ v_1(x; \zeta) + v_E(x; \zeta) + v_M(x; \zeta),$$

$(2.1')$ $v'(x;\overline{\zeta}) = -\overline{\zeta}^{-1}\Pi_0 f(x) + \sum_{j=\pm 1} (\Psi_j')^* [\lambda_j(\eta) - \overline{\zeta}]^{-1}\Psi_j f(x)$

$$+ (\Sigma_E')^* [e(\cdot) - \overline{\zeta}]^{-1}\Sigma_E f(x)$$
$$+ (\Sigma_M')^* [m(\cdot) - \overline{\zeta}]^{-1}\Sigma_M f(x)$$
$$\equiv v_0(x;\overline{\zeta}) + v_{-1}'(x;\overline{\zeta}) + v_1'(x;\overline{\zeta}) + v_E'(x;\overline{\zeta}) + v_M'(x;\overline{\zeta})$$

are solutions of

(2.2) $[\Lambda(D) - \zeta I]v(x;\zeta) = f(x)$, $Bv(x',0;\xi) = 0$,

(2.2') $[\Lambda(D) - \overline{\zeta}I]v'(x;\overline{\zeta}) = f(x)$, $B'v'(x',0;\overline{\xi}) = 0$.

We now take $\zeta = \nu \pm i\varepsilon$, $\nu \neq 0$, $\varepsilon \in (0, \varepsilon_0]$ (with ε_0 so small that $\nu - i\varepsilon_0$ is not in $\sigma(\Lambda)$ in the case $-i\varepsilon$) and pass to the limit $\varepsilon \to 0$ to obtain solutions $v_\pm(x,\nu) = v(x;\nu \pm i0)$, $v_\pm'(v;\nu) = v'(x;\nu \pm i0)$ of

(2.3) $[\Lambda(D) - \nu I]v(x) = f(x)$, $x \in R_+^3$, $Bv(x',0) = 0$,

(2.3') $[\Lambda(D) - \nu I]v'(x) = f(x)$, $x \in R_+^3$, $B'v'(x',0) = 0$.

This forms the content of the limiting-absorption principle. We then determine the asymptotic behavior as $|x| \to \infty$ of each component of the solutions $v_\pm$, $v_\pm'$. This provides the means of specifying uniqueness classes for them.

1. $v_0(x;\nu \pm i0) = -\nu^{-1}\Pi_0 f(x)$. We consider $f \in C_0^\infty(R_+^3, C^6)$ extended by zero to R^3. From (1.12)

(2.4) $\Pi_0 f(x) = (2\pi)^{-3/2}\Phi_3^* P_0\Phi_3 f(x) - 2^{-1}(2\pi)^{-1/2}R(x)$,

$$R(x) = \int_{R_2} \exp(ix'\xi - |\xi||x_3|)|\xi|^{-1}\widetilde{P}_0^r(\xi, -i|\xi|)\phi(\xi)\,d\xi,$$

$$\phi(\xi) = \int_0^\infty \exp(-|\xi|y_3)\Phi_2 f(\xi, y_3)\,dy_3.$$

The first term of $\Pi_0 f(x)$ is a singular integral operator, so (cf. [6], [7])

(2.5) $(2\pi)^{-3/2}\Phi_3^* P_0\Phi_3 f(x) = O(|x|^{-3})$, $|x| \to \infty$.

For $x_3 = 0$ we obtain by integration by parts

(2.6) $|x'|^2|R(x',0)| \leq c(f) < \infty$.

Since $R(x)$ is a harmonic function, (2.6) implies that it can be represented in R^3 by Poisson's formula, but this does not give an estimate which is uniform in $\omega_3 = x_3/|x|$. Proceeding directly from (1.12),

(1.13), (2.4), we have for the *i*th column of R, $x_3 > 0$, supp $f(x) \subset K \times [a, b]$, $0 < a < b < \infty$, $K \subset R^2$ bounded, $i = 1, \dots, 6$,

$$|x_3^2 R_i(x)| \le c(f) x_3^2 \int_0^\infty \exp[-|\xi|(x_3 + y_3)]|\xi|^2 \, d|\xi| \, dy_3$$

$$= c(f) x_3^2 \partial_3^2 \int_a^b (x_3 + y_3)^{-1} \, dy_3$$

$$= c(f)[x_3^2/(x_3 + a)^2 - x_3^2/(x_3 + b)^2] \le c(f) < \infty.$$

Now

$$|x'|^2 |R(x)| = \int_{R_2} [(-\Delta_\xi) \exp(ix'\xi - |\xi|x_3)]|\xi|^{-1} \widetilde{P}_0^r(\xi, -i|\xi|)\phi(\xi) \, d\xi$$

$$- 2i \sum_{k=1}^2 x_k x_3 \int_{R^2} \{\exp(ix'\xi - |\xi|x_3)\xi_k|\xi|^{-2}$$

$$\times \widetilde{P}_0^r(\xi, -i|\xi|)\}\phi(\xi) \, d\xi$$

$$- x_3 \int_{R^2} \exp(ix'\xi - |\xi|x_3)|\xi|^{-2}\widetilde{P}_0^r(\xi, -i|\xi|)\phi(\xi) \, d\xi$$

$$+ x_3^2 R(x)$$

$$\equiv \sum_{k=1}^5 r_j(x).$$

From (2.7) $|r_5(x)| \le c(f) < \infty$. Integration by parts in $r_1(x)$ gives $|r_1(x)| \le c(f) < \infty$. Just as in (2.7)

$$|r_4(x)| \le c(f) x_3 \int_a^b \int_0^\infty \exp[-|\xi|(x_3 + y_3)]|\xi| \, d|\xi| \, dy_3$$

$$= c(f)[x_3/(a + x_3) - x_3/(b + x_3)] < c(f) < \infty.$$

Finally,

$$r_2(x) + r_3(x) = -2x_3 \sum_{k=1}^2 \int_{R^2} [\partial_{\xi_k} \exp(ix'\xi - |\xi|x_3)]\xi_k|\xi|^{-2}$$

$$\times \widetilde{P}_0^r(\xi, -i|\xi|)\phi(\xi) \, d\xi - 2x_3^2 R(x)$$

$$= -2x_3 \sum_{k=1}^2 \int_{R^2} \exp(ix'\xi - |\xi|x_3)\xi_k|\xi|^{-2}$$

$$\times \widetilde{P}_0^r(\xi, -i|\xi|)(\xi, -i|\xi|)\phi(\xi) \, d\xi$$

$$+ 2x_3 \sum_{k=1}^2 \int_{R^2} \exp(ix'\xi - |\xi|x_3)\xi_k|\xi|^{-2}$$

$$\times \partial_{\xi_k} \widetilde{P}_0^r(\xi, -i|\xi|)\phi(\xi) \, d\xi - 2x_3^2 R(x).$$

The last term has already been estimated in (2.7); the estimate for the first four terms is essentially the same as for $r_4(x)$ above. Thus, from (2.7), (2.7′) we have $|x|^2 R(x) \leq c(f) < \infty$. From (2.5) and similar estimates for $D^\beta v_0$ we thus have

$$(2.8) \quad \nu |x|^2 |D^\beta v_0(x; \nu \pm i0)| \leq c(f) < \infty,$$
$$|x| \to \infty, \quad x_3 \geq 0, \quad |\beta| \geq 0.$$

2. $v_1(x; \nu \pm i0)$. Suppose first that $\nu > 0$. From (1.6), (1.8), (1.10)

$$(2.9) \quad \Psi_1' f(\eta) = \chi_\mp(\rho) \check{f}_1(\eta), \quad \check{f}_1(\eta) = P_1(\eta)[\Phi_3 f(\eta) - C_1(\tilde{\eta}) \Phi_3 f(\tilde{\eta})],$$
$$C_1(\eta) \check{f}_1(\eta) = -\check{f}_1(\tilde{\eta}), \quad \tilde{\eta} = (\xi, -\rho).$$

Hence, from (1.11), (2.1), (2.9)

$$(2.10) \quad D^\beta v_1(x; \zeta) = D^\beta \Psi_1^* [\lambda_1(\eta) - \zeta]^{-1} \Psi_1' f(\eta)$$
$$= (2\pi)^{-3/2} \int_{R^3} [\exp(ix\eta)\eta^\beta - \exp(ix\tilde{\eta})\tilde{\eta}^\beta C_1(\eta)]$$
$$\times [c|\eta| - \zeta]^{-1} \chi_\mp(\rho) \check{f}_1(\eta) \, d\eta$$
$$= (2\pi)^{3/2} \int_{R^3} \exp(ix\eta)\eta^\beta [c|\eta| - \zeta]^{-1} \check{f}_1(\eta) \, d\eta.$$

Suppose now that $\nu \in (\nu_0 - \delta, \nu_0 + \delta)$, $\nu_0 - \delta > 0$, and let $\psi \in C_0^\infty(R)$ with supp $\psi \subset \{r : |cr - \nu_0| \leq 4\delta\} \subset (0, \infty)$, $\psi(r) = 1$ for $|cr - \nu_0| \leq 3\delta$. Define $\chi \in C_0^\infty(R^3/\{0\})$ by $\chi(\eta) \equiv \chi(rs) = \psi(r)$, $s \in S^2$. Then for $\zeta = \nu \pm i\varepsilon$ with $\varepsilon \in (0, \varepsilon_0]$ from (2.10)

$$(2.11) \quad D^\beta v_1(x; \zeta) = (2\pi)^{-3/2} \int_{R^3} \exp(ix\eta)\eta^\beta [c|\eta| - \zeta]^{-1} \chi(\eta) \check{f}_1(\eta) \, d\eta$$
$$+ (2\pi)^{-3/2} \int_{R^3} \exp(ix\eta)\eta^\beta [c|\eta| - \zeta]^{-1}$$
$$\times [1 - \chi(\eta)] \check{f}_1(\eta) \, d\eta$$
$$\equiv I(x; \zeta) + J(x; \zeta).$$

From (2.9), (2.11)

$$J(x; \zeta) = \Phi_3^* P_1 \Phi_3 g(x) - \Phi_3^* P_1 C_1(\ddot{\,}) \Phi_3 \tilde{g}(x),$$
$$\Phi_3 g(\eta) = \eta^\beta [c|\eta| - \zeta]^{-1} [1 - \chi(\eta)] \Phi_3 f(\eta),$$
$$\Phi_3 \tilde{g}(\eta) = \eta^\beta [c|\eta| - \zeta]^{-1} [1 - \chi(\eta)] \Phi_3 f(\tilde{\eta}).$$

Since $P_1(\eta)$ and $P_1(\eta)C_1(\tilde{\eta})$ are homogeneous of degree zero, $J(x;\zeta)$ is thus the sum of two singular integral operators. Since g and $\tilde{g}$ are smooth and rapidly decreasing, it now follows easily (cf., e.g., [6], p. 50) that for any $\mu > 0$ uniformly with respect to $\nu \in [\nu_0 - \delta, \nu_0 + \delta]$, $\varepsilon \in (0, \varepsilon_0]$

$$(2.12) \qquad |J(x;\nu \pm i0)| \le c(f)|x|^{-3+\mu}.$$

From (2.9), (2.11) with $s = \eta/|\eta|$

$$(2.13) \quad I(x;\zeta) = (2\pi)^{-3} \int_{R^3} \int_{R^3_+} \exp(ix\eta)\eta^\beta \chi(\eta)[c|\eta| - \zeta]^{-1}P_1(s)$$
$$\times [\exp(-iy\eta) - C_1(\tilde{s})\exp(-iy\tilde{\eta})]f(y)\,dy\,d\eta$$
$$= (2\pi)^{-3} \int_{R^3_+} \int_{R^3} \chi(\eta)\eta^\beta[c|\eta| - \zeta]^{-1}P_1(s)$$
$$\times \{\exp[i(x-y)\eta] - C_1(\tilde{s})\exp[i(x-y)\tilde{\eta}]\}f(y)\,d\eta\,dy$$
$$\equiv (2\pi)^{-3} \int_{R^3_+} D(x,y;\zeta)f(y)\,dy,$$

$$D(x,y;\nu \pm i\varepsilon) = \int_0^\infty [cr - (\nu \pm i\varepsilon)]^{-1}r^{2+\beta}\psi(r)\,dr \int_{S^2} P_1(s)s^\beta$$
$$\times \{\exp[ir(x-y)s] - C_1(\tilde{s})\exp[ir(x-y)\tilde{s}]\}\,dS.$$

Replacing the interval $|cr - \nu_0| \le 2\delta$ by a semicircle of radius $2\delta < \varepsilon_0$ in the lower $(\nu + i\varepsilon)$ or upper $(\nu - i\varepsilon)$ half plane and extending ψ to this half disk by 1, it follows from (2.14) that $I(x;\nu \pm i0)$ exists and is continuous in (x,ν), $\nu \in [\nu_0 - \delta, \nu_0 + \delta(\eta)]$. Hence, from (2.12) $D^\beta v_1(x;\nu \pm i\varepsilon)$ exists and is continuous on $R^3_+ \times [\nu_0 - \delta, \nu_0 + \delta] \times [0, \varepsilon_0]$.

We now use (1.15) to obtain the asymptotic behavior of $I(x;\nu \pm i0)$ as $|x| \to \infty$. We set

$$(2.14) \quad t_j = t_j(x,y) = j|x| - j\omega y, \qquad \tilde{t}_j = \tilde{t}_j(x,y) = j|x| - j\tilde{\omega}y,$$
$$Q_j(\omega) = \exp(-ij\pi/2)P_1(j\omega)(j\omega)^\beta,$$
$$\tilde{Q}_j(\omega) = \exp(-ij\pi/2)P_1(j\omega)C_1(j\tilde{\omega})(j\omega)^\beta,$$
$$j = \pm 1, \quad \omega = x/|x|,$$
$$\tilde{\omega} = (\omega', -\omega_3).$$

then from (1.15), (2.13)

$$D(x, y; \nu \pm i\varepsilon) = (2\pi)|x|^{-1} \sum_{j=\pm 1} \left[Q_j(\omega) \int_0^\infty \frac{r^{1+|\beta|} \psi(r) \exp(irt_j)\, dr}{cr - (\nu \pm i\varepsilon)} \right.$$

$$\left. - \tilde{Q}_j(\omega) \int_0^\infty \frac{r^{1+|\beta|} \psi(r) \exp(ir\tilde{t}_j)\, dr}{cr - (\nu \pm i\varepsilon)} \right]$$

$$+ \int_0^\infty \frac{r^{2|\beta|} \psi(r) q(rx)\, dr}{cr - (\nu \pm i\varepsilon)},$$

$$q(x) = O(|x|^{-2}).$$

From Lemma 6.2 of [5] it follows that uniformly with respect to $\nu \in [\nu_0 - \delta, \nu_0 + \delta]$, $\varepsilon \in [0, \varepsilon_0]$,

$$(2.16) \quad \int_0^\infty [cr - (\nu \pm i\varepsilon)] r^{2+|\beta|} \psi(r) q(rx)\, dr = O(|x|^{-1-\kappa}),$$

$$\kappa \in (0, 1) \text{ arbitrarily close to 1.}$$

It thus remains to evaluate

$$(2.17) \quad I^\pm(t_j, \nu, \varepsilon) = \int_0^\infty [r - (\nu \pm i\varepsilon)]^{-1}] \phi(r) \exp(irt_j/c)\, dr,$$

$$\phi(r) = c^{-2-|\beta|} r^{1+|\beta|} \psi(r/c),$$

and $I^\pm(\tilde{t}_j, \nu, \varepsilon)$. In the usual manner (cf. [5, 7]) we observe that

$$1/[r - (\nu \pm i\varepsilon)] = \pm (2\pi)^{1/2} \Phi_1 \chi_\pm(\cdot) \exp[i(\nu \pm i\varepsilon)\cdot](r),$$

where $\chi_\pm(t)$ are the characteristic functions of the half lines $R_\pm$, and for a function $g_t(x) = g(x + t)$ we have $\Phi_1 g_t(r) = \exp(irt)\Phi_1 g(r)$. By the Parseval equality for the Fourier transform we thus have from (2.17)

$$I^\pm(t_j, \nu, \varepsilon) = \pm i(2\pi)^{1/2} \int_{-\infty}^\infty \exp(irt_j/c) \Phi_1 \chi_\pm(\cdot) \exp[i(\nu \pm i\varepsilon)\cdot](r)$$

$$\times \Phi_1^* \Phi_1 \phi(r)\, dr$$

$$= \pm i(2\pi)^{1/2} \int_{-\infty}^\infty \Phi_1 \{\chi_\pm(\cdot) \exp[i(\nu \pm i\varepsilon)]\}_{t_j/c}(r) \Phi_1^* \Phi_1 \phi(r)\, dr$$

$$= \pm i(2\pi)^{1/2} \int_{-\infty}^\infty \chi_\pm(\tau + t_j/c) \exp(\nu \pm i\varepsilon)(\tau + t_j/c)] \Phi_1 \phi(\tau)$$

$$= \pm i(2\pi)^{1/2} \int_{-\infty}^\infty \chi_\pm(t_j/c - Gt) \exp[i(\nu \pm i\varepsilon)(t_j/c - \tau)] \Phi_1^* \phi(\tau)$$

so that

(2.18) $I^+(t_j, \nu, \varepsilon)$

$$= i(2\pi)^{1/2} \int_{-\infty}^{t_j/c} \exp[i(\nu + i\varepsilon)(t_j/c - \tau)]\Phi_1^*\phi(\tau)\,d\tau,$$

$I^-(t_j, \nu, \varepsilon)$

$$= -i(2\pi)^{1/2} \int_{t_j/c}^{\infty} \exp[i(\nu - i\varepsilon)(t_j/c - \tau)]\Phi_1^*\phi(\tau)\,d\tau$$

from which it follows that $I^\pm(t_j(x,y), \nu, \varepsilon)$, $I^\pm(\tilde{t}_j(x,y), \nu, \varepsilon)$ are bounded and continuous for $x, y \in R_+^3$, $\nu \in [\nu_0 - \delta, \nu_0 + \delta]$, $\varepsilon \in [0, \varepsilon_0]$. Now for $|x|$ large and y in compact sets from (2.15) $jt_j(x,y)$, $j\tilde{t}_j(x,y) > 0$, $j = \pm 1$. Since $\phi(r) = c^{-2-|\beta|}r^{1+|\beta|}\psi(r/c)$, where $\psi(r/c) = 1$ for $|r - \nu_0| < 3\delta$, from (2.15), (2.19) we have

(2.19) $I^+(t_1, \nu, 0)$

$$= i(2\pi)c^{-2-|\beta|} \exp[i\nu(|x| - \omega y)/c]\nu^{1+|\beta|} + O(|x|^{-1}),$$

$I^+(t_{-1}, \nu, 0) = O(|x|^{-1}),$

$I^-(t_{-1}, \nu, 0)$

$$= -i(2\pi)c^{-2-|\beta|} \exp[i\nu(-|x| + \omega y)/c]\nu^{1+|\beta|} + O(|x|^{-1}),$$

$I^-(t_1, \nu, 0) = O(|x|^{-1}),$

where the order relation is uniform with respect to $\nu \in [\nu_0 - \delta, \nu_0 + \delta]$, $\varepsilon \in [0, \varepsilon_0]$, and y in compact sets. Similarly,

(2.20) $I^+(\tilde{t}_1, \nu, 0)$

$$= i(2\pi)c^{-2-|\beta|} \exp(i\nu(|x| - \tilde{\omega}y)/c]\nu^{1+|\beta|} + O(|x|^{-1}),$$

$I^+(\tilde{t}_{-1}, \nu, 0) = O(|x|^{-1}),$

$I^-(\tilde{t}_{-1}, \nu, 0)$

$$= -i(2\pi)c^{-2-|\beta|} \exp[i\nu(-|x| + \tilde{\omega}y)/c]\nu^{1+|\beta|} + O(|x|^{-1})$$

$I^-(\tilde{t}_1, \nu, 0) = O(|x|^{-1}).$

Hence, from (2.14), (2.15), (2.17), (2.19), (2.20)

$(2.21)^+$ $D(x, y; \nu + i0)$
$$= (2\pi)^2 c^{-2-|\beta|}|x|^{-1}P_1(\omega)\nu^{1+|\beta|}\{\exp[i\nu(|x| - \omega y)/c]$$
$$- C_1(\tilde{\omega})\exp[i\nu(|x| - \tilde{\omega}y)/c]\}$$
$$+ O(|x|^{-1-\kappa})$$

$D(x, y; \nu - i0)$
$$= (2\pi)^2 c^{-2-|\beta|}|x|^{-1}P_1(-\omega)\nu^{1+|\beta|}\{\exp[i\nu(-|x| + \omega y)/c]$$
$$- C_1(-\tilde{\omega})\exp[i\nu(-|x| + \tilde{\omega}y)/c]\}$$
$$+ O(|x|^{-1-\kappa}),$$

and thus from (2.11), (2.13), (2.21)

$(2.21)^-$ $D^\beta v_1(x; \nu \pm i0)$
$$= (2\pi)^{-1}|x|^{-1}P_1(\pm\omega)\nu^{1+|\beta|}\exp(\pm i\nu|x|/c)$$
$$\times [\theta(\pm\omega; \nu) - C_1(\pm\tilde{\omega})\theta(\pm\tilde{\omega}; \nu)] + O(|x|^{-1-\kappa}),$$
$$= 2\nu^{1+|\beta|}g_\pm(x; \nu)P_{\pm 1}(\omega)[\theta(\pm\omega; \nu) - C_{\mp 1}(\omega)\theta(\pm\tilde{\omega}; \nu)]$$
$$+ O(|x|^{-1-\kappa}),$$

$$g_\pm(x; \nu) = (4\pi|x|)^{-1}\exp(\pm i\nu|x|/c),$$
$$\theta(\omega; \nu) = c^{-2-|\beta|}\omega^\beta \int_{R_+^3} \exp(-i\omega\nu y/c)f(y)\,dy,$$

$$\nu > 0, x \in (0, 1), |\beta| \geq 0.$$

We note that the first term on the right in (2.21) satisfies both boundary conditions (1.4) and (1.4′) (see (1.7)). The $O(|x|^{-1-\kappa})$ term thus also satisfies the boundary condition (1.4). For positive times $t \exp(-i\nu t)v_1(x; \nu + i0)$ modulo $O(|x|^{-1-\kappa})$ is an outgoing hemispherical wave ($x_3 \geq 0$), while $\exp(-i\nu t)v_1(x; \nu - i0)$ is an incoming hemispherical wave.

For $\nu < 0$ we see as in (2.12) that from (2.11), (2.10) for $\mu > 0$

$$(2.22) \qquad\qquad D^\beta v_1(x; \nu \pm i0) = O(|x|^{-3+\mu})$$

uniformly with respect to $\nu \in [\nu_0 - \delta, \nu_0 + \delta]$, $\varepsilon \in [0, \varepsilon_0]$.

3. $v_{-1}(\dot{x}; \nu \pm i0)$. Suppose first that $\nu < 0$. From (1.6), (1.8), (1.10)

$$(2.23) \qquad \Psi'_{-1}f(\eta) = \chi_\pm \check{f}_{-1}(\eta),$$
$$\check{f}_{-1}(\eta) = P_{-1}(\eta)[\Phi_3 f(\eta) - C_{-1}(\tilde{\eta})\Phi_3 f(\tilde{\eta})],$$
$$C_{-1}(\eta)\check{f}_{-1}(\eta) = -\check{f}_{-1}(\tilde{\eta}), \qquad \tilde{\eta} = (\xi, -\rho).$$

Hence, from (1.11), (2.23)

$$
\begin{aligned}
D^\beta v_{-1}(x; \nu \pm i\varepsilon) &= D^\beta \Psi^*_{-1}[\lambda_{-1}|\eta| - (\nu \pm i\varepsilon)]^{-1}\Psi'_{-1}f(x) \\
&= -D^\beta \Psi^*_{-1}[c|\eta| - (|\nu| \mp i\varepsilon)]^{-1}\Psi'_{-1}f(x) \\
&= -(2\pi)^{-3/2}\int [c|\eta| - (|\nu| \mp i\varepsilon)]^{-1} \\
&\quad \times [\exp(ix\eta)\eta^\beta - \exp(ix\tilde\eta)\tilde\eta^\beta C_{-1}(\eta)]\chi_\pm(\rho)\check f_{-1}(\eta)\,d\eta \\
&= -(2\pi)^{-3/2}\int_{R^3} \exp(ix\eta)\eta^\beta[c|\eta| - (|\nu| \mp i\varepsilon)]^{-1}\check f_{-1}(\eta)\,d\eta.
\end{aligned}
$$

We can thus use the results of part 2 to conclude from (2.21) that for $\nu < 0$.

$$
\begin{aligned}
(2.24)\quad D^\beta v_{-1}(x; \nu \pm i0) \\
= &-(2\pi)^{-1}|x|^{-1}|\nu|^{1+|\beta|}P_{-1}(\mp\omega)\exp(\mp i|\nu||x|) \\
&\times [\theta(\mp\omega; |\nu|) - C_{-1}(\mp\tilde\omega)\theta(\mp\omega; |\nu|)] + O(|x|^{-1-\kappa}) \\
= &\, 2\nu^{1+|\beta|}g_\pm(x; \nu)P_{\pm1}(\omega)[\theta(\pm\omega; \nu) - C_{\mp1}(\omega)\theta(\pm\tilde\omega; \nu)] \\
&+ O(|x|^{-1-\kappa}),
\end{aligned}
$$

where we have used the fact that $C_{-1}(\tilde\omega) = C_1(\omega)$ (see (1.7)). We again observe that the first term on the right satisfies both boundary conditions (1.4), (1.4'). The $O(|x|^{-1-\kappa})$ term satisfies the boundary condition (1.4). For positive times (and negative frequencies) $\exp(-i\nu t)v_{-1}(x; \nu + i0)$ modulo $O(|x|^{-1-\kappa})$ is an outgoing hemispherical wave, while $\exp(-i\nu t)v_{-1}(x; \nu - i0)$ is an incoming hemispherical wave.

For $\nu > 0$, just as in (2.22),

$$
(2.25)\qquad\qquad D^\beta v_{-1}(x; \nu \pm i0) = O(|x|^{-3+\mu}).
$$

4. $v'_1(x; \nu \mp i0)$. From (1.6), (1.8), (1.10)

$$
\Psi_1 f(\eta) = \chi_\mp(\rho)\check f'_1(\eta), \quad \check f'_1(\eta) = P_1(\eta)[\Phi_3 f(\eta) - C'_1(\tilde\eta)\Phi_3 f(\tilde\eta)],
$$
$$
C'_1(\eta)\check f'_1(\eta) = -\check f'_1(\tilde\eta).
$$

Hence, from (1.11), (2.1')

$$
\begin{aligned}
D^\beta v'_1(x; \nu \mp i\varepsilon) &= D^\beta(\Psi'_1)^*[\lambda_1(\eta) - (\nu \mp i\varepsilon)]^{-1}\Psi_1 f(\eta) \\
&= (2\pi)^{-3/2}\int_{R^3}[c|\eta| - (\nu \mp i\varepsilon)]^{-1} \\
&\quad \times [\exp(ix\eta)\eta^\beta - \exp(ix\tilde\eta)\tilde\eta^\beta C'_1(\eta)]\chi_\mp(\rho)\check f'_1(\eta)\,d\eta \\
&= (2\pi)^{-3/2}\int_{R^3}[c|\eta| - (\nu \mp i\varepsilon)]^{-1}\exp(ix\eta)\eta^\beta \check f'_1(\eta)\,d\eta,
\end{aligned}
$$

whence as in part 2 we find for $\nu > 0$

$$(2.26) \quad D^\beta v_1'(x; \nu \mp i0) = (2\pi)^{-1} |x|^{-1} P_1(\mp\omega) \nu^{1+|\beta|} \exp(\mp i\nu |x|)$$
$$\times [\theta(\mp\omega; \nu) - C_1'(\mp\tilde\omega)\theta(\mp\tilde\omega; \nu)]$$
$$+ O(|x|^{-1-\kappa})$$
$$= 2\nu^{1+|\beta|} g_{\mp}(x; \nu) P_{\mp}(\omega) [S\theta(\mp\omega; \nu) - C_1'(\mp\tilde\omega)\theta(\mp\tilde\omega; \nu)]$$
$$+ O(|x|^{-1-\kappa}).$$

Thus, modulo $O(|x|^{-1-\kappa})$ $\exp(i\nu t)v_1'(x; \nu - i0)$ for positive t is an outgoing hemispherical wave, while $\exp(i\nu t)v_1'(x; \nu + i0)$ is an incoming wave. The first term on the right satisfies both boundary conditions (1.4), (1.4′), and the $O(|x|^{-1-\kappa})$ term thus satisfies the boundary condition (1.4′).

Just as in (2.22), for $\nu < 0$

$$(2.27) \qquad\qquad D^\beta v_1'(x; \nu \mp i0) = O(|x|^{-3+\mu}).$$

5. $v_{-1}'(x; \nu \mp i0)$. From (1.6), (1.8), (1.10)

$$\Psi_{-1} f(\eta) = \chi_\pm(\rho) \check{f}_{-1}'(\eta),$$
$$\check{f}_{-1}'(\eta) = P_{-1}(\eta)[\Phi_3 f(\eta) - C_{-1}'(\tilde\eta)\Phi_3 f(\tilde\eta)],$$
$$C_{-1}'(\eta) = -\check{f}_{-1}'(\tilde\eta).$$

Hence, from (1.11), (2.1′) for $\nu < 0$.

$$D^\beta v_{-1}'(x; \nu \mp i0) = D^\beta (\Psi_{-1}'')^* [\lambda_{-1}(\eta) - (\nu \mp i0)]^{-1} \Psi_{-1} f(\eta)$$
$$= - D^\beta (\Psi_{-1}'')^* [c|\eta| - (|\nu| \pm i0)]^{-1} \Psi_{-1} f(\eta),$$

so that as in part 4

$$(2.28) \quad D^\beta v_{-1}'(x; \nu \mp i0) = 2\nu^{1+|\beta|}(-1)^{|\beta|} P_1(\mp\omega) g_{\mp}(x; \nu)$$
$$\times [\theta(\mp\omega; \nu) - C_1'(\pm\omega)\theta(\mp\tilde\omega; \nu)]$$
$$+ O(|x|^{-1-\kappa}).$$

For $\nu > 0$, just as in (2.22),

$$(2.29) \qquad\qquad D^\beta v_{-1}'(x; \nu \mp i0) = O(|x|^{-3+\mu}).$$

6a. $v_S(x; \nu \pm i0)$, $v_S'(x; \nu \pm i0)$, $S = E, M$. The case $\alpha_1 > 0$. These components are all estimated in the same way. We consider, for example, $v_E(x; \nu \pm i0)$. From (1.9)–(1.11), since $e(\xi)$ is in the

lower half plane and $0 \neq \nu \in R$, we can immediately write

$$v_E(x; \nu \pm i0) = \Sigma_E^*[e(\cdot) - (\nu \pm i0)]^{-1}\Sigma_E' f(x)$$

$$= i\mu q(c\alpha\varepsilon)^{-1}(2\pi)^{-1}\int_{R^2} \exp[ix'\xi + \tau_e(\xi)x_3]|\xi|^{-3}$$

$$\times \mathscr{E}(\xi)^t \mathscr{E}(\xi)[e(\xi) - \nu]^{-1} d\xi E \int_0^\infty \exp[i\tau_e(\xi)y_3]\Phi_3 f(\xi, y_3) \, dy_3.$$

Setting $h|\xi| = \operatorname{Im} \tau_e(\xi)$ (see (1.9)) and noting that $\inf_\xi |e(\xi) - \nu| \geq \delta > 0$, we have just as in (2.7)

$$|x_3^2 v_E(x; \nu \pm i0)|$$

$$\leq c(\nu, f)x_3^2\partial_3^2 \int_a^b \int_0^\infty \exp[-h|\xi|(x_3 + y_3)] \, d|\xi| \, dy_3$$

$$\leq c(f) < \infty.$$

We can thus estimate $v(x; \nu \pm i0)$ in exactly the same way as $R(x)$ in part 1. Estimating $D^\beta v(x; \nu \pm i0)$ in a similar way, we thus have

$$(2.30)\ |x|^2|D^\beta v_S(x; \nu \pm i0)| \leq c(\nu, f) < \infty,$$

$$|x|^2|D^\beta v_S'(x; \nu \pm i0)| \leq c(\nu, f) < \infty, \quad S = E, M, |\beta| \geq 0,$$

$$|x| \to \infty, x_3 \geq 0.$$

6b$^+$. The case $\alpha - i\alpha_2$, $\alpha_2 > 0$. In this case $p = \sqrt{(\mu/\varepsilon + \alpha_2^2)^{-1}}$ $q = \alpha_2 p > 0$, and $e(\xi) = cq|\xi|$ lies on the positive semiaxis, while $m(\xi) = -p|\xi|/\varepsilon$ lies on the negative semiaxis (this follows from (1.5), (1.9), and the condition that $\operatorname{Im} \tau_{e,m} \geq 0$). This situation is responsible for the fact that the surface waves decay only like $|x|^{-1/2}$ along the boundary $\{x_3 = 0\}$. We first consider the case $\zeta = \nu \pm i\varepsilon$, $\nu > 0$, $\varepsilon \in (0, \varepsilon_0]$; we choose $\delta > 0$ so that $\nu_0 - 5\delta > 0$, $\nu \in (\nu_0 - \delta, \nu_0 + \delta)$. Let $\psi \in C_0^\infty(R)$, $\operatorname{supp} \psi = \{|e(r) - \nu_0| \leq 4\delta\} = \{|cqr - \nu_0| < 4\delta\} \subset (0, \infty)$, $\psi(r) = 1$ for $|cqr - \nu_0| < 3\delta$, and let $\chi \in C_0^\infty(R^2 \setminus \{0\})$, $\chi(\xi) \equiv \chi(rs) \equiv \psi(r)$, $s \in S^1$.

$$(2.31) \quad D^\beta v_E(x; \zeta) = D^\beta \Sigma_E^*[e(\cdot) - \zeta]^{-1}\Sigma_E f(x)$$

$$= D^\beta \Sigma_E^*[e(\cdot) - \zeta]^{-1}\chi(\cdot)\Sigma_E f(x)$$

$$+ D^\beta \Sigma_E^*[e(\cdot) - \zeta]^{-1}[1 - \chi(\cdot)]\Sigma_E f(x)$$

$$\equiv I_E(x; \zeta) + J_E(x; \zeta).$$

Since for $\xi \in \operatorname{supp}(1 - \chi)$ we have $|e(\xi) - \zeta| \geq 3\delta$, just as in 6a, $J_E(x; \nu \pm i0)$ exists, is continuous on $R_+^3 \times [\nu_0 - \delta_0, \nu + \delta] \times [0, \varepsilon_0]$, and

$$(2.32) \quad |x|^2|J_E(x; \nu \pm i0)| \leq c(\nu, f) < \infty, \quad x_3 \geq 0, |x| \to \infty.$$

With $h = \mu cq/\alpha_2$, $\beta = \beta'\beta_3$, and $c_e = (2\pi)^{-2}\mu q(ih)^{\beta}3(\alpha_2 c\varepsilon)^{-1}$

(2.33) $I_E(x;\zeta)$

$$= c_e \int_{R^2} d\xi \int_{R_+^3} dy \xi^{\beta'} \exp(ix'\xi - h|\xi|x_3)|\xi|^{-3+\beta_3}\mathscr{E}(\xi)^{\bar{t}}\mathscr{E}(\xi)$$

$$\times [e(\xi) - \zeta]^{-1}\chi(\xi)\exp(-iy'\xi - hy_3)Ef(y)$$

$$\equiv c_e \int_{R^3} D(x,y;\zeta)Ef(y)\,dy,$$

where with $\xi = rs \equiv r(\cos\phi_\xi, \sin\phi_\xi)$

$$D(x,y;\zeta) = \int_0^\infty dr(cqr - \zeta)^{-1}\psi(r)r^{2+|\beta|}\exp[-hr(x_3 + y_3)]$$

$$\times \int_{S^1} \{s^{\beta'}\mathscr{E}(s)^{\bar{t}}\mathscr{E}(s)\exp[irs(x' - y')]\}\,d\phi.$$

Since $\psi(r) = 1$ for $|cqr - \nu_0| \leq 2\delta$, replacing this interval by a semicircle in the lower ($\zeta = \nu + i\varepsilon$) or upper ($\zeta = \nu - i\varepsilon$) half plane and extending ψ to the half disk by one, it follows that $I_E(x;\nu \pm i0)$ exists and is continuous on $R_+^3 \times [\nu_0 - \delta, \nu_0 + \delta]$. Hence, $D^\beta v_E(x;\nu \pm i\varepsilon)$ exists and is continuous on $R_+^3 \times [\nu_0 - \delta, \nu_0 + \delta] \times [0, \varepsilon_0]$.

We now estimate $D(x,y;\nu \pm i\varepsilon)$ for large $|x'|$ and fixed $x_3 \in R_+^3$. With $x' = |x'|\gamma$, $\gamma = (\cos\phi_x, \sin\phi_x)$ from (1.16)

$$\int_{S^1} \{\exp(ir|x'|\gamma s)\exp(-irsy')s^{\beta'}\mathscr{E}(s)^{\bar{t}}\mathscr{E}(s)\,d\phi\}$$

$$= (2\pi)^{1/2}|rx'|^{-1/2}\sum_{j=\pm 1}\exp[ijr(|x'| - \gamma y') - ij\pi/4]$$

$$\times (j\gamma)^{\beta'}\mathscr{E}(j\gamma)^{\bar{t}}\mathscr{E}(j\gamma) + q(rx'), q(x')$$

$$= O(|x'|^{-3/2}).$$

and hence

(2.35) $D(x,y;\zeta) = \int_0^\infty dr(cqr - \zeta)^{-1}\psi(r)r^{2+|\beta|}$

$$\times \exp[-hr(x_3 + y_3)]$$

$$\times \left\{(2\pi)^{1/2}|rx'|^{-1/2}\right.$$

$$\times \sum_{j=\pm 1}\exp[ijr(|x'| - \gamma y') - ij\pi/4]$$

$$\left. \times (j\gamma)^{\beta'}\mathscr{E}(j\gamma)^{\bar{t}}(j\gamma) + q(rx')\right\},$$

$$\equiv D_1(x,y;\zeta) + D_2(x,y;\zeta),$$

where

$$D_2(x, y; \zeta)$$
$$= \int_0^\infty dr(cqr - \zeta)^{-1}\psi(r)r^{2+|\beta|}\exp[-hr(x_3 + y_3)]q(rx').$$

Repeating essentially the proof of Lemma 6.2 of [5], we have

(2.36) $\quad |D_2(x, y; \nu \pm i\varepsilon)| \le c(\nu_0, \delta, f)\exp(-a'x_3)(1 + x_3)|x'|^{-1-\kappa'},$
$$|x'| \ge R, x_3 \ge 0, \kappa' \in (0, 1/2) \text{ arbitrarily close to } 1/2,$$
$$a' = \mu(\nu_0 - 4\delta)/\alpha_2 > 0$$

uniformly with respect to $\nu \in [\nu_0 - \delta, \nu_0 + \delta]$, $\varepsilon \in [0, \varepsilon_0]$.
From (2.35)

(2.37) $\quad D_1(x, y; \zeta)$
$$= (2\pi)^{1/2}|x'|^{-1/2}\sum_{j=\pm 1}\exp(-ij\pi/4)(j\gamma)^\beta \mathscr{E}(j\gamma)^{\bar{i}}\mathscr{E}(j\gamma)$$
$$\times \int_0^\infty dr(cqr - \zeta)^{-1}\psi(r)r^{3/2+|\beta|}\exp[-hr(x_3 + y_3)]$$
$$\times \exp[ijr(|x'| - \gamma y')]$$
$$\equiv (2\pi)^{1/2}|x'|^{-1/2}\sum_{j=\pm 1}\exp(-ij\pi/4)(j\gamma)^{\beta'}\mathscr{E}(j\gamma)^{\bar{i}}\mathscr{E}(j\gamma)$$
$$\times I^\pm(t_j, x_3, y_3; \nu, \varepsilon),$$

$$I^\pm(t_j, x_3, y_3; \zeta) = \int_0^\infty dr(r - \zeta)^{-1}\phi(r, x_3, y_3)\exp(irt_j),$$
$$\zeta = \nu \pm i\varepsilon, \quad t_j = j(|x'| - \gamma y')/cq, \quad j = \pm 1,$$
$$\phi(r, x_3, y_3) = (cq)^{-5/2-|\beta|}\psi(r/cq)\exp[-\mu r(x_3 + y_3)/\alpha_2]r^{3/2+|\beta|}$$
$$\equiv \lambda(r)\exp[-\mu r(x_3 + y_3)/\alpha_2],$$
$$\operatorname{supp}\lambda(r) \subset [\nu_0 - 4\delta, \nu_0 + 4\delta].$$

Hence,

(2.38) $\quad I^+(t_j, x_3, y_3; \nu, \varepsilon)$
$$= i(2\pi)^{1/2}\int_{-\infty}^{t_j}\exp[i(\nu + i\varepsilon)(t_j - \tau)]\Phi_1^*\phi(\tau, x_3, y_3)\,d\tau,$$
$$I^-(t_j, x_3, y_3; \nu, \varepsilon)$$
$$= -i(2\pi)^{1/2}\int_{t_j}^\infty\exp[i(\nu - i\varepsilon)(t_j - \tau)]\Phi_1^*\phi(\tau, x_3, y_3)\,d\tau,$$

from which it follows that $I^\pm(t_j, x_3, y_3; \nu, \varepsilon)$ are bounded and continuous for x_3, $y_3 \in R_+^3$, $\nu \in [\nu_0 - \delta_0, \nu_0 + \delta]$, $\varepsilon \in [0, \varepsilon_0]$. With $\tilde{a} = \nu_0 - 5\delta > 0$, $r - \tilde{a} > 0$ for $r \in \operatorname{supp} \lambda$

$$(2.39) \quad \Phi_1^* \phi(\tau, x_3, y_3) = (2\pi)^{-1/2} \int e^{i r \tau} \lambda(r) \exp[-h r(x_3 + y_3)/cq] \, dr$$
$$\equiv \exp[-\mu \tilde{a}(x_3 + y_3)/\alpha_2] \pi(\tau, x_3, y_3),$$

$$\pi(\tau, x_3, y) = (2\pi)^{-1/2} \int e^{i r \tau} \lambda(r) \exp[-\mu(r - \tilde{a})(x_3 + y_3)/\alpha_2] \, dr.$$

It follows readily on integrating by parts that $\tau^2 |\pi(\tau, x_3, y_3)| \leq c(\nu_0)$. Hence, from (2.38), (2.39)

$$(2.40) \quad I^\pm(t_{\pm 1}, x_3, y_3; \nu, \varepsilon)$$
$$= \pm(2\pi) \exp(i\nu t_{\pm 1})\phi(\nu) + \exp(-\mu \tilde{a} x_3/\alpha_2) q^\pm(x', y),$$
$$|q^\pm(x', y)| \leq c^\pm(\nu_0)|x'|^{-1}, \quad |x'| \to \infty.$$
$$I^\pm(t_{\mp 1}, x_3, y_3; \nu, 0) = \exp(-\mu \tilde{a} x_3/\alpha_2) \tilde{q}^\pm(x', y),$$
$$|\tilde{q}^\pm(x', y)| \leq \tilde{c}^\pm(\nu_0)|x'|^{-1}, \quad |x'| \to \infty, \ \tilde{a} = \nu_0 - 5\delta > 0.$$

Hence, from (2.37)

$$D_1(x, y; \nu \pm i0)$$
$$= \pm i(2\pi)^{3/2}|x'|^{-1/2} \exp(\mp i\pi/4) \exp(i\nu t_{\mp})\lambda(n)(j\gamma)^{\beta'}$$
$$\times \mathscr{E}(\pm \gamma)^{\overline{l}}(\pm \gamma) + Q^\pm(x, y),$$

$$|Q^\pm(x, y)| \leq c^\pm(\nu_0, \beta) \exp(-a x_3)|x'|^{-3/2}, \quad a = \mu(\nu_0 - 5\delta)/\alpha_2,$$
$$x_3 > 0, \ |x'| \to \infty,$$

and from this and (2.35), (2.36) we have

$$(2.41) \quad D(x, y; \nu \pm i0)$$
$$= c^\pm(\nu)|x'|^{-1/2} \exp[\pm i\nu(|x'| - \gamma y')/cq - \mu\nu(x_3 + y_3)/\alpha_2]$$
$$\times (\pm \gamma)^{\beta'} \mathscr{E}(\pm \gamma)^{\overline{l}} \mathscr{E}(\pm \gamma) + P^\pm(x, y, \nu, \beta).$$
$$c^\pm(\nu) = \pm i(2\pi)^{3/2}(cq)^{-1}(\nu/cq)^{3/2+|\beta|} \exp(\mp(i\pi/4)),$$

$$|P^\pm(x, y; \nu, \beta)| \leq c(\nu_0, \beta, f) \exp(-a x_3)|x'|^{-1-\kappa'},$$
$$|x'| \text{ large}, \quad x_3 \geq 0, \quad a = \mu(\nu_0 - 5\delta) > 0,$$
$$\kappa' \in (0, 1/2) \text{ arbitrarily close to } 1/2.$$

Thus, from (2.31), (2.32), (2.33), (2.41)

$$(2.42) \quad D^\beta v_E(x; \nu \pm i0)$$
$$= c_E^\pm(\nu, \beta) \exp(\pm i\nu|x'|/cq - \mu\nu x_3/\alpha_2)|x'|^{-1/2}\mathscr{E}(\pm\gamma)$$
$$\times {}^t\mathscr{E}(\pm\gamma)\theta_E(\pm\gamma, \nu) + R_E^\pm(x, \nu, \beta, \quad \nu > 0, \ \alpha_2 > 0,$$

$$\theta_E(\pm\gamma, \nu) = (\gamma)^{\beta'} \int_{R^3} \exp(\mp i\nu\gamma y'/cq - \mu\nu y_3/\alpha_2)Ef(y)\,dy,$$

$$|R_E^\pm(x, \nu, \beta)| \leq c(\nu_0, \beta; f) \exp(-ax_3)(1 + x_3)|x'|^{-1-\kappa'}.$$

$$c_E^\pm(\nu, \beta) = \pm i(2\pi)^{-1/2}\mu(\alpha_2 c^2\varepsilon)^{-1}(\nu/cq)^{3/2+\beta}(i\mu cq/\alpha_2)^{\beta_3}$$
$$\times \exp(\mp i\pi/4),$$

$$a = \mu(v_0 - 5\delta)/\alpha_2 > 0, \quad \kappa' \in (0, 1/2) \text{ arbitrarily close to } 1/2,$$
$$x' = |x'|\gamma, x_3 \geq 0, \quad |x'| \text{ large},$$

uniformly with respect to $\nu \in [\nu_0 - \delta, \nu_0 + \delta]$, $\nu_0 - 5\delta > 0$, $\delta = \delta(\nu_0)$.

REMARK 1. Writing $v_E(x; \nu \pm i0) = \overset{\circ}{v}_E(x; \nu \pm i0) + R_E^\pm(x; \nu)$, we see that $\exp(-i\nu t)\overset{\circ}{v}_E(x; \nu - i0)$ for $t > 0$ and each $x_3 \geq 0$ is an outgoing cylindrical wave, while $\exp(-i\nu t)\overset{\circ}{v}_E(x; \nu - i0)$ is an incoming cylindrical wave. Each decays like $|x'|^{-1/2}$ for $x_3 = 0$. The leading term $\overset{\circ}{v}_E(x; \nu \pm i0)$ satisfies the boundary condition (1.4) and hence so also does $R_E^\pm(x; \nu)$. Further, $[\Lambda(D) - \nu I]\overset{\circ}{v}(x; \nu \pm i0) = 0$ modulo a term decaying like $\exp(-\mu\nu x_3/\alpha_2)|x'|^{-3/2}$, and hence $[\Lambda(D) - \nu I]v_E(x; \nu \pm i0) \in \mathscr{X}$ as it must, since $[\Lambda(D) - \nu I]v_E(x; \nu \pm i0) = \Sigma_E^*\Sigma_E f \in \mathscr{X}$ (see (2.31)). We observe also that with $A(D)$ of (0.1) for $\gamma \in S^1$

$$(2.43) \qquad\qquad {}^t\mathscr{E}(\pm\gamma)A(\gamma, 0)\mathscr{E}(\pm\gamma) = \pm 2cq/\mu.$$

From (1.9)–(1.11) we have

$$(2.44) \quad D^\beta v_M(x; \zeta) = D^\beta \Sigma_M^*[m(\cdot) - \zeta]^{-1}\Sigma_M f(x)$$
$$= c_m \int_{R^3} \int_{R^2} \{\exp(ix'\xi - \alpha_2 p|\xi|x_3)(\xi)^{\beta'}|\xi|^{-3+\beta_3}\mathscr{M}(\xi)$$
$$\times {}^t\mathscr{M}(\xi)[m(\xi) - \zeta]^{-1}\exp(-i\xi y' - \alpha_2 p|\xi|x_3)Ef(y)\,d\xi\,dy,$$

$$m(\xi) = -p|\xi|/\varepsilon, \quad c_m = (2\pi)^{-2}p\alpha_2\varepsilon(ip\alpha_2)^{\beta_3}, \quad \zeta = \nu \pm i\varepsilon.$$

Since $p > 0$ for $\alpha_2 > 0$, in the case $\nu > 0$ we have $|m(\xi) - \zeta| \geq \text{const} > 0$, and, as in 6a, $D^\beta v_M(x; \nu \pm i0)$ exists and is continuous on $R_+^3 \times [\nu_0 - \delta, \nu_0 + \delta] \times [0, \varepsilon_0]$, and for $|x|$ large

$$(2.45) \qquad |x|^2|D^\beta v(x; \nu \pm i0)| \leq c(\nu, f) < \infty, \quad \nu > 0, \quad \alpha_2 > 0.$$

In the case $\nu < 0$, $|e(\xi) - \zeta| \geq \text{const} > 0$, so the same argument applies to $D^\beta v_E(x; \zeta)$, and for large $|x|$

$$(2.46) \qquad |x|^2 D^\beta v_E(x; \nu \pm i0)| \leq c(\nu, f) < \infty, \quad \nu < 0, \alpha_2 > 0.$$

As for $D^\beta v_M(x; \zeta)$, $\zeta = \nu \pm i\varepsilon$, $\nu < 0$, we proceed from (2.43) exactly as above in (2.31)–(2.42) and conclude that $D^\beta v_M(x; \nu \pm i\varepsilon)$ exists and is continuous on $R_+^3 \times [\nu_0 - \delta, \nu_0 + \delta] \times [0, \varepsilon_0]$, and for $x' = |x'|\gamma$, $|x'|$ large, $\nu \in [\nu_0 - \delta, \nu_0 + \delta] \subset (-\infty, 0)$, $|\nu_0| - 5\delta > 0$

$$(2.47) \quad D^\beta v_M(x; \nu \pm i0)$$
$$= c_M^\pm(\nu, \beta)|x'|^{-1/2} \exp(\pm i\nu\varepsilon|x'|/p + \alpha_2\varepsilon\nu x_3)\mathscr{M}(\mp\gamma)$$
$$\times {}^{\bar{t}}\mathscr{M}(\mp\gamma)\theta_M(\mp\gamma; \nu) + R_M^\pm(x; \nu), \quad \nu < 0, \alpha_2 > 0,$$

$$\theta_M(\mp\gamma; \nu) = \int_{R_+^3} \exp(\pm i\nu\varepsilon\gamma y'/p + \alpha_2\varepsilon\nu y_3)Ef(y)\,dy.$$

$$|R_+^\pm(x; \nu)|$$
$$\leq c(\nu, f, \beta)\exp(-bx_3)(1 + x_3)|x'|^{-1-\kappa'}, \quad \kappa' \in (0, 1/2),$$

$$b = \alpha_2\varepsilon_0(|\nu| - 5\delta) > 0,$$
$$c_M^\pm = \pm i(2\pi)^{-1/2}\alpha_2\varepsilon^2(\varepsilon|\nu|/p)^{3/2+|\beta|}(i\alpha_2 p)^{\beta_3}\exp(\pm i\pi/4).$$

REMARK 2. The full analogue of Remark 1 holds for $v_M(x; \nu \pm i0)$, $\nu < 0$. In particular,

$$(2.48) \qquad\qquad {}^{\bar{t}}\mathscr{M}(\mp\gamma)A(\gamma, 0)\mathscr{M}(\mp\gamma) = \pm 2p/\varepsilon^2.$$

Further, from (1.14) with $t = 0$ and $S = \{0, \pm 1, E, M\}$

$$f = \Pi_0 f + \Psi_1^* \Psi_1 f + \Psi_{-1}^* \Psi_{-1} f + \Sigma_E^* \Sigma_E^* f + \Sigma_M^* \Sigma_M f$$
$$= \sum_{j \in S} \Pi_j f,$$

where the Π_j are mutually orthogonal orthoprojectors in $\mathscr{X}$ [2, 3]. Thus, writing $v_R(x; \nu \pm i0) = v_0(x; \nu) + v_1(x; \nu \pm i0) + v_{-1}(x; \nu + i0)$ for the "Raumwelle" and $v_S(x; \nu \pm i0) = v_E(x; \nu \pm i0) + v_M(x; \nu \pm i0)$ for the surface wave, we see in all cases from (2.2) that

$$\mathscr{X} \ni [\Lambda(D) - \nu I]v_R(x; \nu \pm i0) = \sum_{j=0,\pm 1} \Pi_j f(x)$$

is orthogonal to

$$\mathscr{X} \ni [\Lambda(D) - \nu I]v_S(x; \nu \pm i0) = \Pi_E f(x) + \Pi_M f(x).$$

6b⁻. The case $\alpha = i\alpha_2$, $\alpha_2 < 0$. In this case $p = -\sqrt{(\mu/\varepsilon + \alpha_2)} < 0$ and $q = -\alpha_2 p < 0$, so $m(\xi) = -p|\xi|/\varepsilon \in [0, \infty)$, $e(\xi) = cq|\xi| \in (-\infty, 0]$ (this follows from (1.5), (1.9) and the condition $\operatorname{Im} \tau_{e,m} \geq 0$); the roles of the TE- and TM-waves are thus reversed. Mutatis mutandis, however, the arguments are exactly the same as in 6b⁺, so we shall simply state the results.

For $\nu > 0$ $D^\beta v_{M,E}(x; \nu \pm i\varepsilon)$ exist and are continuous on $R_+^3 \times [\nu_0 - \delta, \nu_0 + \delta] \times [0, \varepsilon_0]$, $\nu_0 - 5\delta > 0$. For large $|x'|$ and $x_3 \geq 0$

(2.49) $D^\beta v_M(x; \nu \pm i0)$
$$= \tilde{c}_M^\pm(\nu, \beta)|x'|^{-1/2} \exp(\pm i\nu\varepsilon|x'|/|p| - |\alpha_2|\varepsilon\nu x_3)\mathscr{M}(\pm\gamma)$$
$$\times {}^t\mathscr{M}(\pm\gamma)\tilde{\theta}_M(\pm\gamma; \nu) + \tilde{R}_M^\pm(x; \nu), \quad \nu > 0, \ \alpha_2 < 0,$$

$$\tilde{\theta}_M(\pm\gamma; \nu) = \int_{R^3} \exp(\mp i\nu\varepsilon\gamma y'/|p| - |\alpha_2|\varepsilon\nu y_3)Ef(y)\,dy,$$

$$|R_M^\pm(x; \nu)| \leq c(\nu_0, f)\exp(-bx_3)|x'|^{-1-\kappa'},$$

$$\tilde{c}_M(\nu, \beta) = \pm i(2\pi)^{1/2}|\alpha_2|(i\alpha_2 p)^\beta 3(\varepsilon\nu/|p|)^{3/2+|\beta|}\exp(\mp i\pi/4),$$
$$b = |\alpha_2|\varepsilon(\nu_0 - 5\delta) > 0, \quad \kappa' \in (0, 1/2).$$

(2.50) $|x|^2|D^\beta v_E(x; \nu \pm i0)| \leq c(\nu_0, f) < \infty, \quad \nu > 0, \ \alpha_2 < 0.$

For $\nu < 0$ $D^\beta v_{M,E}(x; \nu \pm i\varepsilon)$ exist and are continuous on $R_+^3 \times [\nu_0 - \delta, \nu_0 + \delta] \times [0, \varepsilon_0]$, $|\nu_0| - 5\delta > 0$. For large $|x'|$ and each $x_3 \geq 0$

(2.51) $D^\beta v_E(x; \nu \pm i0)$
$$= \tilde{c}_E^\pm(\nu, \beta)\exp(\pm i\nu|x'|/c|q| + \mu\nu x_3/|\alpha_2|)|x'|^{-1/2}$$
$$\times \mathscr{E}(\mp\gamma){}^t\mathscr{E}(\mp\gamma)\tilde{\theta}_E(\mp\gamma, \beta) + \tilde{R}_E^\pm(x, \nu, \beta),$$
$$\nu < 0, \ \alpha_2 < 0,$$
$$\tilde{\theta}_E(\mp\gamma, \nu) = (\mp\gamma)^{\beta'}\int \exp(\mp i\nu\gamma y'/c|q| + \mu\nu y_3/|\alpha_2|)Ef(y)\,dy$$
$$|\tilde{R}_E^\pm(x; \nu, \beta) \leq c(\nu_0, \beta, f)\exp(-ax_3)(1 + x_3)|x'|^{-1-\kappa'},$$
$$a = \mu(|\nu_0| - 5\delta)/|\alpha_2| > 0, \quad \kappa' \in (0, 1/2).$$

(2.52) $|x|^2 |D^\beta v_M(x; \nu \pm i0)| \leq c(\nu_0, \beta, f) < \infty, \quad \nu < 0, \alpha_2 < 0.$

The results of this section can now be summarized as follows.

THEOREM 2.1. *For any* $0 \neq \nu \in R$ *and* $f \in C_0^\infty(R_+^3)$ *there exist smooth solutions* $v_\pm(x; \nu) = v(x; \nu \pm i0)$, $v_\pm'(x; \nu) = v'(x; \nu \pm i0)$ *of* (2.3), (2.3'). *For* $\alpha \equiv 0$ *and* $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1 > 0$, $\alpha_2 \neq 0$, *they have the asymptotic behavior as* $|x| \to \infty$ *in* R_+^3

(2.53) $v(x; \nu \pm i0)$
$$= 2\nu g_\pm(x; \nu) P_{\pm 1}(\omega)[\theta(\pm\omega; \nu) - C_{\pm 1}(\tilde{\omega})\theta(\pm\tilde{\omega}; \nu)]$$
$$+ O(|x|^{-1-\kappa})$$

(2.54) $v'(x; \nu \pm i0)$
$$= 2\nu g_\pm(x; \nu) P_{\pm 1}(\omega)[\theta(\pm\omega; \nu) - C_{\pm 1}'(\tilde{\omega})\theta(\pm\tilde{\omega}; \nu)]$$
$$+ O(|x|^{-1-\kappa})$$

where

(2.56) $g_\pm(x; \nu) = (4\pi|x|)^{-1} \exp(\pm i\nu|x|/c),$

$$\theta(\omega; \nu) = c^{-2} \int_{R_+^3} \exp(-i\nu\omega y/c) f(y) \, dy,$$

$$\omega = x/|x|, \quad \tilde{\omega} = (\omega', -\omega_3),$$

$\kappa \in (0, 1)$ *is arbitrarily close to* 1, $P_{\pm 1}$ *are given in* (1.3), *and* $C_{\pm 1}$, $C_{\pm 1}'$ *are given in* (1.7). *Modulo* $O(|x|^{-1-\kappa})$ *for positive* t

$$\exp(-i\nu t) v_+(x, \nu), \quad \exp(i\nu t) v_-'(x; \nu)$$

are outgoing hemispherical waves, while

$$\exp(-i\nu t) v_-(x; \nu), \quad \exp(i\nu t) v_+'(x; \nu)$$

are incoming hemispherical waves. The order relations are uniform with respect to $x_3 \geq 0$. *For* $\alpha = i\alpha_2$, $\alpha_2 \neq 0$, $v_\pm(x; \nu) = v_R(x; \nu \pm i0) + v_S(x; \nu \pm i0)$ *where* $v_R(x; \nu \pm i0)$ *has the asymptotic behavior* (2.53).

For $x_3 \geq 0$, $|x'|$ large, and $\nu > 0$

(2.57)
$$v_S(x; \nu \pm i0)$$
$$= |x'|^{-1/2} \exp[\nu(\pm i|x'| - \tilde\tau x_3)/s] S(\pm\gamma) \theta_S(\gamma, \nu; f) + R_S^\pm(x; \nu),$$

$s, S = e, E$ for $\alpha_2 > 0$, $s, S = m, M$ for $\alpha_2 < 0$, $e = cq > 0$,
$m = -p/\varepsilon > 0$, $\quad \tilde\tau_e = \mu c q / \alpha_2$, $\quad \tilde\tau_m = \alpha_2/p$,
$|R_S^\pm(x; \nu)| \leq \exp(-\lambda_s x_3)(1 + x_3)|x'|^{-1-\kappa'}$,
$$\lambda_s = \text{const} > 0, \quad \kappa' \in (0, 1/2),$$

while for $\nu < 0$

(2.58) $\quad v_S(x; \nu \pm i0)$
$$= |x'|^{-1/2} \exp[\nu(\mp i|x| - \tilde\tau_s x_3)/s] S(\mp\gamma) \tilde\theta(\gamma, \mu; f)$$
$$+ \tilde{R}^\pm(x; \nu),$$

$s, S = m, M$, *for* $\alpha_2 > 0$, $s, S = e, E$ *for* $\alpha_2 < 0$.

$|\tilde{R}_S^\pm(x; \nu)| \leq \text{const} \exp(-i\tilde\lambda_s x_3)(1 + x_3)|x'|^{-1-\kappa'}$,
$$\tilde\lambda_S = \text{const} > 0, \kappa' \in (0, 1/2).$$

The exact values of the smooth, scalar functions of $\gamma \in S^1$, θ, $\tilde\theta$ and of the constants λ, $\tilde\lambda$ can be read from (2.42), (2.47), (2.49), (2.51). The function $(\Lambda - \nu I)v_S \in \mathscr{X}$ is orthogonal in $\mathscr{X}$ to $(\Lambda - \nu I)v_R \in \mathscr{X}$. For $\nu, t > 0$ modulo $O(|x|^{-1-\kappa'}) \exp(-i\nu t)v_S(x; \nu + i0)$ is an outgoing cylindrical wave, while $\exp(-i\nu t)v_S(x; \nu - i0)$ is an incoming cylindrical wave.

3. The uniqueness theorem and the principle of limiting amplitude. In this section we prove uniqueness of solutions of (2.3), (2.3′) in particular radiation classes which contain the solutions constructed in §2. We then show via the principle of limiting amplitude that the unique solutions of §2 are the physically interesting solutions.
We define

(3.1) $\quad \mathscr{R}_\pm = \{w_\pm \in C^1(R_+^3; C^6) \colon Bw_\pm(x,', 0) = 0,$
$$w_\pm(x) = |x|^{-1} \exp(\pm i\nu|x|)P_{\pm 1}(\omega)\beta(\omega, \nu; w_\pm)$$
$$+ O(|x|^{-1-\kappa}), |x| \to \infty, \kappa \in (0, 1/2),$$
$$\beta \text{ smooth on } S^2 \text{ for each } \nu\}.$$

The class $\mathscr{R}'_\pm$ is defined in the same way with B replaced by B'. For

$\nu > 0$ we define the classes of functions $\mathscr{R}_\pm^S$, $S = E, M$, as follows: $w_\pm \in R_\pm^S$ iff

(3.2) (1) $w_\pm \in C^1(R_+^3, C^6)$, and $w_\pm = w_\pm^R + w_\pm^S$, $BW_\pm^{R,S} = 0$ on $\{x_3 = 0\}$;

  (2) $W_\pm^R \in \mathscr{R}_\pm$ and for $x_3 \geq 0, |x'|$ large

   $$v_\pm^S = |x'|^{-1/2} \exp[(\nu(\pm i|x'| - \tilde{\tau}_s x_3)/s)] S(\pm\gamma)\theta_s(\gamma; \nu)$$
   $$+ R_s^\pm(x; \nu).$$

   $$|R_s^\pm(x; \nu)| \leq \text{const} |x'|^{-1-\kappa'} h(x_3), \quad h \in L_2(R_+), \kappa' \in (0, 1/2);$$

  (3) $(\Lambda - \nu I)w_\pm^R \in \mathscr{X}$ is orthogonal to $(\Lambda - \nu I)v_\pm^S \in \mathscr{X}$;

  (4) for each $0 < R < \infty$

   $$\lim_{L\to\infty} \int_{|x'|\leq R} dx' \, {}^{\bar{t}}v_\pm^S(x', L)A_3 v_\pm^S(x; L) \to 0,$$

   and ${}^{\bar{t}}S(\pm\gamma)A(\gamma, 0)S(\pm\gamma) = \text{const} \neq 0, \gamma \in S^1$.

For $\nu < 0$ the class $\mathscr{R}_\pm^S \omega$ is defined in the same way except that in (2) $\pm i|x'| \to \mp i|x'|$ and $S(\pm\gamma) \to S(\mp\gamma)$.

REMARK. Condition (4) for $v_S(x; \nu \pm i0)$, $S = E, M$, in the present case is automatically satisfied, since

$$^{\bar{t}}v_S(x', L; \nu \pm i0)A_3 v_S(x', L; \nu \pm i0) = 0$$

for any $L > 0$ (this follows from (2.31) and the ensuing argument, since ${}^{\bar{t}}S(\xi)A_3 S(\xi) = 0, S = E, M$). In more general situations, however, (e.g., elasticity) (4) is satisfied but not trivially as here. The second part of condition (4) is satisfied because of (2.43), (2.48). Otherwise, the fact that the $v(x; \nu \pm i0)$ constructed in §2 belong to $\mathscr{R}_\pm$ or $\mathscr{R}_\pm^S$ follows from Theorem 2.1.

THEOREM 3.1. *Suppose $w_\pm$, $w_\pm'$ are solutions of (2.3), (2.3') with $f \in C_0^\infty(R_+^3, C^6)$ and $0 \neq \nu \in R$. In the nonselfadjoint case w_+ is unique in $\mathscr{R}_+$ and w_-' is unique in $\mathscr{R}_-$ (these are the outgoing solutions). In the selfadjoint case $\alpha \equiv 0$ $w_\pm$ are unique in $\mathscr{R}_\pm$. In the selfadjoint case $\alpha = i\alpha_2$, $\alpha_2 \neq 0$, $w_\pm$ are unique in $\mathscr{R}_\pm^S$.*

COROLLARY. *If $\alpha \equiv 0$ or $\alpha = i\alpha_2$, $\alpha_2 \neq 0$, the solutions $v_\pm(x; \nu)$ of (2.3) constructed in §2 are unique in $\mathscr{R}_\pm$ or $\mathscr{R}_\pm^S$ respectively. In the case $\alpha_1 > 0$ $v_+(x; \nu)$ is unique in $\mathscr{R}_+$, and $v_-'(x; \nu)$ is unique in $\mathscr{R}_-'$.*

Proof of Theorem 3.1. We first consider the cases $\alpha \equiv 0$ or $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1 > 0$. Thus, suppose $w, w' \in C^1(R^3, C^6)$ and for $\nu \neq 0$ $(\Lambda - \nu I)w = 0$, $(\Lambda - \nu I)w' = 0$ in R_+^3, $Bw(x', 0) = 0$, $B'w'(x', 0) = 0$. Integration by parts in the intersection of R_+^3 with a ball of radius R then gives for $u = w, w'$

$$0 = (u, [\Lambda - \nu I])_{B_R} - ([\Lambda - \nu I]u, u)_{B_R}$$

$$= i \int_{|x'| < R} {}^t\overline{u}(x', 0)A_3 u(x', 0) \, dx'$$

$$- i \int_{R_+^3 \cap \{|x| = R\}} {}^t\overline{u}(x)A(\omega)u(x) \, dS_R(x)$$

where A_3 is the coefficient matrix of D_3 in (0.1) and $\omega = x/|x|$. Since $Bw(x', 0) = 0$, $B'w'(x', 0) = 0$ and $A_3 u = {}^t(u_5, -u_4, 0, -u_2, u_1, 0)$, we thus have

(3.3)
$$\int_{R_+^3 \cap \{|x| = R\}} {}^t\overline{w}(x)A(\omega)w(x) \, dS_R(x)$$

$$= -2\alpha_1 \int_{|x'| < R} [|w_4(x', 0)|^2 + |w_5(x', 0|^2] \leq 0$$

(3.3')
$$\int_{R_+^3 \cap \{|x| = R\}} {}^t\overline{w'}(x)A(\omega)w'(x) \, dS_R(x)$$

$$= 2\alpha_1 \int_{|x'| \leq R} [|w'_4(x', 0)| + |w'_5(x', 0)|^2] \geq 0.$$

If now $w = w_\pm \in \mathscr{R}_\pm$, $w' = w'_\pm \in \mathscr{R}'_\pm$, then from (3.1), (3.3), (3.3') and the fact that $A(\omega)P_{\pm 1}(\omega) = \pm P_{\pm 1}(\omega)$, we have

(3.4) $\pm \int_{S^2} {}^t\overline{[P_{\pm 1}(\omega)\beta(\omega, \nu; w_\pm)]}P_{\pm 1}(\omega)\beta(\omega, \nu; w_\pm) \, dS + O(R^{-\kappa})$

$$= -2\alpha_1 \int_{|x'| \leq R} [|w_4(x', 0)|^2 + |w_5(x', 0)|^2] \, dx' \leq 0,$$

(3.4')

$\pm \int_{S^2} {}^t\overline{[P_{\pm 1}(\omega)\beta(\omega, \nu; w'_\pm)]}P_{\pm 1}(\omega)\beta(\omega, \nu; w'_\pm) \, dS + O(R^{-\kappa})$

$$= 2\alpha_1 \int_{|x'| \leq R} [|w'_4(x', 0)|^2 + |w'_5(x', 0)|^2] \, dx' \geq 0.$$

Letting $R \to \infty$, it follows from (3.4) in the selfadjoint case $\alpha \equiv 0$ that $w_\pm = O(|x|^{-1-\kappa})$, whence, since $\kappa > 1/2$, $w_\pm \in \mathscr{X}$ and is thus

an eigenfunction. Now zero is the only eigenvalue and $\nu \neq 0$, so that $w_\pm \equiv 0$. The assertions for w_+ and w'_- in the nonselfadjoint case follow in the same way.

We now suppose that $w_\pm \in \mathscr{R}_\pm^S$ and $(\Lambda - \nu I)w_\pm(x) = 0$ in R_+^3, $\nu \neq 0$, $w_\pm = w_\pm^R + w_\pm^S$. Condition (3) of (3.2) implies that $(\Lambda - \nu I)w_\pm^R = 0$ and $(\Lambda - \nu I)w_\pm^S(x) = 0$ individually. It follows just as above that $W_\pm^R \in \mathscr{X}$. Now let $C_R^L = \{x \in R_+^3 : |x'| \leq R, \ 0 \leq x_3 \leq L\}$. Then, since $Bw_\pm^S(x', 0) = 0$ implies ${}^{\bar{t}}w_\pm^S(x', 0)A_3 w_\pm^S(x', 0) = 0$, we have for $\gamma \in S^1$

$$
\begin{aligned}
0 = {}&(w_\pm^S, [\Lambda - \nu I]w_\pm^S)_{C_R^L} - ([\Lambda - \nu I]w_\pm^S, w_\pm^S)_{C^L} \\
&- i \int_{|x'| \leq R} {}^{\bar{t}}w_\pm^S(x', L)A_3 w_\pm^S(x', L) \\
&- i \int_0^L \int_{|x'| = R} {}^{\bar{t}}w_\pm^S(x)A(\gamma, 0)w_\pm^S(x)\, dx'\, dx_3.
\end{aligned}
$$

Letting $L \to \infty$ and using condition (4) of (3.2), it follows that

$$
0 = \int_0^\infty \int_{|x'| = R} {}^{\bar{t}}w_\pm^S(x)A(\gamma, 0)w_\pm^S(x)\, dx'\, dx_3.
$$

Now from condition (2) of (3.2) we have

$$
\begin{aligned}
(3.5) \quad 0 = &\int_{S^1} \int_0^\infty \exp(-2\tau x_3/s)|\theta_S(x; \nu)|^2\, {}^{\bar{t}}S(\pm\gamma) \\
&\hspace{4cm} \times A(\gamma, 0)S(\pm\gamma, 0)\, dx_3\, d\phi \\
&+ \int_0^\infty dx_3 \int_{|x'| = R} \exp[(\mp i|x'| - \tau_S x_3)/s]\bar{\theta}(\gamma; \nu)\, {}^{\bar{t}}S(\pm\gamma) \\
&\hspace{4cm} \times A(\gamma, 0)R_S^\pm(x; \nu)R^{1/2}\, d\phi \\
&+ \int_0^\infty dx_3 \int_{|x'| = R} \exp[(\pm i|x'| - \tau_S x_3)/s]\, {}^{\bar{t}}R_S^\pm(x; \nu)\theta_S(\gamma; \nu) \\
&\hspace{4cm} \times A(\gamma, 0)S(\pm\gamma)R^{1/2}\, d\phi \\
&+ \int_0^\infty dx_3 \int_{|x'| = R} {}^{\bar{t}}R_S^\pm(x; \nu)A(\gamma, 0)R_S^\pm(x; \nu)R\, d\phi.
\end{aligned}
$$

From (3.2) and Cauchy's inequality the second and third terms are bounded by const $R^{-1/2-\kappa'}|h|^2_{L_2(R)_+}$. The last term is bounded by const $R^{-1-2\kappa'}|h|^2_{L_2(R)_+}$. Thus, letting $R \to \infty$ in (3.5) and integrating on x_3,

we have

$$0 = \int_{S^1} |\theta_S(\gamma;\nu)|^{2\bar{t}} S(\pm\gamma) A(\gamma,0) S(\pm\gamma) \, d\phi.$$

From the second part of condition (4) it now follows that $\theta_S(\gamma;\nu) = 0$ for a.e. $\gamma \in S^1$. Since $R_S^\pm(x;\nu) \in \mathscr{X}$, we thus find that also $w_S^\pm \in \mathscr{X}$. Hence, $w_\pm = w_\pm^R + w_\pm^S \in \mathscr{X}$, so, as above, $w \equiv 0$. This completes the proof of Theorem 3.1.

Because of the simple structure of the spectrum, in §2 we were able to construct solutions of (2.3), (2.3′) from both sides of the real axis even in the nonselfadjoint case. Our uniqueness proof, however, works only for solutions obtained from the side of the real axis in the resolvent set. There is some obscure justice in this: the outgoing solutions w_+, w'_- are the physically interesting solutions, since time-harmonic incoming solutions are obtained via limiting amplitude by going backward in time which is impossible in the nonselfadjoint case. We proceed to demonstrate all this.

Let $u(x,t)$, $u'_3(x,t)$ be the solutions in $\mathscr{X}$ of problems (0.2), (0.3) with $f \in C_0^\infty(R_+^3, C^6)$ with $u(x,0) = 0$, $u'(x,0) = 0$. Then by Duhamel's principle we have

$$(3.6) \qquad u(x,t) = i \int_0^t [S(t-\tau)f](x) \exp(-i\nu\tau) \, d\tau,$$

$$(3.6') \qquad u'(x,t) = -i \int_0^t [S^*(t-\tau)f](x) \exp(i\nu\tau) \, d\tau.$$

By a change of variables we have from (1.14)

$$(3.7) \quad \exp(i\nu t)u(x,t) = i \int_0^t \exp(i\nu\tau)[S(t)f](x) \, d\tau$$

$$= i \int_0^t \exp(i\nu\tau)[\Pi_0 f(x) + \Psi_1^* \exp(-ic|\cdot|\tau)\Psi_1' f(x)$$

$$+ \Psi_{-1}^* \exp(ic|\cdot|\tau)\Psi_{-1}' f(x)$$

$$+ \Sigma_E^* \exp(-ie(\cdot)\tau)\Sigma_E' f(x)$$

$$+ \Sigma_M^* \exp(-im(\cdot)\tau)\Sigma_M' f(x)] \, d\tau$$

$$\equiv w_0(x,t) + w_1(x,t) + w_{-1}(x,t) + w_E(x \cdot t) + w_M(x \cdot t),$$

$$(3.7')\quad \exp(-i\nu t)u'(x,t) = -i\int_0^t \exp(-i\nu\tau)[S^*(\tau)f](x)\,d\tau$$

$$= -i\int_0^t \exp(-i\nu\tau)[\Pi_0 f(x) + (\Psi_1')^* \exp(ic|\cdot|\tau)\Psi_1 f(x)$$

$$+ (\Psi_{-1}')^* \exp(-ic|\cdot|\tau)\Psi_{-1} f(x)$$

$$+ (\Sigma_E')^*(i\overline{e}(\cdot)\tau)\Sigma_E f(x)$$

$$+ (\Sigma_M')^* \exp(i\overline{m}(\cdot)\tau)\Sigma_M f(x)]\,d\tau$$

$$\equiv w_0'(x,t) + w_1'(x,t) + w_{-1}'(x,t) + w_E'(x,t) + w_M'(x\cdot t).$$

It is clear that the limit as $t \to +\infty$ of $w_0(x,t)$ does not exist unless $\Pi_0 f(x) = 0$ which we henceforth assume. As in (2.9), (2.10) with integration by parts

$$-c^2\tau^2\Psi_1^* \exp(-ic|\cdot|\tau)\Psi_1' f(x)$$

$$= (2\pi)^{-3/2}\int_{S^2} dS \int_0^\infty \exp(ic|\eta|\tau)\partial_{|\eta|}^2$$

$$\times [\exp(ix\eta)\check{f}_1(\eta)|\eta|^2]\,d|\eta|$$

has finite modulus, so there exists ($\nu > 0$)

$$(3.8)\quad \lim_{t\to\infty} w_1(x,t)$$

$$= i\int_0^\infty \exp(i\nu\tau)\Psi_1^* \exp(-ic|\cdot|\tau)\Psi_1' f(x)\,d\tau$$

$$= i(2\pi)^{-3/2}\int_0^\infty \int_{R^3} \exp[i\tau(\nu - c|\eta|)]\exp(ix\eta)\check{f}_1(\eta)\,d(\eta)\,d\tau$$

$$= i(2\pi)^{-3/2}\lim_{\varepsilon\to 0}\int_0^\infty \exp(-\varepsilon\tau)\,d\tau \int_{R^3} \exp[i\tau(\nu - c|\eta|)]$$

$$\times \exp(ix\eta)\check{f}_1(\eta)\,d\eta$$

$$= (2\pi)^{-3/2}\lim_{\varepsilon\to 0}\int_{R^3} \exp(ix\eta)[c|\eta| - (\nu + i\varepsilon)]^{-1}\check{f}_1(\eta)\,d\eta$$

$$= v_1(x;\nu + i0)$$

of part 2, §2. Continuing in this manner, we see that there exists the limit as $t \to +\infty$ of $\exp(i\nu t)u(x,t) = v(x;\nu + i0)$ where $v(x;\nu + i0)$ is the solution of Theorem 2.1.

In exactly the same way as in (3.6) there exists

$$\lim_{t\to\infty} -i\int_0^t [\exp(-i\nu\tau)(\Psi_1')^* \exp(ic|\cdot|\tau)\Psi_1 f](x)\,d\tau$$

$$= v_1'(x;\nu - i0)$$

of part 4, §2, and, continuing in this manner, we conclude from $(3.7')$ that there exists the limit as $t \to \infty$ of $\exp(-i\nu t)u'(x, t)$ which is equal to $v'(x; \nu - i0)$, the solution of Theorem 2.1.

Further, in the selfadjoint cases

$$\lim_{t \to -\infty} w_1(x, t)$$

$$= -i(2\pi)^{-3/2} \int_{-\infty}^{0} \int_{R^3} \exp[i\tau(\nu - c|\eta|)] \exp(ix\eta) \check{f}_1(\eta) \, d\eta \, d\tau$$

$$= \lim_{\varepsilon \to 0} \int_{R^3} [c|\eta| - (\nu - i\varepsilon)]^{-1} \exp(ix\eta) \check{f}_1(\eta) \, d\eta$$

$$= v_1(x; \nu - i0)$$

of part 2, §2, and in this way from (3.7) we see that there exists the limit as $t \to -\infty$ of $\exp(i\nu t)u(x, t)$ which is equal to $v(x; \nu - i0)$ where the latter is the solution of Theorem 2.1. We summarize these results as follows.

THEOREM 3.2. *Let $f \in C_0^\infty(R_+^3, C^6)$ be in the complement of the null space of Λ, and let $u(x, t)$, $u'(x, t)$ be the solutions in $\mathscr{X}$ of (0.2), (0.3) with $u(x, 0) = 0$, $u'(x, 0) = 0$. Then in the selfadjoint case*

$$\lim_{t \to \pm \infty} \exp(i\nu t)u(x, t) = v(x; \nu \pm i0)$$

where $v(x; \nu \pm i0)$ are the solutions of (2.3) considered in Theorem 2.1. In the nonselfadjoint cases

$$\lim_{t \to \infty} \exp(i\nu t)u(x, t) = v(x; \nu + i0),$$

$$\lim_{t \to \infty} \exp(-i\nu t)u'(x, t) = v'(x; \nu - i0),$$

where the right sides are the outgoing solutions of (2.3), (2.3)' of Theorem 2.1.

Concluding remarks. In the applied literature it is often assumed that the source is a time-harmonic point source, e.g., an oscillating dipole. The spatial part of the response is then sought as a solution of the Helmholtz equation (a Hertz potential), and a prescription is given for constructing the electric and magnetic fields from this potential. What this really amounts to is a specialization of the Green function for the original problem for Maxwell's equations. In the present case the Green functions can be read off from the material above (as the kernel of $\mathscr{G}_\pm$ in the expression $v_\pm = \mathscr{G}_\pm f$), but a simple expression (without transforms) is obtained only in the case $\alpha = 0$, i.e., the case of the classical boundary condition. It seems worthwhile to present this expression.

For Maxwell's equations in all of R^3 the outgoing $(+)$ and incoming $(-)$ Green functions for the time dependence $\exp(-i\nu t)$ are $(\varepsilon = \mu = 1)$

$$G_{\pm}(x, y; \nu) = \left(\nu I + A(D_x) + \nu^{-1}\begin{bmatrix} \partial_x \otimes \partial_x & 0 \\ 0 & \partial_x \otimes \partial_x \end{bmatrix}\right) g_{\pm}(x - y; \nu),$$

where $g_{\pm}(x; \nu)$ are the functions of (2.32), the matrix $[\partial \otimes \partial]_{ij} = \partial_i \partial_j$, and on functions $f = {}^t(f^1, f^2) \in C_0^{\infty}(R^3, C^6)$ in $\mathscr{D}'$

$$\text{rot}_x\, g_{\pm}(x - \cdot; \nu)(f^i) = g_{\pm}(x - \cdot; \nu)(\text{rot}\, f^i),$$
$$\partial_x \otimes \partial_x g_{\pm}(x - \cdot; \nu)(f^i) = g_{\pm}(x - \cdot; \nu)(\partial \otimes \partial f^i), \qquad i = 1, 2.$$

It is straightforward to verify that

$$G_+(x, y; \nu)\begin{pmatrix} \nu a \\ 0 \end{pmatrix} = \begin{pmatrix} \nu^2 g_+(x - y; \nu)a + \partial \otimes \partial g_+(x - y; \nu)a \\ -i\nu\, \text{rot}\, g_+(x - y; \nu)a \end{pmatrix}$$

is the usual response to an electric dipole at the point $y \in R^3$ with direction $a \in R^3$.

For R_+^3 and the classical boundary condition $(\alpha = 0)$ the Green functions are

$$\mathscr{G}_{\pm}(x, y; \nu) = G_{\pm}(x, y; \nu) - R_{\pm}(x, y; \nu),$$

$$R_{\pm}(x, y; \nu)$$
$$= \left(\nu I + A(D_x) + \nu^{-1}\begin{bmatrix} \partial_x \otimes \partial_x & 0 \\ 0 & \partial_x \otimes \partial_x \end{bmatrix}\right) Q g_{\pm}(x, -\tilde{y}; \nu),$$
$$\tilde{y} = (y', -y_3)$$

where Q is the matrix of (1.7).

We leave to the reader as an interesting exercise the verification that $\mathscr{G}_{\pm}(x, \cdot; \nu)(f)$ has the asymptotic behavior (2.53) and that the response with the classical boundary condition to the electric dipole above is

$$\mathscr{G}_+(x, y; \nu)\begin{pmatrix} \nu a \\ 0 \end{pmatrix}.$$

REFERENCES

[1]   D. Eidus, *The limiting absorption principle for the diffraction problem with two unbounded media*, Comm. Math. Phys., **107** (1986), 29–39.
[2]   D. S. Gilliam and J. R. Schulenberger, *Electromagnetic waves in a three-dimensional conducting half space*, J. Math. Anal. Appl., **89**, No. 1, (1982), 129–185.
[3]   ——, *On the structure of surface waves*, Adv. in Appl. Math., **4** (1983), 212–243.

[4]   M. Matsumura, *Comportement des solutions de quelques problèmes mixtes pour certains systèmes hyperboliques symétriques à coefficients constants*, Publ. RIMS Kyoto, Ser. A, **4**, No. 2, (1968), 309–359.

[5]   ———, *Asymptotic behavior at infinity for Green functions of first order systems with characteristics of nonuniform multiplicity*, Publ, RIMS Kyoto, **12**, No. 2, (1976), 317–377.

[6]   S. G. Mikhlin, *Multidimensional singular integrals and integral equations*, Oxford, Pergamon Press (1965).

[7]   J. R. Schulenberger and C. H. Wilcox, *A Rellich uniqueness theorem for steady-state wave propagation in inhomogeneous, anisotropic media*, Arch. Rat. Mech. Anal., **41**, No. 1, (1971), 18–45.

[8]   J. R. Schulenberger, *Boundary waves on perfect conductors*, J. Math. Anal. Appl., **66**, No. 3, (1978), 514–549.

[9]   B. R. Vainberg, *Asymptotic methods in equations of mathematical physics*, Izd. Mosk. Univ., (1982).

[10] S. Wakabayashi, *The principle of limiting amplitude for symmetric hyperbolic systems of first order in the half space R_+^n*, Publ. RIMS Kyoto, **11**, (1975), 149–162.

427 E. NORTON
TUCSON, AZ 85719