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DEFORMING VARIETIES OF k-PLANES OF PROJECTIVE COMPLETE INTERSECTIONS

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We consider the variety F of k-dimensional linear projective subspaces lying on a generic projective complete intersection S. Under general assumptions involving k, the multidegree and the dimension of S, we prove that F is connected, smooth, and its local deformations come from deformations of S.

Introduction. Linear varieties lying on a projective variety have been considered in several contexts.

A classical instance, going back to Cayley [**6**], is that of a smooth cubic surface. There are twenty-seven lines on such a surface, and, as observed later, the incidence preserving permutations of this set of lines form a group isomorphic to the Weyl group of a root system of type E_6. It is also the monodromy group of the global family of smooth cubics and the Galois group of the corresponding enumerative problem (see [**12**]).

Similar results (involving the root system D_{2k+3}) hold for the k-planes contained in a smooth $2k$-dimensional intersection of two quadrics ([**14**, **16**]).

Beyond the enumerative level, and besides homogeneous-rational varieties such as Grassmannians or linear spaces lying on a smooth quadric, a first example should be the Fano surface of lines contained in a cubic threefold ([**11**]). The Abel-Jacobi map induces an isomorphism from the Albanese variety of the Fano surface to the intermediate Jacobian of the cubic threefold and one has a global Torelli theorem ([**7**, **19**]).

With planes instead of lines, but generically this time, the analogous statements hold true for cubic fivefolds ([**8**, **10**]).

Nor should cubic fourfolds be neglected here: their varieties of lines are irreducible symplectic projective fourfolds ([**3**]) which play an important role in the proof of the global Torelli theorem ([**20**]).

We also mention the variety of k-planes contained in a smooth $(2k+1)$-dimensional intersection of two quadrics: it is an Abelian variety isomorphic with the intermediate Jacobian of the given intersection of quadrics ([**9**, **16**]).

All these varieties may be realized as zero loci of sections of certain homogeneous vector bundles over Grassmannians ([**1**, **18**]). This circumstance makes the Schubert calculus relevant, for instance, in computing Chern numbers; it also reduces questions about connectivity, regularity, etc., as well as deformations to questions about the cohomology of homogeneous vector bundles.

Our main concern will be to set up a general framework for a calculus with weights, such that the theorem of Bott [**5**] become expressive in this context—a perspective we initially used in [**4**].

Specific computations enabled Wehler to deal with small deformations of Fano surfaces: he showed, namely, that all of them are induced by deformations of the corresponding cubic threefolds ([**21**]). This result is here extended to a large class (Theorem 5.3). Similarly (Theorem 4.1), we extend (and give an alternative proof for) the connectedness result of Barth and Van de Ven concerning lines on hypersurfaces ([**2**]).

1. Varieties of k-planes. We shall consider projective k-planes contained in a complete intersection $S = S_n(d)$ of dimension n and multidegree $d = (d_1, \dots, d_r)$ in the projective space $P = P_{n+r}$ over the complex field C.

Let $\mathcal{O}_P(m)$ denote the mth tensor power of the hyperplane line bundle on P and let S be given as the variety of zeros $Z(s) = S$ of a section $s \in H^0(P, E)$, where $E = \bigoplus_{t=1}^{r} \mathcal{O}_P(d_t)$.

Denote by $G = G(k+1, n+r+1)$ the Grassmann variety of projective k-planes in P, i.e. $(k+1)$-planes in C^{n+r+1}, and let $\Gamma \subset P \times G$ be the subvariety defined by the incidence relation $\Gamma = \{(x, \pi) | x \in \pi\}$, with canonical projections:

$$\begin{array}{ccc} & \Gamma & \\ {}^{p}\swarrow & & \searrow^{q} \\ P & & G \end{array}$$

p represents Γ as a $G(k, n+r)$-bundle over P and q represents Γ as a P_k-bundle over G. Accordingly, we have isomorphisms: $H^0(P, E) \xrightarrow{\sim} H^0(\Gamma, p^*E) \xrightarrow{\sim} H^0(G, q_*p^*E)$.

If $0 \to \tau = \tau_{k+1} \to G \times C^{n+r+1} \to Q = Q_{n+r-k} \to 0$ denotes the canonical exact sequence of vector bundles over the Grassmannian G, we have a natural identification: $q_*p^*\mathcal{O}_P(m) = S^m(\tau^*) =$ the mth symmetric tensor power of the dual tautological bundle.

Put $\mathscr{E} = q_* p^* E$.

Let Φ be the isomorphism indicated above:

$$\Phi : H^0(P, E) \xrightarrow{\sim} H^0(G, \mathscr{E}) = \bigoplus_{t=1}^{r} H^0(G, S^{d_t}(\tau^*)).$$

To $s \in H^0(P, E)$, defining the variety $Z(s) = S$, we thus associate $\Phi(s) \in H^0(G, \mathscr{E})$, defining the variety of zeros $Z(\Phi(s)) = F_k(S) = F$, which consists of all k-planes contained in $S \subset P$.

REMARK 1.1. The rank of $\mathscr{E}$ is $\sum_{t=1}^{r} \binom{d_t+k}{k}$, and we expect F to be non-empty for $\dim G - \operatorname{rk} \mathscr{E} \geq 0$, i.e. for

$$(A_0) \qquad (k+1)(n+r-k) - \sum_{t=1}^{r} \binom{d_t+k}{k} \geq 0.$$

This will presently be seen to be true, provided S is not a quadric, in which case the assumption $n \geq 2k$ is needed. Note that, if S is neither a quadric, nor a linear space, condition (A_0) already implies $n > 2k$.

2. Dimension and smoothness in the generic case. Let $V = H^0(P, E)$ and consider the subvariety $I \subset G \times V$ defined by: $I = \{(s, \pi) | s|_\pi = 0\}$, with projections:

$$G \xleftarrow{\ \alpha\ } I \xrightarrow{\ \beta\ } V$$

α represents I as a sub-vector-bundle of $G \times V \to G$, which shows that I is smooth, while β is proper and the fibre over $s \in V$ is precisely $Z(\Phi(s))$.

Confirming our Remark 1.1, we have:

PROPOSITION 2.1. *If* $\dim G - \operatorname{rk} \mathscr{E} \geq 0$, β *is onto, provided* $n \geq 2k$ *in the case of quadrics.*

Proof. If we find a k-plane π in S, with S smooth along π, and such that the normal bundle $N_{\pi/S}$ has $H^1(\pi, N_{\pi/S}) = 0$, the proposition will follow from Kodaira's criterion for stability of compact submanifolds **[15]**.

We consider the exact sequence:

$$(1) \qquad 0 \to N_{\pi/S} \to N_{\pi/P} \to N_{S/P}|_\pi \to 0.$$

We have:

$$N_{\pi/P} = \bigoplus^{n+r-k} \mathscr{O}_\pi(1) \quad \text{and} \quad N_{S/P}|\pi = \bigoplus_{t=1}^{r} \mathscr{O}_\pi(d_t).$$

Let π be given by $x_{k+1} = \cdots = x_{n+r} = 0$, for homogeneous coordinates $(x_0 \colon \cdots \colon x_{n+r})$, so that $s \in H^0(P, E)$, $s|_\pi = 0$ will be given by r homogeneous polynomials $(s_1, \dots, s_r)$ of the form

$$s_t = \sum_{i=k+1}^{n+r} x_i \cdot p_t^{(i)} + r_t \tag{2}$$

where

$$p_t^{(i)} = \sum_\mu c_{t\mu}^{(i)} \cdot x^\mu, \tag{3}$$

$$\mu = (\mu_0, \dots, \mu_k),\ x^\mu = x_0^{\mu} \cdots x_k^{\mu_k},\ |\mu| = \mu_0 + \cdots + \mu_k = d_t - 1$$

and every monomial in r_t contains a product $x_i x_j$ with $i \geq j > k$.

Since we may suppose $n \geq 2k$, the condition that S be smooth along π is satisfied for generic s. (For example, the following matrix of partial derivatives

$$\left(\frac{\partial s_t}{\partial x_i}(x)\right)_{i \geq k+1}, \qquad x \in \pi$$

may be produced:

$$\begin{pmatrix} x_0^{d_1-1} & \dots & x_k^{d_1-1} & 0 & 0 & \dots \\ 0 & x_0^{d_2-1} & \dots & x_k^{d_2-1} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0\dots 0 & x_0^{d_r-1} & \dots & \dots & x_k^{d_r-1} \dots \end{pmatrix}).$$

We represent a global section of $N_{\pi/P}$ by a matrix

$$a = (a_{ij})_{0 \leq j \leq k < i \leq n+r},$$

so that the map $H^0(N_{\pi/P}) \xrightarrow{\sigma} H^0(N_{S/P}|\pi)$ induced from (1) is described by

$$A \to \left(\sum_{j \leq k < i} a_{ij} \cdot p_t^{(i)} \cdot x_j\right)_{1 \leq t \leq r} \in H^0\left(\bigoplus_{t=1}^{r} \mathscr{O}_\pi(d_t)\right). \tag{4}$$

Looking at monomial coefficients in (4) and using (3), one obtains that σ is a surjection if and only if the linear system (with indeterminates a_{ij})

$$(5) \quad \sum_{j\leq k<i} a_{ij}\cdot c^{(i)}_{t,\nu(j)} = 0, \qquad t=1,\dots,r,\ \nu=(\nu_0,\dots,\nu_k),\ |\nu|=d_t$$

where

$$\nu(j)=\nu-(0,\dots,1,0,\dots,0) \quad \text{and} \quad c^{(i)}_{t,\nu(j)}=0 \quad \text{for } \nu(j) \text{ improper}$$

has maximal rank, namely $\sum_{t=1}^r \binom{d_t+k}{k} = \operatorname{rk}\mathscr{E} = R$.

For generic s, this is actually the case. To see it, consider the lexicographic order on the set of column-indices $\{(i,j)|0\leq j\leq k<i\leq n+r\}$ and look at the $R\times R$ matrix given by the first R columns. Its determinant is a polynomial in $c^{(i)}_{t,\mu}$, with $|\mu|=d_t-1$. It is not difficult to check that this polynomial is different from zero. Consider, for example, the lexicographic order on the set of indices (i,t,μ) affecting the coefficients $c^{(i)}_{t,\mu}$. Now order the monomials in the expression of the above determinant according to the rule: $m_1>m_2$ if the smallest index (i,t,μ) for which $c^{(i)}_{t,\mu}$ occurs in m_1 with exponent p_1 and in m_2 with exponent $p_2\neq p_1$, we have $p_1>p_2$. The greatest monomial in this ordering will have perforce coefficient 1 or -1, since in each row, the choice of $c^{(i)}_{t,\mu}$ entering this monomial is prescribed.

Thus, for generic s, S is smooth along π and $H^1(\pi, N_{\pi/S})=0$.

Corollary 2.2. *The projective k-planes contained in a generic complete intersection $S_n(d)$ of dimension n and multidegree $d=(d_1,\dots,d_r)$ in P_{n+r} define a smooth subvariety $F_k(S_n(d))$ of $G(k+1,n+r+1)$ of codimension $\sum_{t=1}^r\binom{d_t+k}{k}$, provided that $(k+1)(n+r-k)\geq\sum_{t=1}^r\binom{d_t+k}{k}$ and $S_n(d)$ is not quadric, in which last case $n\geq 2k$ is required.*

Remark 2.3. The variety of lines $F_1(S_n(3))$ of a cubic hypersurface $S_n(3)\subset P_{n+1}$ is smooth if the cubic is smooth, but in general, the smoothness of $S_n(d)$ does not imply that of $F_k(S_n(d))$ (cf. **[12]**, **[18]**).

3. Weights. In what follows, we take $\dim G\geq \operatorname{rk}\mathscr{E}$ (and $n\geq 2k$ for quadrics), and assume the complete intersection $S=S_n(d)$ to be such that the codimension of $F = F_k(S)$ in $G = G(k+1, n+r+1)$ be precisely rk $\mathcal{E}$. Generically, this is the case (Corollary 2.2).

Let J_F denote the sheaf of ideals defining F on G.

The Koszul complex of (the section of $\mathcal{E} = q_* p^* E$ defining) J_F gives, for any holomorphic vector bundle M on G, spectral sequences:

$$(6) \qquad H^p\left(G, M \otimes \bigwedge^{q} \mathcal{E}^*\right) \Rightarrow H^{p-q}(F, M|_F),$$

$$H^p\left(G, M \otimes \bigwedge^{q+1} \mathcal{E}^*\right) \Rightarrow H^{p-q}(G, M \otimes J_F), \qquad q \geq 0.$$

If M is a homogeneous vector bundle, we may use the theorem of Bott [**5**, Th. IV$'$] for dealing with the groups on the left. To this purpose, we use the following description of the Grassmann manifold $G(k+1, n+r+1)$:

$\mathrm{SL}(n+r+1, C)$, which is the universal cover of $\mathrm{Aut}(P_{n+r}) = \mathrm{PGL}(n+r+1, C)$, has Lie algebra $\mathrm{sl}(n+r+1, C) = \{A = (a_{ij}) | \operatorname{tr} A = 0\}$. Take as Cartan subalgebra $h = \{A | a_{ij} = 0 \text{ for } i \neq j\}$. This gives root spaces $L_{ij} = C \cdot E_{ij}$ $(i \neq j)$ where E_{ij} has zeros everywhere except the (i, j) entry.

The Killing form identifies the corresponding roots α_{ij} with $E_{ii} - E_{jj}$ $(i \neq j)$ so that the root system A_{n+r} may be viewed as embedded in a euclidean space with orthonormal basis $e_i = E_{ii}$, $i = 1, \dots, n+r+1$, the roots being represented by vectors α orthogonal to $e_1 + \dots + e_{n+r+1}$ and of square-norm $(\alpha, \alpha) = 2$ (cf. [**36**, p. 64]).

Put $\alpha_s = \alpha_{s+1,s} = e_{s+1} - e_s$. $\{\alpha_s | s = 1, \dots, n+r\}$ gives a basis of the root system A_{n+r}.

If U_{k+1} denotes the subgroup of $\mathrm{SL}(n+r+1, C)$ consisting of the transformations which preserve the linear space $\{x_{k+2} = \dots = x_{n+r+1} = 0\} \subset C^{n+r+1}$ with coordinates $(x_1, \dots, x_{n+r+1})$, the Lie algebra u_{k+1} of U_{k+1} will contain h, all the negative roots $(\alpha_{ij}, i < j)$ and all positive roots not involving α_{k+1} when expressed in terms of the given basis.

We have $G(k+1, n+r+1) = \mathrm{SL}(n+r+1, C)/U_{k+1}$, which is the description we shall use.

Let us now investigate the weights associated to various homogeneous vector bundles over $G = G(k+1, n+r+1)$.

Such a bundle is defined by a holomorphic representation $\rho\colon U_{k+1} \to \mathrm{GL}(N, C)$ and the weights are taken with respect to h.

(a) Consider first the tautological bundle τ over G. It corresponds to the natural representation of U_{k+1} on the invariant subspace $\{x_{k+2} = \cdots = x_{n+r+1} = 0\}$.

Let β_s denote the weight characterized by

$$(\beta_s, \alpha_t) = 0 \text{ for } t \neq s \quad \text{and} \quad (\beta_s, \alpha_s) = \tfrac{1}{2}(\alpha_s, \alpha_s) = 1.$$

An elementary computation then gives the weights of

$$\tau_{k+1}\colon t_1 = -\beta_1, t_2 = \beta_1 - \beta_2, \dots, t_{k+1} = \beta_k - \beta_{k+1}.$$

(b) The line bundle $\det(\tau^*_{k+1})$, which gives the Plücker embedding of $G(k+1, n+r+1)$, has therefore associated weight: β_{k+1}.

(c) The tangent bundle of G: θ_G is given by the adjoint representation of U_{k+1} on $\mathrm{sl}(n+r+1, C)/u_{k+1}$. Consequently, its weights are precisely the positive roots involving α_{k+1} in their expression, namely α_{ij}, $i > k+1 \geq j$.

(d) $\mathscr{E}^* = \bigoplus_{m=1}^{r} S^{d_m}(\tau^*_{k+1})^*$ and (a) immediately gives that its weights are of the form:

$$\sum_{i=1}^{k+1} a_i t_i = (a_2 - a_1)\beta_1 + (a_3 - a_2)\beta_2 + \cdots + (a_{k+1} - a_k)\beta_k - a_{k+1}\beta_{k+1}$$

with $a_i \in N$, $\sum_{i=1}^{k+1} a_i = d_m$ for some $m \leq r$.

We now draw up a table of scalar products of positive roots and various weights, which will be relevant in estimating indices of weights.

δ is half the sum of all positive roots.

$\omega = \sum_{i=1}^{k+1} a_i t_i$, $a_i \in Z$ (motivated by (d) above and the spectral sequences (6)).

$1 \leq m \leq k$.

We anticipate here the type of reasoning to be used in the sequel. Given a homogeneous vector bundle over G, defined by a representation $U_{k+1} \to \mathrm{GL}(N, C)$, we first produce a filtration with consecutive quotients corresponding to irreducible representations of U_{k+1}. Such an irreducible representation determines a highest weight, say ρ. This ρ has to be one of the weights of the original representation and further satisfy $(\rho, \alpha_s) \geq 0$ for all $s \neq k+1$.

In our computations ρ will be either of type ω or $\omega + \alpha_{n+r+1,m}$ $(m \leq k+1)$.

In order to obtain the vanishing of $H^s(G, \rho)$, it will suffice either to ascertain the singularity of the weight $\rho + \delta$ or to prove: $s < \mathrm{index}(\rho + \delta)$.

In this context, the main feature of our table of products is that $(\alpha_{t,m}, \rho + \delta)$ increases by 1 when t increases by 1, except the last step for $\rho = \omega + \alpha_{n+r+1,m}$ $(m \leq k+1)$.

TABLE 1

	Conditions	δ	ω	$\alpha_{n+r+1,m}$	$\alpha_{n+r+1,k+1}$
α_p	$p \neq m-1, m$ $p \leq k$	1	$a_{p+1} - a_p$	0	$p < k$ 0 $p = k$ −1
α_{m-1}		1	$a_m - a_{m-1}$	−1	0
α_m		1	$a_{m+1} - a_m$	1	$m < k$ 0 $m = k$ −1
α_q	$k+1 \leq q \leq n+r$	1	0	$q < n+r$ 0 $q = n+r$ 1	0 1
$\alpha_{t,k+1}$	$t > k+1$	$t-k-1$	$-a_{k+1}$	$t < n+r+1$ 0 $t = n+r+1$ 1	1 2
$\alpha_{t,m}$	$t > k+1$	$t-m$	$-a_m$	$t < n+r+1$ 1 $t = n+r+1$ 2	0 1
$\alpha_{t,p}$	$t > k+1 > p$ $p \neq m$	$t-p$	$-a_p$	$t < n+r+1$ 0 $t = n+r+1$ 1	0 1

Note also that for $1 \leq p \leq k+1$, $(\alpha_{k+2,p}, \rho+\delta) < (\alpha_{k+2,p-1}, \rho+\delta)$ since $(\alpha_{p-1}, \rho) \geq 0$.

4. Connectedness. Suppose

$$(\mathrm{A}_1) \qquad \dim F = \dim G - \operatorname{rk} \mathscr{E} \geq 1.$$

F is connected if and only if $H^0(\mathscr{O}_F) = C$.

We have $H^s(G, \bigwedge^s \mathscr{E}^*) \Rightarrow H^0(\mathscr{O}_F)$; therefore the vanishing of $H^s(G, \bigwedge^s \mathscr{E}^*)$ for $s > 0$ will imply the connectedness of F.

According to our method, described at the end of §3, we examine $H^s(G, \rho)$, with ρ an irreducible representation of U_{k+1} with highest weight (again denoted ρ) among the weights of $\bigwedge^s \mathscr{E}^*$. Thus $\rho = \omega = \sum_{i=1}^{k+1} a_i t_i$ and we know (see Table 1):

(1) $a_{k+1} \geq a_k \geq \cdots \geq a_q \geq 0$;

(2) $\rho + \delta$ is either singular or of index $u(n+r-k)$, $1 \leq u \leq k$ ($u = k+1$ is excluded because $\operatorname{rk} \mathscr{E} < \dim G$).

Suppose therefore $s = u(n+r-k)$.

For $\rho + \delta$ to have index s, we must have $(\alpha_{t,p}, \rho+\delta) > 0$ for $p = 1, \dots, k+1-u$; in particular: $a_{k+1-u} \leq u$.

Now remember that ρ is a weight of $\bigwedge^s \mathscr{E}^*$, thus a sum of s weights of $\mathscr{E}^*$, each weight counted at most as many times as the dimension of its eigenspace. There are (multiplicities included) $\sum_{m=1}^{r}\binom{d_m+u-1}{u-1}$ weights involving only t_i, $i > k+1-u$. Adding any other weight increases some a_j, $j \leq k+1-u$; thus we must not add more than $u(k+1-u)$ such weights. This will be clearly impossible if n satisfies the following conditions:

$$(\mathrm{C}_u) \qquad \sum_{m=1}^{r}\binom{d_m+u-1}{u-1} + u(k+1-u) < u(n+r-k) = s$$

with u running from 1 to k.

Now, use (repeatedly) the formula:

$$(7) \qquad \frac{1}{q+1}\binom{d_m+q}{q} - \frac{1}{q}\binom{d_m+q-1}{q-1} = \frac{d_m-1}{q(q+1)}\binom{d_m+q-1}{q-1}$$

to show that if some $d_m \geq 3$, or at least two degrees in d are ≥ 2, then (C_u), $1 \leq u \leq k$, is a consequence of our assumption (A_1). Note that (C_1) reads: $n > 2k$.

We have therefore:

THEOREM 4.1. *Let $S = S_n(d_1, \ldots, d_r)$ be a complete intersection in P_{n+r} and $F = F_k(S)$ its variety of projective k-planes. Suppose*

$$\dim F = (k+1)(n+r-k) - \sum_{m=1}^{r}\binom{d_m+k}{k} \geq 1,$$

or, in case S is a quadric, suppose $n > 2k$.

Then F is connected.

REMARK 4.2. For a smooth quadric $S = S_{2k}(2)$, $F_k(S)$ consists of two isomorphic (hermitian symmetric) connected components.

This should rather be viewed as the exception which confirms the rule: $S_{2k}(2)$ is a homogeneous (hermitian symmetric) space (of rank one) in its own right, and the generating k-planes of the two families in $F_k(S)$ correspond to Schubert cycles which are not homologically equivalent.

REMARK 4.3. There is a simple formula for the canonical bundle of $F = F_k(S_n(d))$, when smooth.

Let $\mathcal{O}_G(1)$ denote the positive generator of $\mathrm{Pic}(G)$, restricting to $\mathcal{O}_F(1)$ on F.

Set

$$K = \sum_{m=1}^{r}\binom{d_m+k}{k+1} - (n+r+1).$$

Then $K_F = \mathcal{O}_F(K)$.

5. Deformations. In this section we assume that $F = F_k(S_n(d))$ has the "right" codimension and dimension at least two:

$$(\mathrm{A}_2) \qquad \dim F = \dim G - \operatorname{rk} \mathscr{E} \geq 2.$$

Our purpose is to produce conditions on (n, d, k) which ensure the completeness of the natural deformation of F, parametrized by a neighborhood of the section $\Phi(s) \in H^0(G, \mathscr{E})$ defining F. Notice that the family of complete intersections to which $S_n(d)$ belongs (parametrized by a neighbourhood of $s \in H^0(P, E) \cong H^0(G, \mathscr{E})$, i.e. the "same" base) is itself complete (see **[4]**, **[17]**, **[21]**).

A sufficient condition for completeness is the vanishing of $H^1(G, \mathscr{E} \otimes J_F)$ and $H^1(F, \theta_G|_F)$. This is a general result for varieties defined by sections in a vector bundle (see **[21]**).

We look therefore at the spectral sequences (6) abutting to the above two groups.

(5.1) Take first $H^s(G, \mathscr{E} \otimes \bigwedge^s \mathscr{E}^*)$, $s \geq 1$.

We obtain vanishing conditions for these groups as we did for $H^s(G, \bigwedge^s \mathscr{E}^*)$ in §4.

Let $D = \max_{1 \leq m \leq r}(d_m)$. Filtering and taking highest weights will produce as above weights $\rho = \omega = \sum_{i=1}^{k+1} a_i t_i$, with $(\alpha_p, \rho) \geq 0$ for $p \leq k$.

Since ρ is the sum of a weight ω' of $\mathscr{E}$ and a weight ω'' of $\bigwedge^s \mathscr{E}^*$, adding ω' to $\omega'' = \sum_{i=1}^{k+1} a_i'' t_i$ decreases some of its coefficients a_i'', diminishing their sum by at most D.

This means that our sufficient conditions (C_u), $1 \leq u \leq k$, for the vanishing of $H^s(G, \bigwedge^s \mathscr{E}^*)$, $s \geq 1$, become, by the same type of reasoning, sufficient conditions (C_u^D), $1 \leq u \leq k$, for the vanishing of $H^s(G, \mathscr{E} \otimes \bigwedge^s \mathscr{E}^*)$, once we add D to the left hand side of each inequality:

$$(\mathrm{C}_u^D) \qquad \sum_{m=1}^{r} \binom{d_m + u - 1}{u - 1} + u(k + 1 - u) + D < u(n + r - k).$$

(5.2) Consider now $H^{s+1}(G, \theta_G \otimes \bigwedge^s \mathscr{E}^*)$, $s \geq 0$. For $s = 0$, we have $H^1(G, \theta_G) = 0$, because G is rigid **[5]**. Suppose $s \geq 1$.

Again, using a filtration (actually, the representations we are dealing with are all completely reducible) and successive quotients corresponding to irreducible representations of U_{k+1}, we find that the highest weight ρ associated to such a representation is necessarily of the form $\rho = \omega + \alpha_{t,m}$, with $\omega = \sum_{i=1}^{k+1} a_i t_i$ a weight of $\bigwedge^s \mathscr{E}^*$,

$t > k+1 \geq m$ (cf. §3 (c)), and further conditions: $(\rho, \alpha_q) \geq 0$ for all $q \neq k+1$, which imply in particular $t = n+r+1$.

Take therefore $\rho = \omega + \alpha_{n+r+1,m}$ $(m \leq k+1)$ and consider the series of integers: $(\rho+\delta, \alpha_{t,p})$ with $p \leq k+1$ fixed and t increasing from $k+2$ to $n+r+1$. If $\rho+\delta$ is non-singular, this series of non-zero integers will keep the same sign, except possibly at the last step $t = n+r+1$, when it might "jump" precisely over zero (see Table 1).

Now let p decrease from $k+1$ to 1 and notice the relations of the starting values in each series:

$$(\rho+\delta, \alpha_{k+2,k+1}) < (\rho+\delta, \alpha_{k+2,k}) < \cdots < (\rho+\delta, \alpha_{k+2,1}).$$

This means that we might encounter non-vanishing cohomology $H^{s+1}(G, \rho)$ at most for $s+1$ or s a multiple of $n+r-k$, say $u(n+r-k)$ $(u < k+1$ by our assumption $\operatorname{rk} \mathscr{E} \leq \dim G - 2)$.

For the coefficients a_i in $\omega = \sum_{i=1}^{k+1} a_i t_i$, we have either:

(1) $a_{k+1} > a_k \geq \cdots \geq a_1$ for $m = k+1$, or

(2) $a_{k+1} \geq \cdots \geq a_{m+1}$; $a_{m+1} + 1 \geq a_m > a_{m-1} \geq \cdots \geq a_1$ for $m \leq k$.

Since ω is a weight of $\bigwedge^s \mathscr{E}^*$, it appears that (C_u^2) above is a sufficient condition for the vanishing of $H^{s+1}(G, \rho)$.

Now, one may verify that the combination of (A_2) and (C_1^D) above implies (C_u^D) for $1 \leq u \leq k$.

First, suppose $d_m \geq 2$, which is no restriction of generality. Making use of the identity (7) in §4 and the fact that the right hand side in (7) clearly increases with q, the following implications obtain:

(i) If $k \geq 2$, $(\mathrm{A}_2) \Rightarrow (\mathrm{C}_k^D)$ as soon as $\sum_{m=1}^r (d_m^2 - 1) > 3D + 2$, i.e. $d \neq (2), (2, 2), (3), (2, 3)$; and for $n > 6$ also for $d = (2, 3)$.

(ii) If $u > 1$, $(\mathrm{C}_{u+1}^D) \Rightarrow (\mathrm{C}_u^D)$ for $\sum_{m=1}^r (d_m^2 - 1) \geq D + 6$, i.e. $d \neq (2), (2, 2), (3)$.

Finally, for $d = (2), (2, 2), (3)$ or $(2, 3)$, a direct check shows that $(\mathrm{A}_2)\&(\mathrm{C}_1^D) \Rightarrow (\mathrm{C}_u^D)$.

Summing-up, we obtain:

THEOREM 5.3. *Let $S = S_n(d_1, \ldots, d_r)$ be a complete intersection in P_{n+r} and suppose that its variety of k-planes $F = F_k(S)$ satisfies*

$$(\mathrm{A}_2) \qquad \dim F = (k+1)(n+r-k) - \sum_{m=1}^{r} \binom{d_m + k}{k} \geq 2.$$

If $n > 2k + D$, where $D = \max_{1 \leq m \leq r}(d_m)$, then every small deformation of F is induced by a (small) deformation of S.

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